Palindromic Ramsey Theory

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Abstract

In this paper the researcher studies palindromic ramsey theory on the integers. Given an integer coloring of any length the researcher studies the restrictions imposed by mirroring the first half of the coloring onto the second half. Let $pdw(k_1, k_2, \ldots, k_r, r)$ be defined as the smallest integer n, such that every palindromic r-coloring contains some k_i-term monochromatic progression. This paper includes proofs for lower bounds on palindromic colorings and explicit colorings that avoid a k_i-term monochromatic progression are given. New and exact values are computed for $pdw(k_1, k_2, \ldots, k_r, r)$ and this paper includes the algorithm used. New open questions are proposed in the conclusion.

1 Introduction

Ramsey theory began in a seminal 1928 paper [10] by Frank Ramsey. The theory has a nice philosophical interpretation: one is searching for order in randomness. In particular, Ramsey theory attempts to identify which constraints one can place on a set to guarantee that a certain property holds. There has been extensive research done within the field, dealing with various sets and various constraints. Techniques used in researching Ramsey theory have mostly been counting arguments such as the Pigeonhole principle [5], that have allowed mathematicians to put upper bounds on the growth rate of Ramsey problems. A classic example of the Pigeon Hole principle is that if you have thirteen people in a room, you know that at least two people out of the thirteen have the same birthday month. More generally, the Pigeonhole Principle claims that there is a size of a set for which certain conditions will have to hold, like the example above. This is just one of the tools used in order to help put bounds on the problem.

Consequences of Ramsey theory stretch into numerous other mathematical fields such as geometry, number theory, and graph theory [11], all of which have major consequences in technological advancements. Studying this phenomenon can allow mathematicians to better understand random structures that appear especially often in computer science [7]. Finding patterns in randomness quickly and efficiently is one of the most popular research topics in computer science given that it could provide insight into the P vs. NP problems, such as the traveling salesman problem [6]. The Traveling Salesman Problem asks, when given a list of cities and the distance between the cities, what is the shortest route that visits each city only once and returns back to the original city. These problems seek to identify how "hard" a problem is to solve with a computer, and have challenged researchers for decades. The Traveling Salesman problem alone has applications in general planning, studying DNA sequences, and the building of modern microchips. [9] The more that is known about Ramsey theory the more tools researchers have to identify the complexity of such problems. There are many useful tools for studying such problems that can be found in [3] and [4]. These are classic resources for approaching combinatorial problems that contain the basics of studying such problems. Erdos and Renyi [2] studied the evolution of random graphs and used some interesting techniques that today are crucial tools in combinatorics research. The main contribution of Erdos and Renyi is the use of statistical and heuristic arguments for being able to say that certain conditions will hold. Calculating the probability that a certain condition will hold and figuring out how many different ways this condition could occur is key to studying random structures. Approaching problems in this manner is extremely insightful because it helps mathematicians understand and have reason for why there must be some order in any random structure. These methods are employed in the research done on the problem. Some useful notation is as follows.

Notation 1.1.

1. A k-term monochromatic arithmetic progression will be abbreviated k-TMAP.

2. $[N] := \{1, 2, \ldots, N\}$.

3. "Arithmetic progression" will be abbreviated AP.

4. A bound on a palindromic coloring for 2 colors and k_1 and k_2 length progressions that must be avoided in each color respectively will be denoted as $pdw(k_1, k_2, 2)$.

5. A bound on a palindromic coloring for r colors and a k length progression that must be avoided in all colors will be denoted as $pdw(k, r)$.

6. A bound on a palindromic coloring for r colors and a k_i length progression that must be avoided in a specific color will be denoted as $pdw(k_1, k_2, \ldots, k_r, r)$.

Theorem 1.2. (van der Waerden)

For every $k, r \in \mathbb{N}$ there is some $N = N(k, r)$ such that every r-coloring of $[N]$ contains a k-TMAP. The smallest such N will be denoted $w(k, r)$.

$$\underbrace{\underline{\bullet}\ \underline{\bullet}\ \underline{\dots}\ \underline{\bullet}}_{d\text{ times}}\ \ \underbrace{\overline{\bullet}\ \overline{\bullet}\ \overline{\dots}\ \overline{\bullet}}_{d\text{ times}}\ \ \underbrace{\underline{\bullet}\ \underline{\bullet}\ \underline{\dots}\ \underline{\bullet}}_{d\text{ times}}\ \ \underbrace{\overline{\bullet}\ \overline{\bullet}\ \overline{\dots}\ \overline{\bullet}}_{d\text{ times}}\ \ \cdots \tag{1}$$

This coloring has several possible MAPs, one of which is the d-TMAP for the underline and overline colorings. Depending on how long this pattern repeats, it is known that one can find k-TMAPs for terms between the groups of colorings in the above figure. For example, the first term is underlined and there is another underlined term of distance $2d$ away from the first term, and then another that is $2d$ away from that term, and so on until the pattern ends.

There has been very little work done on Palindromic Ramsey theory. This is a subset of basic Ramsey theory and the research strongly relates to the work done by Van der Waerden [12]. Van der Waerden's work (1.2) is a strong foundation from which the researcher will seek to establish bounds on the growth rate of the studied problem. Van der Waerden established some loose bounds on general Ramsey theory coloring problems, which informs the researcher of when one can expect to find a k-TMAP within any random coloring of a set.

The problem studied by the researcher was first introduced in a paper by Ahmed, Kullmann and Snevily [1] and deals with Palindromic Ramsey theory. By applying van der Waerden's theorem one can show that there exists some N for which every palindromic coloring of the set $\{1, 2, 3, \dots, N\}$ has a k-TMAP. If there exists such an N, then certainly there exists a smallest N with this property. In this project the researcher works to find bounds on this smallest N.

Definition 1.3. A **palindromic coloring** of a set is where the first half of the coloring is reflected over the midpoint, thus imposing a level of order and specificity to a random coloring with no restrictions. An example of such a coloring is below.

$$\underline{1}\ \ \overline{2}\ \ \underline{3}\ \ 4\ \ 5\ \ \underline{6}\ \ \overline{7}\ \ \underline{8} \tag{2}$$

In [1], Ahmed, Kullmann, and Snevily computed some bounds for palindromic colorings of the form $pdw(3, k_2, 2)$ for small n. In the following section, the researcher proves a general lower bound and uses computed values from [1] as well as computes new values to study the trend of valid colorings.

2 Methods

In order to be able to find bounds on the growth rate of when palindromic colorings must contain a k-TMAP, it is important to understand how the two mirrored halves interact with one another. Throughout the research process some interesting behaviors of palindromic colorings that severely hinder the growth rate of $pdw(k_1, k_2, \ldots, k_r, r)$ were discovered.

Lemma 2.1. Given any even length set colored palindromically, if one color contains within its set the two center elements and no other elements that are adjacent, then there will exist no k-TMAP in that color that contains the center elements as middle elements of the k-TMAP.

Proof of Lemma 2.1. It is known that in any k-TMAP where $k \geq 3$, there is a 3-TMAP within the k-TMAP. Therefore, showing that you can never construct a 3-TMAP containing the center elements of the coloring as middle elements of the k-TMAP, then this excludes the possibility of having larger k-TMAPs that work with the given constrictions of a palindromic coloring. For any 3-TMAP, the end points of the AP must be of different parity when numbering the integers in the method used in the coloring below. However, in palindromic colorings of even length, when you mirror an element in the first half to the second half, the parity of the related element remains the same. Consider the coloring below, one can observe that 3 is the mirroring of -3 and 6 is the mirroring of -6, and so on.

$$\mathbf{-8}\quad \textit{-7}\quad \mathbf{-6}\quad \textit{-5}\quad \textit{-4}\quad \mathbf{-3}\quad \textit{-2}\quad \mathbf{-1}\quad \mathbf{1}\quad \textit{2}\quad \mathbf{3}\quad \textit{4}\quad \textit{5}\quad \mathbf{6}\quad \textit{7}\quad \mathbf{8} \tag{3}$$

The only time one would use the center elements is if the coloring were to have two elements in both halves that are equidistant from the same center element in the same color. This means that the only elements in palindromic colorings that would ever contain the center elements as the middle of a 3-TMAP is two elements that are mirror images of one another or an element that is adjacent to the mirrored element. The lemma restricts monochromatically coloring elements that are adjacent to one another, if that same color is also used to color the center elements. It was mentioned earlier that the parity between two mirrored elements in an even length palindromic coloring is the same, thus there does not exist a center element in an even length coloring that is equidistant from two monochromatically colored elements. □

Corollary 2.2. Lemma 2.1 also implies, that in a coloring that has no adjacent monochromaticly colored elements, the distance between an element and its respective mirrored element is always odd. Thus, if in either half of the coloring the distance between monochromatically colored elements is even, then there cannot possibly be an AP that crosses the center, since the distance from any element to a similarly colored element in the other half of the coloring is odd.

It is important to note that this does not mean that a k-TMAP that goes across the center of the set cannot be found. In (3) it can be seen that -8, -3, and 3 make a 3-TMAP. Center elements may also be used as endpoints of a k-TMAP, for example in the above coloring -1, 3, and 6 make a 3-TMAP.

Lemma 2.3. For any palindromic coloring of even length where the number 1 is the first element in the second half of the palindrome, coloring numbers of the form n and $3n - 1$ in the same color creates a 4-TMAP, where $n \geq 1$.

Proof of Lemma 2.3. In even length palindromic colorings, the researcher showed that for any number $n \geq 1$, the mirrored element related to n is of distance $2n - 1$ away from n. In (3) this can be observed, for example the distance between 2 and -2 is $2(2) - 1 = 3$. The distance between two numbers n and $3n - 1$ is the difference between the values. So $3n - 1 - n = 2n - 1$. So by monochromatically coloring elements n and $3n - 1$ a 4-TMAP is created as a result of the palindrome. In (3), it can be seen that -5, -2, 2, and 5 are all colored blue and therefore create a 4-TMAP. □

Lemma 2.4. For any palindromic coloring of odd length where the number 0 is the center element of the palindrome, coloring numbers of the form n and $3n$ in the same color creates a 4-TMAP, where $n \geq 1$. The proof is the same as the one for Lemma 2.3.

A common method to approaching problems like these is to look for systematic ways to build colorings that avoid the given k-TMAPs. For general Ramsey theory, there currently aren't any good systematic constructions that provide good bounds for the problem. The major problem in looking for patterns when moving from avoiding a k-TMAP to a $(k + 1)$-TMAP, is that one can often find k-TMAPs within a good coloring that does not have a $(K + 1)$-TMAP. So patterns found in colorings that do not contain k-TMAPs are often not found in colorings not containing $(k + 1)$-TMAPs. After studying the palindromic colorings for sometime, it became evident that these colorings also do not have repeating patterns for increasing k. However, in the Results section below, Theorem 3.1 demonstrates how to construct systematic colorings that do not contain k-TMAPs. The theorem creates a general and currently best known lower bound for all colorings of the form $pdw(k, r)$.

The next approach was to use the algorithm to compute small values and look at the amount of valid colorings that exist for varying lenths n. A valid coloring is defined as one that does not contain a k-TMAP for any k. Specifically the researcher observed and recorded the data for colorings of the form $pdw(3, k_2, 2)$ so that it can be compared with the results from [1]. By studying the amount of valid colorings on n while fixing the values for $pdw(k_1, k_2, \ldots, k_r, r)$, it is possible to view the problem in a different light. The researcher can study the growth rate of the valid amount of colorings and observe the trend of how slowly the amount of valid colorings reduces as n increases.

3 Results

The above graph shows the amount of valid colorings for $pdw(3,4,2)$ on a set of n integers. An interesting property that can be noticed from this graph is that the amount of valid colorings on a set of n integers corresponds to the parity of n. All odd length colorings follow one trend, while even length colorings follow a different trend. In general, it is noted in [1] that there is a gap between the size at which odd length colorings run out of valid colorings and when even length colorings no longer contain valid colorings. This is due to the fact that the center elements play an important role forcing k-TMAPs in even colorings more so than in odd colorings. Given that for even length colorings, the center element is mirrored, and so by definition is more likely to create more new K-TMAPs as is stated in Lemma 2.1. In odd length colorings, there is only one center element and this element is not subject to mirroring. This trend can be seen more clearly in the following graph. The researcher computed the values of n up to the largest that the computer could handle. For $pdw(3,8,2)$, which is represented in the graph below, finding the amount of valid colorings past $n = 41$ is extremely time consuming even with current technology given the complexity of Ramsey theory.

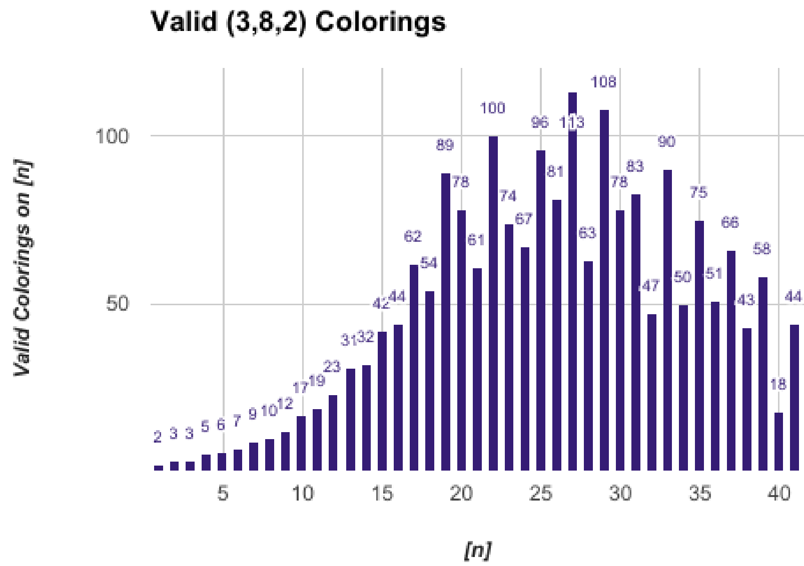

Figure 1: This graph illustrates that amount of valid colorings on a 2-colored set of n integers such that there are no 3-TMAPs in one color and no 8-TMAPs in the other.

In general, a trend can be seen that the amount of valid colorings peaks somewhat earlier than where one would expect $\frac{pdw(3,8,2)}{2}$ to be. This trend is likely to become more evident with the increasing length of k-TMAP's that the colorings must avoid. In comparison to the growth rate of general Ramsey theory, the growth rate of palindromic colorings as k increase seems to be much slower. The code for the program used was written in Java and can be found in the appendix below. The following theorem comes as a result of the culmination of the proven theorems and lemma previously in this paper.

Theorem 3.1. For all $r \geq 2$ where r is even, $pdw(k, r) > 2r(k-1)$.

$$\mathit{-8}\quad \mathbf{-7}\quad \underline{-6}\quad \overline{-5}\quad \mathit{-4}\quad \mathbf{-3}\quad \underline{-2}\quad \overline{-1}\qquad \overline{1}\quad \underline{2}\quad \mathbf{3}\quad \mathit{4}\quad \overline{5}\quad \underline{6}\quad \mathbf{7}\quad \mathit{8} \tag{4}$$

Proof of Theorem 3.1. When constructing palindromic colorings, one needs to avoid creating k-TMAPS within any half of the coloring and make certain that there aren't any k-TMAPs that come about as a result of the interaction between the two mirrored halves. Using the previous lemmas, it is known that in even length colorings every element is an odd distance away from it's mirrored counterpart. This implies that if every color only colors elements of one parity, then the mirrored elements will all be of an odd distance from any of the original elements. Then by coloring elements consecutively by color, like in (4), then all elements in one color or of the same parity in any half since r must be even. This method of coloring limits interaction between the mirrored halves because in any half, monochromatically colored elements have a distance of r between them. Also, this method of coloring insures there are no monochromatically colored elements adjacent to one another besides the center elements which allows us to apply Lemma 2.1 to such colorings. This allows the placement of $k-1$ copies of r uniquely colored elements and not have any k-TMAPs between the two mirrored halves. By placing only $k-1$ copies, there cannot be any k-TMAPs in any half of the coloring. Thus this method of construction contains no k-TMAPs and since one half of the palindrome is of length $r(k-1)$, the full length palindrome is of length $2r(k-1)$. □

4 Future Work

Theorem 3.1 is the best known general lower bound for $pdw(k, r)$. The researcher plans to continue researching bounds on the van der Waerden numbers and to further improve the current results. The next step is to find a systematic way of stacking permutations of $\{1, 2, \ldots, r\}$ to the current construction in order to improve Theorem 3.1.

Another approach that may yield interesting results is using the bounds on the general van der Waerden colorings to place bounds on palindromic colorings with more colors. It is very likely that there is a correlation between the growth rate of the regular van der Waerden numbers and the palindromic van der Waerden numbers.

5 Acknowledgments

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6 Appendix

```java
import java.util.Scanner;

/**
 * Driver class containing main() and taking user input
 */
public class Driver {

    public static void main(String[] args) {

        Scanner sc = new Scanner(System.in);
        int start, end, p, q;
        int mode = 0;
        SetOfNums nums;
        String rerun; // User input to rerun or exit the program
        String modeStr; // User input for mode
        String stepOne = "STEP ONE: Specify the initial set of numbers";
        String tempNewline = "\n"; // Cosmetic effect; changes like stepOne

        System.out.println("Arithmetic Progression Checker v2.3.1
                \n\nFor a set of sequential, contiguous integers " +
                "[n, n + 1, ..., k - 1, k], palindromic partitions without arithmetic progressions of length p
                and q, " +
                "if any, will be returned in 'mode 1'. In 'mode 2', only the first progressionless partition
                will be" + " returned.\n");
        System.out.println("Type \"mode 1\" or \"mode 2\" to select mode: ");

        do {
            modeStr = sc.nextLine();
            // Admonishment
            while (!(modeStr.equals("mode 1")) && !(modeStr.equals("mode 2")) && !(modeStr.length() == 6)) {
                System.out.println("Enter either \"mode 1\" (returns all progressionless partitions) or" +
                        " \"mode 2\" (returns only the first progressionless partition).");
                modeStr = sc.nextLine();
            }

            mode = Character.getNumericValue(modeStr.charAt(5));
            System.out.println("Running in mode " + mode + "." + tempNewline);
            System.out.println(stepOne);
            System.out.println("============================================\n");
            System.out.println("Enter starting number n (greater than 0): ");
            start = sc.nextInt();

            // Admonishment
            while (!(start > 0)) {
                System.out.println("The starting number must be greater than 0. Please re-enter.");
                start = sc.nextInt();
            }

            System.out.println("Enter ending number k (greater than 0): ");
            end = sc.nextInt();

            // Admonishment
            while (!(end > 0)) {
                System.out.println("The ending number must be greater than 0. Please re-enter.");
                end = sc.nextInt();
            }

            System.out.println("\nSTEP TWO: Specify the arithmetic progression lengths");
            System.out.println("====================================================\n");
            System.out.println("Enter first progression length p (greater than 0): ");
            p = sc.nextInt();

            // Admonishment
            while (!(p > 0)) {
                System.out.println("Progression lengths must be greater than 0. Please re-enter.");
                p = sc.nextInt();
            }

            System.out.println("Enter second progression length q (greater than 0): ");
            q = sc.nextInt();
```

```
            // Admonishment
            while (!(q > 0)) {
                System.out.println("Progression lengths must be greater than 0. Please re-enter.");
                q = sc.nextInt();
            }

            // Rock 'n roll
            nums = new SetOfNums(start, end, p, q, mode);
            System.out.println("\nWorking...");
            nums.buildPartitionSchemes();
            // Afterparty
            System.out.println(nums.getResults());
            System.out.println("Would you like to run another check (y/n)?");
            rerun = sc.next().toLowerCase();

            // Admonishment
            while (rerun.charAt(0) != 'y' && rerun.charAt(0) != 'n' &&
             rerun.charAt(0) != 'm' && rerun.length() != 6) {
                System.out.println("Please use either \"y\" or \"n\" to run another check or exit.");
                rerun = sc.next().toLowerCase();
            }

            stepOne = "\nSTEP ONE: Specify the initial set of numbers";
            tempNewline = "";
        }

        while (rerun.charAt(0) == 'y');
        System.out.println("Exiting...");
        sc.close();
    }
}
```

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