UNIVERSITY OF CALIFORNIA, SAN DIEGO

Flips and Juggles

A Dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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The Dissertation of Jay Cummings is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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University of California, San Diego

2016

DEDICATION

To mom and dad.

EPIGRAPH

We've taught you that the Earth is round,

That red and white make pink.

But something else, that matters more –

We've taught you how to think.

— Dr. Seuss

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It's been said that if you are the smartest person in a room, then you are in the wrong room; I'm lucky to have always been in the right rooms. And because of everyone mentioned thus far, every step has been a joy.

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Chapter 1 is a version of the material appearing in "Juggling card sequences", *arXiv preprint* arXiv:1504.01426 (2015), co-authored with Steve Butler, Fan Chung and Ron Graham. The author was the primary investigator and author of this paper.

Chapter 2 is a version of the material appearing in "Edge flipping in the complete graph", *Advances in Applied Mathematics* 69 (2015): 46-64, co-authored with Steve Butler, Fan Chung and Ron Graham. The author was the primary investigator and author of this paper.

ABSTRACT OF THE DISSERTATION

Flips and Juggles

by

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Doctor of Philosophy in Mathematics

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Professor Ron Graham, Chair
Professor Jacques Verstraëte, Co-Chair

In this dissertation we study juggling card sequences and edge flipping in graphs, as well as some related problems. Juggling patterns can be described by a sequence of cards which keep track of the relative order of the balls at each step. This interpretation has many algebraic and combinatorial properties, and we place particular focus on discovering connections between this model and other studied structures and sequences. We begin with the juggling card properties of traditional juggling patterns, and their enumerative connections to Stirling numbers. We then study the case where multiple balls are thrown at once, a problem with connections to

arc-labeled digraphs and boson normal ordering. Next we examine crossings in juggling cards. The first such case connects neatly to Dyck paths and Narayana numbers, while the later cases require new approaches. Lastly we examine a randomized model of juggling.

Edge flipping in graphs is a randomized coloring process on a graph. In particular: fix a graph G and repeatedly choose an edge from $E(G)$ (uniformly at random, with replacement). After each selection, with probability p color the vertices of the edge blue; otherwise color these vertices red. This induces a well-behaved random walk on the state space of all red/blue colorings of the complete graph and so has a stationary distribution. We derive this stationary distribution for the complete graph. We then study more classes of graphs and asymptotics of some special cases. We conclude with two related problems. First we study graph builds, which is is a graph construction interpretation of edge flipping which has recently garnered interest in its own right. We then introduce hyperedge flipping in t-uniform hypergraphs and discuss some future work.

Chapter 1

Juggling Cards

1.1 Introduction

It is traditional for mathematically-inclined jugglers to represent various juggling patterns by sequences $T = (t_1, t_2, \ldots, t_n)$ where the t_i are natural numbers. The connection to juggling being that at time i a juggling ball is thrown, and that juggling ball is thrown high enough that it comes down t_i time steps later at time $i + t_i$. This is usually drawn as in Figure 1.1.

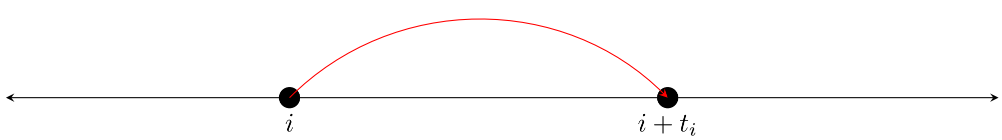


Figure 1.1: A juggling throw of distance t_i

The usual convention is that the sequence T is repeated indefinitely; i.e., it is periodic, so that the expanded pattern is actually $(\ldots, t_1, t_2, \ldots, t_n, t_1, t_2, \ldots, t_n, \ldots)$.

A sequence T is said to be a *juggling sequence*, or *siteswap sequence*, provided that it never happens that two balls come down at the same time. For example,

$(3,4,2)$ is a juggling sequence (see Figure 1.2), while $(4,3,2)$ is not.

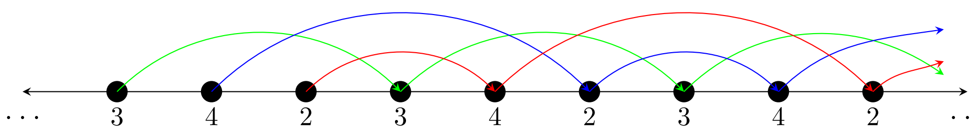


Figure 1.2: The juggling sequence $(3,4,2)$

It is known [5] that a necessary and sufficient condition for T to be a juggling sequence is that all the quantities $i+t_i \pmod{n}$ are distinct. For a juggling sequence $T=(t_1,t_2,\ldots,t_n)$, its *period* is defined to be n. A well known property is that the number of balls b needed to perform T is the average $b=\frac{1}{n}\sum_{i=1}^{n} t_i$ [25]. It is also known that the number of juggling sequences with period n and at most b balls is b^n (cf. [4, 5]; our convention assumes that a ball is thrown at every time step). We will prove this result in Section 1.1.1.

There is an alternative way to represent periodic juggling patterns, a variation of which was first introduced by Ehrenborg and Readdy [9]. For this method, certain *cards* are used to indicate the relative ordering of the balls (with respect to when they will land) as the juggling pattern is executed. In this way, the first representation can be viewed as a *temporal* model of juggling patterns, while the second can be viewed as a *spatial* model of juggling.

In this chapter we will explore various algebraic and combinatorial properties associated with these order sequences. It will turn out that there are a number of unexpected connections with a wide variety of combinatorial structures. In the remainder of this section we will introduce these juggling card sequences, and then in the ensuing sections we will count the number of juggling card sequences that induce a given ball ordering, count the number of juggling card sequences that do not change the ordering and have a fixed number of crossings, and look at the probability that the induced ordering consists of a single cycle.

1.1.1 Juggling card sequences

We will represent juggling patterns by the use of *juggling cards*. Sequences of these juggling cards will describe the behavior of the balls being juggled. In particular, the set of juggling cards produce the juggling diagram of the pattern.

Throughout the chapter, we will let b denote the number of balls that are available to be juggled. We will also have available to us a collection of cards $\mathcal{C}$ that can be used. In the setting when at each time step one ball is thrown, we can represent these by $C_1, C_2, \ldots, C_b$ where C_i indicates that the bottom ball in the ordering has now dropped into our hand and we now throw it so that relative to the other balls it will now be the i-th ball to land. Visually we draw the cards so that there are b levels on each side of the card (numbered *1*,*2*,...,b from bottom to top) and b tracks connecting the levels on the left to the levels on the right by the following: level *1* connects to level i; level j connects to level $j-1$ for $2 \leq j \leq i$; level j connects to level j for $i+1 \leq j \leq b$. An example of the cards when $b = 4$ is shown in Figure 1.17.

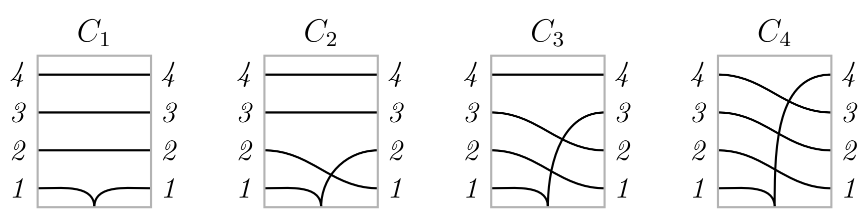


Figure 1.3: Cards for $b = 4$

As we juggle the b balls, $1, 2, \ldots, b$, move along the track on the cards. For each card C_i the relative ordering of the balls changes and corresponds to a permutation π_{C_i}. Written in cycle form this permutation is $\pi_{C_i} = (i\ i{-}1\ \ldots\ 2\ 1)$. In particular, a ball starting on level j on the left of card C_i will be on level $\pi_{C_i}(j)$ on the right of card C_i.

A sequence of cards, A, written by concatenation, i.e., $C_{i_1}C_{i_2}\ldots C_{i_n}$, is a *juggling card sequence* of length n. The n cards of A are laid out in order so that the levels match up. The balls now move from the left of the sequence of cards to the right of the sequence of cards with their relative ordering changing as they move. The resulting final change in the ordering of the balls is a permutation denoted π_A, i.e., a ball starting on level i will end on $\pi_A(i)$. We note that $\pi_A = \pi_{C_{i_1}}\pi_{C_{i_2}}\cdots\pi_{C_{i_n}}$. We will also associate with juggling card sequence A the arrangement $[\pi_A^{-1}(1), \pi_A^{-1}(2), \ldots, \pi_A^{-1}(b)]$, which corresponds to the resulting ordering of the balls on the right of the diagram when read from bottom to top.

As an example, in Figure 1.4 we look at $A = C_3C_3C_2C_4C_3C_4C_3C_2C_2$ (note we allow ourselves the ability to repeat cards as often as desired). For this juggling card sequence we have $\pi_A = (1\ 2\ 4\ 3)$ and corresponding arrangement $[3, 1, 4, 2]$. We have also marked the ball being thrown at each stage under the card for reference.

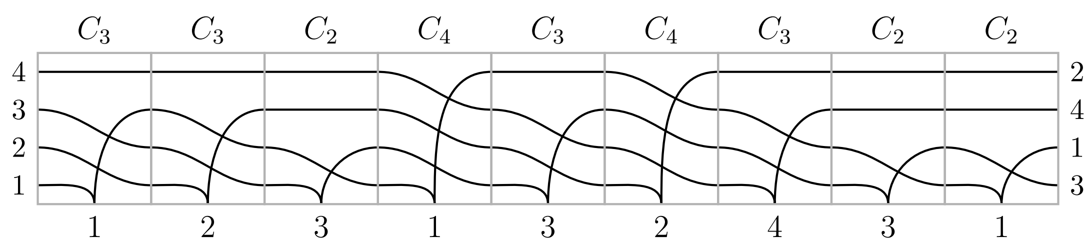


Figure 1.4: A juggling card sequence A; below each card we mark the ball thrown

From the juggling card sequence we can recover the siteswap sequence by letting t_i be the number of cards traversed starting at the bottom of the ith card until we return to the bottom of some other card. For example, the siteswap pattern in Figure 1.4 is $(3, 4, 2, 5, 3, 10, 5, 2, 2)$.

With this observation, we see that the set of length n, b ball siteswap sequences is in bijection with the set of length n, b ball juggling card sequences. We already mentioned a theorem that says that the number of length n siteswap sequences with at most b balls is exactly b^n. This theorem was once non-trivial to prove, but with juggling cards it is trivial.

Theorem 1. *[5, 25] The number of length n juggling sequences using at most b balls is b^n.*

Proof. Note that all juggling patterns using at most b balls can be described using juggling cards with b tracks, since the top tracks need not be used. We have observed that the answer can be found by asking how many different length n juggling card sequences are there, when using these cards with b tracks. Since there are b options for each card, the answer is b^n. □

We can also increase the number of balls thrown at one time, which is known as *multiplex* juggling. In the more general setting we will denote the cards C_S where $S = (s_1, s_2, \ldots, s_k)$ is an ordered subset of $[b]$. Each card still has levels $\mathit{1}, \mathit{2}, \ldots, b$, but now for $1 \le j \le k$ the ball at level i goes to level s_i and the remaining balls then fill the available levels in the unique way that preserves their order. As an example, the cards $C_{2,5}$ and $C_{5,2}$ are shown in Figure 1.5 for $b = 5$.

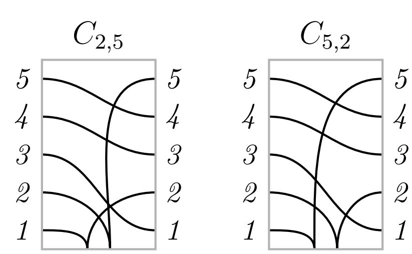


Figure 1.5: Two cards where $|S| = 2$

As before, we can combine these together to form juggling card sequences A which induce permutations π_A and corresponding arrangements. An example of a juggling card sequence composed of cards C_S with $|S| = 2$ is shown in Figure 1.6, which has final arrangement $[3, 4, 2, 5, 1]$. We note that it is also possible to form juggling card sequences which have differing sizes of S.

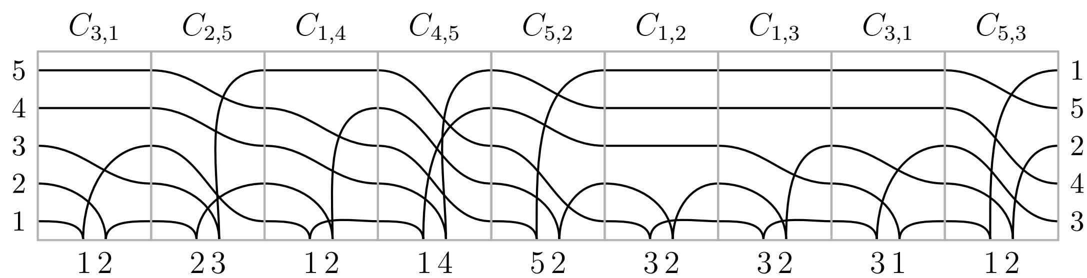


Figure 1.6: A juggling card sequence A; below each card we mark the balls thrown

1.2 Throwing one ball at a time

In this section we will consider two enumeration problems. The first is that of enumerating juggling card sequences of length n using cards drawn from a collection of cards $\mathcal{C}$ with the final arrangement corresponding to the permutation σ. We will denote the number of such sequences by $JS(\sigma, n, \mathcal{C})$.

The second enumeration problem is that of determining the number of length n sequences for cards from the collection $\mathcal{C}$ in which the final permutation is σ and every ball is thrown at least once. The number of such sequences is denoted by $\widetilde{\mathcal{JS}}(\sigma, n, \mathcal{C})$.

In the first subsection we will study $JS(\sigma, n, \mathcal{C})$, for which the following definition will be helpful.

Definition 1. Let σ be a permutation of $1, 2, \ldots, b$. Then $L(\sigma)$ is the largest ℓ such that $\sigma(b-\ell+1) < \cdots < \sigma(b-1) < \sigma(b)$. Alternatively, $L(\sigma)$ is the largest ℓ so that $b-\ell+1, \ldots, b-1, b$ appear in increasing order in the arrangement for σ.

As an example, the final arrangement in Figure 1.4 has $L(\sigma) = 2$ and the final arrangement in Figure 1.6 has $L(\sigma) = 3$.

The key idea for our approach will be that with information about what balls are thrown we can "work backwards." In particular, we have the following.

Lemma 2. *Given a single card, if we know the ordering of balls on the right hand side of the card* and *we know which balls are thrown, then we can determine the card C_S and the ordering of the balls on the left hand side of the card. Moreover, there is a unique card realizing this.*

Proof. Suppose that $i_1, i_2, \ldots, i_\ell$ are the balls, in that order, which are thrown. Then the card is C_S where $S = (s_1, s_2, \ldots, s_\ell)$ and s_j is the location of ball i_j in the ordering of the balls (i.e., where the ball i_j moved). The ordering of the left hand side starts $i_1, i_2, \ldots, i_\ell$ and the remaining balls are then determined by noting that their ordering must be preserved. □

1.2.1 The unrestricted case

We now work through the case when one ball at a time is thrown. We call this the unrestricted case because in the following section we will demand additionally that each ball is thrown at least once.

Theorem 3. *Let b be the number of balls and $\mathcal{C} = \{C_1, \ldots, C_b\}$. Then*

$$JS(\sigma, n, \mathcal{C}) = \sum_{k=b-L(\sigma)}^{b} \left\{ {n \atop k} \right\},$$

where $\left\{ {n \atop k} \right\}$ *denotes the Stirling numbers of the second kind.*

Proof. We establish a bijection between the partitions of $[n]$ into k nonempty subsets $[n] = X_1 \cup X_2 \cup \cdots \cup X_k$ where $b - L(\sigma) \le k \le b$ and juggling card sequences of length n using cards from $\mathcal{C}$ with the final arrangement corresponding to σ. Because such partitions are counted by the Stirling numbers of the second kind, the result will then follow.

Starting with a partition we first reindex the sets so that the minimal elements are in increasing order, i.e., $\min X_i < \min X_j$ for $i < j$. We now place n *blank* cards, mark the final arrangement corresponding to σ on the right of the final card, and then under the i-th card we write j if and only if $i \in X_j$.

We interpret the labeling under the cards as the ball that is thrown at that card, in particular we will have that k of the balls are thrown. We can now apply Proposition 2 iteratively from the right hand side to the left hand side to determine the cards in the juggling card sequence, where we update our ordering as we move from right to left.

We claim that the final ordering that we will end up with on the left hand side is $[1, 2, \ldots, b]$ so that this is a juggling card sequence which should be counted. Looking at the proof of the proposition we see that at each step the only ball which changes position in the ordering is the ball which is thrown, and in that case the ball was thrown from the bottom of the ordering. We now have two observations to make:

- For the k balls that will be thrown they will move into the first k slots in the ordering, and by the assumption of our indexing we have that the first k balls are ordered, i.e., for $1 \le i < j \le k$ the first occurrence when going from left to right of i is before the first occurrence of j so that i will move below j.

- The remaining balls will not have their relative ordering change. However, by our assumption on k we have that $k+1, \ldots, b$ are already in the proper ordering.

This establishes the map from partitions to juggling card sequences. To go in the other direction, we take a juggling card sequence of length n using our cards from $\mathcal{C}$, write down which ball is thrown under each card, and then form our sets for the partition by letting X_i be the location of the cards where ball i is thrown. Because $\sigma(b - L(\sigma)) > \sigma(b - L(\sigma) + 1)$ it must be that at some time that the ball $b - L(\sigma)$ was thrown and therefore the number of sets in our partition is at least $b - L(\sigma)$. This finishes the other side of the bijection and the proof. $\square$

For the partition of $[9] = \{1,4,9\} \cup \{2,6\} \cup \{3,5,8\} \cup \{7\}$ with final arrangement $[3,1,4,2]$, the juggling card sequence which will be formed is the one given in Figure 1.4.

1.2.2 Throwing each ball at least once

Because of the physical genesis for this problem, it is reasonable to demand that each ball is shown at least once. Recall that we defined $\widetilde{\mathcal{JS}}(\sigma, n, \mathcal{C})$ to be the number of length n sequences for cards from the collection $\mathcal{C}$ in which the final permutation is σ and every ball is thrown at least once.

Theorem 4. *Let b be the number of balls and $\mathcal{C} = \{C_1, \ldots, C_b\}$. Then*

$$\widetilde{JS}(\sigma, n, \mathcal{C}) = \left\{ {n \atop b} \right\},$$

where $\left\{ {n \atop k} \right\}$ denotes the Stirling numbers of the second kind.

One can easily modify the structural proof of Theorem 3 to prove this theorem. Here we provide two other proofs using the recurrence properties of Stirling numbers.

The arguments are important to gain intuition which will then be used to create the second bijection in Section 1.6.1. As you will see, some steps in the proof are embellished in order to provide this intuition.

Proof. Fix σ and call a sequence "b-valid" if $\mathcal{C} = \{C_1, \ldots, C_b\}$, each ball is thrown at least once and the final permutation is σ. For simplicity, let $S(n, b)$ denote the number of length n, b-valid sequences. We will show that this quantity satisfies the recurrence relation $S(n+1, b) = b \cdot S(n, b) + S(n, b-1)$, which is a well-known recurrence for Stirling numbers. Consider the b-valid arrangements of length $n+1$ in which Ball 1 bounces exactly once; card 1 is this bounce. Moreover, we know that this first card is a b-card; otherwise every ball above Ball 1 would never bounce, as doing so means that Ball 1 had to have bounced again.

Consider all $(b-1)$-valid arrangements of the final n cards. Each of these ends with Ball 2 on the bottom level, Ball 3 above it, ..., Ball b on level $b-1$ and Ball 1 on level b. I claim that each of these arrangements can be uniquely paired with an arrangement of n final cards which ends with the identity permutation of the balls.

Consider the following example in which $b = 4$ and $n = 5$. Here is one $(b-1)$-valid sequence of n cards:

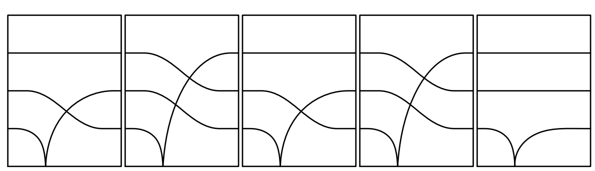


Figure 1.7: Card order: 2,3,2,3,1. Ball order: 1,2,1,3,1.

Each ball's final bounce must finish above Ball 1, because if neither ball will bounce again, then their relative order can not change. To get our new card sequence, we simply identify the location of each ball's final bounce, and adjust this card to ensure that their new bounce puts the ball above Ball 1, while still maintaing the same relative order with every other ball as in the sequence we started with (if card i was previously used, then the new card with be card $i+1$).

This is always possible since this ball is guaranteed to land above all balls that have more bounces to go, and in the correct relative order among the balls which are done bouncing. In the above, the second, fourth and fifth cards corresponded to last-bounces, so in the below (after the addition of Ball 1's b Card throw) the third, fifth, and sixth cards will correspond to the cards which are altered so that each ball's final throw will land above Ball 1 (shown in red).

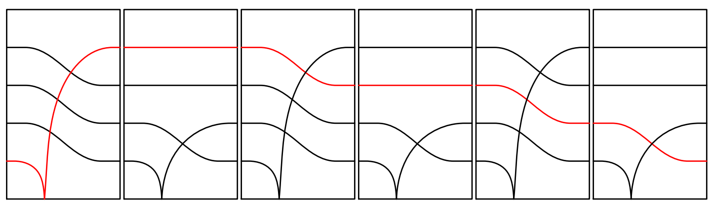


Figure 1.8: Card order: 4,2,4,2,4,2

The process is reversible, too. Given the b-valid, length $n+1$ sequence above in which Ball 1 bounces exactly once, we can remove this ball and the result is the $(b-1)$-valid arrangement of n cards that we started with. Therefore there are precisely $S(n, b-1)$ arrangements in this case.

Now consider the case in which Card 1 bounces more than once. After its first throw it will land on one of b levels, say level ℓ, inducing a permutation π of the balls. There are $S(n, b)$ choices for the final n cards to conclude with the permutation π. Moreover, in each of these Ball 1 will bounce again. We once again use this sequence to create another in which each ball ends up where it should.

The idea is the same as above. Each time a ball takes its final bounce, ensure that it lands above Ball 1 (the balls with number greater than ℓ will do this automatically), while also respecting the relative order in our original card sequence.

Again, this process is invertible. Given a b-valid, length $n+1$ sequence in which Ball 1 bounces more than once, we can create a $(b-1)$, length n sequence by first identifying what ℓ is, and then each time a ball with number less than ℓ takes its final bounce, alter the card so that the relative order with each ball except Ball 1 is maintained, and Ball 1's relative order is flipped. Once again, this will amount to replacing card i with card $i+1$. This is clear if start with the first first such ball to commit its final bounce, and proceed forward in time.

Therefore each of the b possible initial bounces for Ball 1 results in $S(n,b)$ arrangements of cards. Thus this case has $b \cdot S(n,b)$ arrangements. The base case is clear, so we have established the recurrence $S(n+1,b) = b \cdot S(n,b) + S(n,b-1)$. $\square$

Below is a second proof. Again, we will add additional details to help provide intuition.

Proof. The Stirling numbers satisfy a recurrence usually written as

$$\left\{ {n+1 \atop b+1} \right\} = \sum_{j=b}^{n} \binom{n}{j} \left\{ {j \atop b} \right\},$$

but let's begin by rewriting it as $\left\{ {n+1 \atop b+1} \right\} = \sum_{i=0}^{n-b} \binom{n}{i} \left\{ {n-i \atop b} \right\}$. Let's partition the set of $(b+1)$-valid arrangements of length $n+1$ by the number of times Ball 1 bounces. Say it bounces $i+1$ times; then there are $n-i$ cards in which a ball other than Ball 1 bounces.

If we take any such arrangement of cards, remove Ball 1 and straighten out the other tracks accordingly, we are left with a b-valid, length $n-i$ arrangement A of cards for which $\pi(A) = \mathrm{id}$.

Consider one of the $S(n-i,b)$ different arrangements of these cards. We think about these cards as being for balls 2 though b. Our goal is to alter these cards so that each new card respects the relative order of cars 2 through b in the card it is replacing, but also include Ball 1 properly so that it bounces when it should. We claim that there is a unique way to do this. It suffices to show that between a consecutive pair of Ball 1 bounce locations, that there is a unique way to add Ball 1 to the mix.

Among the i cards in which Ball 1 bounces, if two of these are adjacent, say at positions j and $j+1$, then it is clear that there is precisely one card choice for position j: a 1-Card. Now assume that there are $m \geq 1$ cards between two bounce locations for Ball 1. Instead of dictating where to add an additional track to the b-valid sequence as we did in proof 1, let's still imagine that we have just b tracks. Instead, our $(b+1)^{\text{st}}$ ball will pick a track to "ghost-ride", meaning that it will follow the path of another ball. In reality our $(b+1)^{\text{st}}$ ball will be either right above this track or right below it, but we won't decide which just yet. Note that a ball can potentially bounce between our ghost ball and the track it is following.

Remember, our ghost ball will be Ball 1. It bounced right before this sequence of m cards (the height of its bounce is our choosing), it does not bounce at all during these m cards, and it must be at level 1 at the end of the m^{th} card (as it must bounce immediately after).

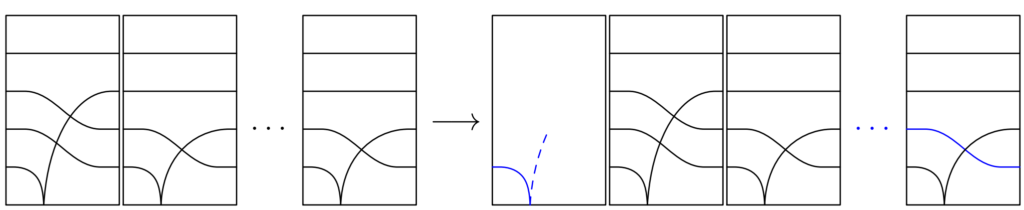


Figure 1.9: The ghost ball

Now, one of the b balls will of course end up at level 1 at the end of the m

cards. If this ball does not bounce at all within the m cards, then our job is easy: let our ghost ball ride this track, and choose it to in reality be "right below" this ghost-ridden track. Note this is equivalent to instead choosing to ghost ride the track below, while in reality being "right above" this track.

Whenever a ball bounces to the level of the track that the ghost ball is riding, or to the level below, we have to choose whether or not the bounced ball will land between the ghost ball and the track it's riding. We are allowed to alter the cards provided the relative order of balls 2 through b remain intact. If you do not follow the track that you have been ghost riding and drop when the track stayed level or vice versa, then you will instead now be ghost riding the track that just bounced up.

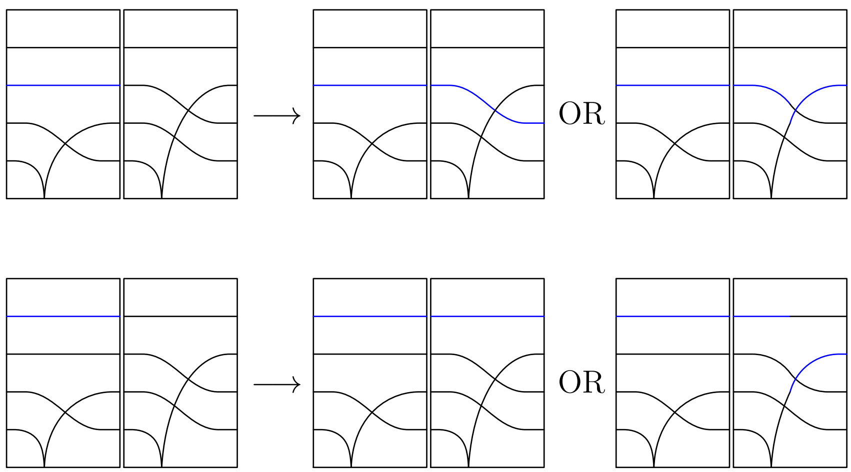


Figure 1.10: One choice in a decision sequence

Call the selection of the initial track along with each stay-up-or-drop-down choice a *decision sequence*. It is clear that different decision sequences induce different card sequences, our goal now is to show that there exists a decision sequence which will cause our ghost ball to arrive at level 1 at the conclusion of the m cards.

Consider two different ghost balls which at some point make their first differing decision. After this, the balls will be on different levels. Since these balls are not allowed to bounce, the only way they could ever realign is by their responses to balls bouncing up to one of their levels. This, however, does not help. If the top ball has the option to move down, the bottom ball must be forced to move down. Therefore the differing decision ensures that the bottom ball will be forced to bounce before the top ball. Consequently, at most one decision sequence could result in a ball being ready to bounce after the m cards.

So we must only show that such a decision sequence is possible. For this, simply note that each differing decision only changes the balls next bounce location by 1, and you can bounce within one card by initially choosing track 1 to ghost ride. Therefore, provided there is a decision sequence which allows you to bounce after more than m cards, then we have shown that you can bounce immediately, as late as you need, and every point in-between. In particular, you can bounce after precisely m cards.

So it suffices to show that you can wait arbitrarily long to bounce. To see this, assume we have an infinite b-valid arrangement of cards. Locate the highest L at which there are infinitely many L-cards. If $L = b$, then simply immediately thrown to level L, choose to stay up at level L until you have passed the m^{th} card, and then elect to drop down each time you have the choice.

Otherwise, let k_0 be the first position where an L-card is played and moreover there will never again be an M-card where $M > L$. Next find the nearest past position k_1 at which point there was an M_1-card where $M_1 \geq L + 1$. Then find the nearest past position k_2 before that at which point there was an M_2-card where $M_2 \geq L + 2$. Continue until you have a sequence $k_0 > k_1 > \cdots > k_{b-L}$. These are going to be the positions at which we descend.

First throw to level b. Then when at position k_{b-L} drop to level $b - 1$. At position k_{b-L-1}, drop to level $b-2$. We are never force to drop down because each descent point was chosen to be the final one at the level Ball 1 is then at. Once Ball 1 arrives at position k_0, it can then remain on this level until it passes position m, at which point it can the descend down to bounce again. This concludes the proof. $\square$

1.3 Throwing $m \geq 2$ balls at a time

In this section we consider the case where one throws multiple balls at once. Jugglers have accomplished this feat for some simple patterns in which they throw two balls at each time step, but the fun beyond that is, as of now, reserved just for mathematicians.

When you throw i balls at once, what you are doing is taking the bottom i balls and keeping their relative order intact, taking the top $b-i$ balls and keeping their relative order intact, and then shuffling these two stacks together in some way. You are essentially cutting the deck and shuffling the halves. One consequence of this is that cutting the deck i balls from the bottom is the same as cutting it $b-i$ balls from the top. Therefore we can see that there is a natural correspondence between throwing i balls at a time and throwing $b-i$ balls at a time. By reflecting each card over the horizontal we obtain a reflected arrangement where many of the properties of the original arrangement still hold. Here is an example where $i=2$ and $b=5$.

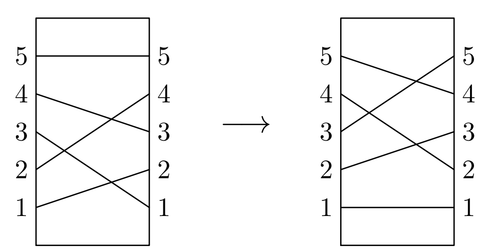


Figure 1.11: Example of a horizontal flip

In the above example card $C_{2,4}$ was transformed into card $C_{1,3,5}$. By regarding the subscripts as sets we note that in general card C_S is transformed into card $C_{[b]\setminus S}$.

Suppose A is an arrangement of n cards of height b where i balls are thrown at a time and $\pi(A) = \sigma$. By reflecting each card we obtain a new sequence A' of cards of height b where $b - i$ balls are thrown at once and $\pi(A) = \sigma'$ where σ' is the "horizontally-reflected" permutation of σ, namely $\sigma'(j) = b - \sigma(b - i + 1) + 1$. Note that if $\sigma = \mathrm{id}$ then also $\sigma' = \mathrm{id}$.

Furthermore, in Section 1.6 we will discuss in depth a juggling card's "crossing number," which is the number of points in which paths cross in the picture of that card. This number is invariant under this horizontal flipping procedure.

Moreover, unless Ball 1 is thrown in each card in A, if A has the property that each ball is thrown at least once (a b-valid condition) then A' also has this property. To see this, simply observe that the condition that Ball L be thrown in A is equivalent to the condition that at some point Ball L occupies one of the bottom i tracks. Likewise, Ball L being thrown in A' is equivalent to this ball at some point occupying one of the top $b - i$ tracks of A. Furthermore, since the order of the thrown balls is maintained, all of the bottom i balls will reach one of the top $b - i$ tracks if and only if Ball 1 does. And ensuring that these bottom i balls are thrown in A' is enough to ensure that all the balls do, since the others are thrown in the very first card.

1.3.1 A digraph approach

The proof of Theorem 3 readily generalizes to the setting where we throw m balls at a time. What we need to do is to find the appropriate way to generalize the Stirling numbers of the second kind.

Definition 2. Given n and k let $X = \{x_1, x_2, \ldots, x_k\}$ and let $\left\{ {n \atop k} \right\}_m$ be the number of ways, up to relabeling the x_i, to form $Y_1, Y_2, \ldots, Y_n$ so that $Y_j = (x_{j_1}, \ldots, x_{j_m})$ is an ordered subset of X and each x_i is in at least one Y_j.

We note that $\left\{ {n \atop k} \right\}_1 = \left\{ {n \atop k} \right\}$. This can be seen by observing that each Y_i is a single entry and then we form our partition by grouping the indices of the Y_i which agree. We claim that this gives the appropriate generalization, which we demonstrate now.

Theorem 5. *Let b be the number of balls and $\mathcal{C}$ be the collection of all cards for which m balls are thrown. Then*

$$JS(\sigma, n, \mathcal{C}) = \sum_{k=b-L(\sigma)}^{b} \left\{ {n \atop k} \right\}_m .$$

Proof. Suppose we are given $Y_1, \ldots, Y_n$ with $Y_j = (x_{j_1}, \ldots, x_{j_m})$ an ordered subset of $\{x_1, \ldots, x_k\}$. Then we first concatenate the Y_j together and remove all but the *first* occurrence of each x_i leaving us with a list Y'. By our assumptions we have that Y' consists of $x_1, \ldots, x_k$ in some order. For $Y_1, \ldots, Y_n$, we now replace $x_1, \ldots, x_k$ by $1, \ldots, k$ by replacing x_i with j if x_i is in the j-th position of Y'. (This process is equivalent to the reindexing carried out in the special case when one ball is thrown at a time.)

We now have $Y_1, \ldots, Y_n$ with each consisting of m distinct numbers drawn from $\{1, \ldots, k\}$ with the property that if $i < j$ then i appears before j (i.e., in the sense that if the first occurrence of i is in Y_p and the first occurrence of j is in Y_q and then either $p < q$ or $p = q$ and i appears in the list before j in Y_p). We now put down n blank cards, write down the arrangement corresponding to σ on the right side of the last card and write Y_i under the ith card for all i. The remainder of the proof

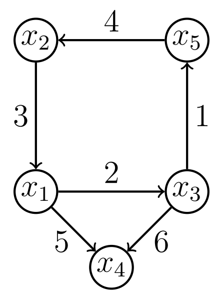


Figure 1.12: An edge labeled multi-digraph

then proceeds as before, i.e., we can now work from right to left and determine the card used at each stage by Proposition 2. The resulting process gives a valid juggling sequence because the initial arrangement will have the first k balls in order (by our work on reindexing) and the final balls inherit their order, which by assumption were already in the correct order.

The map in the other direction is carried out as before, i.e., given a juggling card sequence under each card we write the balls which are thrown and use these to form $Y_1, \ldots, Y_n$ which contribute to the count of $\left\{ {n \atop k} \right\}_m$ for some appropriate k. $\square$

The value $\left\{ {n \atop k} \right\}_2$ is found by counting sets of ordered pairs. In particular, this counts the number of multi-digraphs with n *labeled* edges and k vertices. This leads to a bijection between these digraphs and juggling sequences for a given σ, provided $k \geq b - L(\sigma)$. As an example consider the edge-labeled directed graph shown in Figure 1.12.

Using the edge labeling we can now form the sets so that $Y_1 = (x_3, x_5)$, $Y_2 = (x_1, x_3)$, $Y_3 = (x_2, x_1)$, $Y_4 = (x_5, x_2)$, $Y_5 = (x_1, x_4)$ and $Y_6 = (x_3, x_4)$. Concatenating and then removing all but the first occurrences of an x_i we get the following:

$$(x_3, x_5, x_1, x_3, x_2, x_1, x_5, x_2, x_1, x_4, x_3, x_4) \to (x_3, x_5, x_1, x_2, x_4) = Y'$$

So we replace x_3, x_5, x_1, x_2, x_4 by $1, 2, 3, 4, 5$ respectively. If we now set the final arrangement to be $[4, 5, 2, 1, 3]$ then we get the corresponding juggling card sequence shown in Figure 1.13.

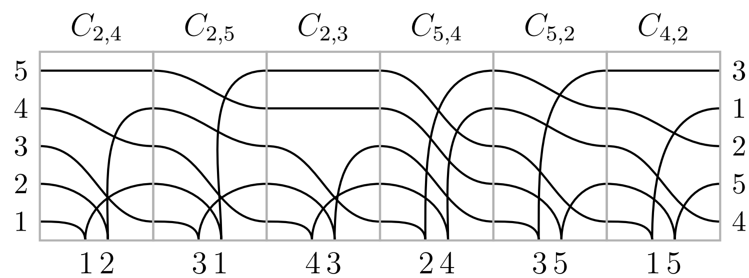


Figure 1.13: The juggling card sequence corresponding to the digraph from Figure 1.12 and final arrangement $[4, 5, 2, 1, 3]$

The numbers $\left\{ {n \atop k} \right\}_m$ have appeared recently in the literature in connection with the so-called *boson normal ordering problem* arising in statistical physics [2, 14]. This ordering problem will be discussed further in Section 1.4.1. The sequence $\left\{ {n \atop k} \right\}_2$ is `A078739` in the OEIS [17].

For general m it has been observed [7] that $\left\{ {n \atop k} \right\}_m$ is the number of ways to properly color the graph nK_m using exactly k colors, i.e., each Y_i is the coloring on the i-th copy of K_m, and by definition all k colors must be used.

If we denote the *falling factorial* $x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$, then the ordinary Stirling numbers $\left\{ {n \atop k} \right\}$ act as *connection coefficients* between $x^{\underline{n}}$ and x^n by means of the formula (e.g., see [11])

$$x^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}.$$

In particular, they satisfy the recurrence:

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\},$$

and have the explicit representation

$$\left\{ {n \atop k} \right\} = \frac{(-1)^k}{k!} \sum_{i=1}^{k} (-1)^i \binom{k}{i} i^n.$$

The $S_m(n,k) = \left\{ {n \atop k} \right\}_m$ satisfy analogs of these three relationship. Namely, as connection coefficients

$$(x^{\underline{m}})^n = \sum_{k=m}^{mn} \left\{ {n \atop k} \right\}_m x^{\underline{k}},$$

satisfying a recurrence

$$\left\{ {n+1 \atop k} \right\}_m = \sum_{i=0}^{m} \binom{k+i-m}{i} m^{\underline{i}} \left\{ {n \atop k+i-m} \right\}_m,$$

and with the explicit representation

$$S_m(n,k) = \left\{ {n \atop k} \right\}_m = \frac{(-1)^k}{k!} \sum_{i=m}^{k} (-1)^i \binom{k}{i} (i^{\underline{m}})^n.$$

1.3.2 A differential equations approach

We now present a second way to count the number of sequences A of length $n+1$ where the final permutation $\pi(A)$ is some fixed σ and each ball is required to be thrown at least once.

Suppose that after the first card, i of these k balls are not thrown again; call such a ball *inactive*, and otherwise call a ball *active*. As we have discussed before, those

i inactive balls must land above all other balls since all balls begin the arrangement being active and clearly an active ball can not be above an inactive ball. Moreover, their relative arrangement at the top must match $\pi(A)$, and hence the choice of where they go is determined.

As fleshed out in the proofs in Section 1.2.2, the remaining cards are now in a 1-to-1 correspondence with the set of k-at-a-time arrangements A' of length n where the cards now have height $b-i$, each ball is thrown at least once and $\pi(A')$ is the appropriate restriction of $\pi(A)$. So if $S_k(n+1,b)$ is the number of possibilities for A, then $S_k(n,b-i)$ is the number of possibilities for A'. With $\binom{k}{i}$ choices for which of the first k balls became inactive and $\binom{b-i}{k-i}$ choices for where the remaining balls are inserted, we get the recurrence

$$S_k(n+1,b) = \sum_{i=0}^{k} \binom{k}{i}\binom{b-i}{k-i} S_k(n,b-i).$$

In particular, when $k=1$ this collapses down to the Stirling recurrence, showing that $S_1(n,b)$ are the Stirling numbers. When $k=2$ we get the recurrence

$$S_2(n+1,b) = \binom{b}{2} S_2(n,b) + 2(b-1)S_2(n,b-1) + S_2(n,b-2).$$

This set of recurrences can be solved via the differential equations method. Define

$$A_k(x,y) := \sum_{n,b\geq 0} S_k(n,b)x^n y^b.$$

Then

$$[x^n y^b]\frac{\partial A_k}{\partial x} = (n+1)S_k(n+1,b)$$

$$
\begin{aligned}
&= (n+1)\sum_{i=0}^{k}\binom{k}{i}\binom{b-i}{k-i}S_k(n,b-i)\\
&= (n+1)\sum_{i=0}^{k}[x^n y^{b-i}]\binom{k}{i}\frac{1}{(k-i)!}y^{k-i}\frac{\partial^{k-i}A_k}{\partial^{k-i}}\\
&= [x^n y^b]\sum_{i=0}^{k}\binom{k}{i}\frac{n+1}{(k-i)!}y^k\frac{\partial^{k-i}A_k}{\partial y^{k-i}}
\end{aligned}
$$

Summing over n and b gives

$$
\frac{\partial A_k}{\partial x} = \sum_{i=0}^{k}\binom{k}{i}\frac{n+1}{i!}y^k\frac{\partial^i A_k}{\partial y^i}
$$

For example when $k = 2$ we get

$$
\frac{\partial A_2}{\partial x} = \frac{n+1}{2}y^2\frac{\partial^2 A_2}{\partial y^2} + 2(n+1)y^2\frac{\partial A_2}{\partial y} + (n+1)y^2 A_2.
$$

The initial conditions can now be easily found, and in each case we leave it to the reader to plug in what was found in the previous section to verify that it agrees. One could also solve these equations independently.

1.4 Throwing different numbers of balls at different times

We have restricted our analysis to the case when each card in our collection throws the same number of balls. We can relax this restriction and allow ourselves to throw differing number of balls at each step. For example, we could insist that at the

i-th time step that m_i balls are thrown.

Under the card in the i-th position we place a sequence $Y_i = (y_{i,1}, y_{i,2}, \ldots, y_{i,m_i})$. We then concatenate the labels as before to give a mapping from the $y_{i,j}$ to $[k]$ to give a compatible ball assignment to the card positions. Then we work from right to left and recover the unique juggling card sequence which corresponds to this collection of ordered sets.

We briefly turn away from this perspective, though, to study another to better understand this problem.

1.4.1 Boson normal ordering

If two sequences of sets of structures have the same number of structures in each set, then oftentimes there is a deeper connection between the structures. In Section 1.3.1 we introduced this connection, and in this section and the next we explore it more deeply.

Quantum physics partially studies means of creation and annihilation. To represent this, physicists and mathematicians describe a creation operator $a^\dagger$ and an annihilation operator a. Sentences on the alphabet $\{a^\dagger, a\}$ are of importance, which is complicated by their principe characteristic: these operators do not commute. However, we do have

$$aa^\dagger = a^\dagger a + 1.$$

The ordering of a string of operators is significant to determine the corresponding physical properties, so scientists needed a method to tell when two (sums of) strings are the same. It was therefore natural to demand a natural ordering on these generators. The chosen ordering is quite natural indeed: When writing a string, use the property $aa^\dagger = a^\dagger a + 1$ repeatedly until each all of the annihilation operators are on to the right of the creation operators, in each word in the resulting sum.

Example 1. We find the normal ordering of $a^\dagger a a^\dagger a a^\dagger a$.

$$\begin{aligned}
a^\dagger a a^\dagger a a^\dagger a &= a^\dagger (a a^\dagger)(a a^\dagger) a \\
&= a^\dagger (a^\dagger a + 1)(a^\dagger a + 1) a \\
&= a^\dagger a^\dagger a a^\dagger a a + 2 a^\dagger a^\dagger a a + a^\dagger a \\
&= a^\dagger a^\dagger (a^\dagger a^\dagger + 1) a a + 2 a^\dagger a^\dagger a a + a^\dagger a \\
&= a^\dagger a^\dagger a^\dagger a a a + 3 a^\dagger a^\dagger a a + a^\dagger a.
\end{aligned}$$

Or, written differently,

$$(a^\dagger a)^3 = (a^\dagger)^3 a^3 + 3 (a^\dagger)^2 a^2 + a^\dagger a.$$

And if you have been studying enumerative combinatorics just a little bit too long, you might notice a pattern among the three coefficients: 1,3,1. Indeed, there is a pattern there, as the following well-known theorem states.

Theorem 6. *[8]*

$$(a^\dagger a)^n = \sum_{k=1}^{n} S(n,k) (a^\dagger)^k a^k,$$

where $S(n,k)$ is the n,k-Stirling number of the second kind.

We have already encountered Stirling numbers many times in this chapter as the number of length n, k-ball juggling card sequences in which each ball is thrown at least once. But the interesting thing is that the juggling problem, as well as this annihilation problem, each have a generalization, and in fact this enumerative connection between the two problems is maintained when we move to the generalization!

We generalize the juggling problem by throwing multiple balls at once, and we generalize this annihilator problem by instead finding the normal ordering of $((a^\dagger)^m a^m)^n$. Indeed,

Theorem 7. *[8]*

$$((a^\dagger)^m a^m)^n = \sum_{k=1}^{n} S_m(n,k)(a^\dagger)^k a^k,$$

where $S_m(n,k)$ is the n,k-generalized m^{th} Stirling number, as described in Section 1.3.1.

If one is allowed to throw a different numbers of balls at different times, the analysis again generalizes in a similar way, and corresponds to the coefficients of a normal ordering of another word on $\{a, a^\dagger\}$. These details are not completely developed for our situation, but there is much progress made in [8].

Also in [8], the authors make the connection we previously made to graph colorings (although they phrase it in terms of vertex partitions). In it they prove that the generalized Stirling numbers $S_m(n,k)$ count the number of k-colorings of mK_n. From their analysis it is clear that the number of length n, k-ball juggling card sequences in which m_i balls are thrown on card i is equal to the number of k-colorings of $\cup_{i=1}^{n} K_{m_i}$.

For more details on this specifically, please see [8]. To learn more about norming ordering (mostly unrelated to the juggling problem), please see [13, Ch. 10].

1.5 Preserving ordering while throwing

In the preceding section when we threw multiple balls at one time, we did not worry about preserving the ordering of the balls which were thrown. The goal of this section is to add the extra condition that the relative order of the thrown balls is preserved, e.g., for $m = 2$ our set of cards will be the set of $\binom{b}{2}$ cards given by $\{C_{i,j} : 1 \leq i < j \leq b\}$. We will see that this situation is more complicated than the one in the preceding section.

To begin the analysis, we start with a 2-cover of the set $[n]$. This is a collection of k (not necessarily distinct) subsets S_i of $[n]$ with the property that each element j of $[n]$ occurs in exactly two of the S_i. We can represent a 2-cover by a $k \times n$ matrix M where for $1 \leq i \leq k$, $1 \leq j \leq n$, we have $M(i,j) = 1$ if $j \in S_i$, and $M(i,j) = 0$ otherwise. For each set S_i we will associate a *virtual ball* x_i. For $1 \leq j \leq n$, we define the 2-element set $B_j = \{x_i : j \in S_i\}$. In other words, $x_i \in B_j$ if and only if $M(i,j) = 1$. The interpretation is that at time j, the two virtual balls $x_i \in B_j$ will be the balls that are thrown at that time.

We now produce the (unique) mapping between the actual balls and the virtual balls x_i. To do this, we define a partial order on the x_i as follows: x_u is less than x_v, written as $x_u \prec x_v$, if among all the $B_i \neq \{x_u, x_v\}$, x_u occurs *before* x_v (i.e., with a lower indexed B_i). If there are no such B_i, we say that x_u and x_v are equivalent.

Example 2. A 2-cover of [7] with five subsets is given by the following matrix.

$$M = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

We have labeled the rows of M with the x_i and the columns with the B_j. Thus, we see that x_2 and x_4 are equivalent, so that the partial order on the x_i is

$$x_1 \prec x_3 \prec x_2 \equiv x_4 \prec x_5.$$

If in the current arrangement we have that u is below v, then v cannot be thrown before u (though it might possibly be at the same time). Therefore the partial order on the x_i determines how the balls are positioned relative to one another. The partial order doesn't specify anything about the relative order of equivalent x_i but because such pairs are always thrown together, their relative order never changes during the process of traversing all the cards in the sequence.

In Figure 1.14 we show the sequence generated by the 2-cover from M, where we assume the finishing arrangement of the x_i is from bottom to top x_4, x_1, x_5, x_3, x_2. This choice was arbitrary, except that the initial and terminal orders of the equivalent pair x_2 and x_4 must be the same, since there is a unique initial sequence which can have the x_i in B_j being thrown at time j, namely, the sequence that is consistent with the partial order $\prec$ on the x_i. To determine the appropriate cards needed for the required throwing patterns it is simply a matter of starting at the right hand side and choosing the cards sequentially which achieve the required throws. In Figure 1.14, we have also have indicated the corresponding cards $C_{i,j}$ which accomplish the indicated throws.

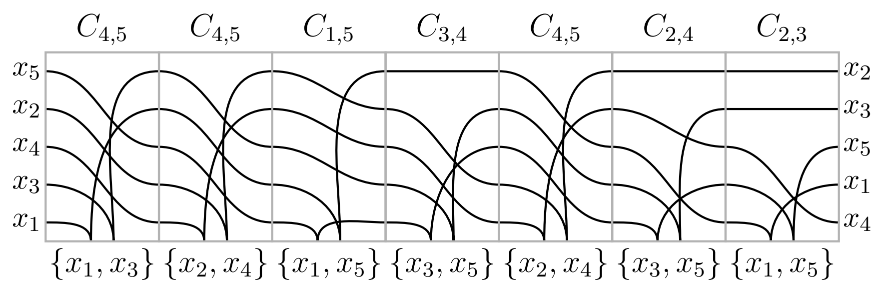


Figure 1.14: A card sequence for the matrix M

If we now make the identification $x_1 \to 1, x_3 \to 2, x_4 \to 3, x_2 \to 4, x_5 \to 5$, then we have the picture shown in Figure 1.15.

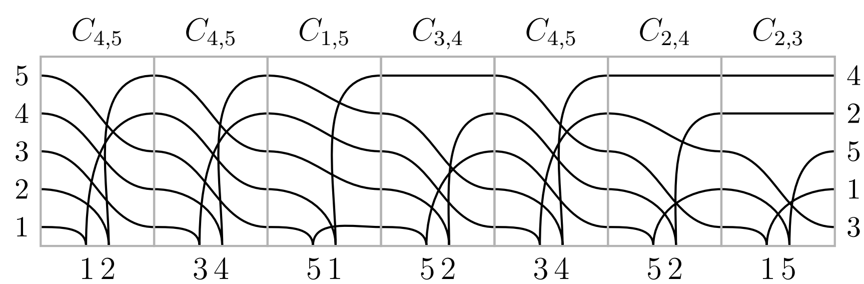


Figure 1.15: A card sequence for the matrix M using actual balls

We can achieve any permutation σ of the balls $\{1, 2, 3, 4, 5\}$ starting in increasing order provided only that $\sigma(2)$ is below $\sigma(4)$.

For general n and k, given a 2-cover of $[n]$ with k sets $S_1, \ldots, S_k$, there is an induced partial order on the sets (or what we called virtual balls). For any terminal permutation σ which preserves the relative order of equivalent balls, there is a unique sequence of cards which achieves this permutation.

As pointed out in [6], there is a direct correspondence between 2-covers of $[n]$ with k subsets and multigraphs $G(n,k)$ having k vertices and n labeled edges. In the case of graphs, the vertices of G will be $\{x_1, x_2, \ldots, x_k\}$. We insert the edge $\{x_r, x_s\}$ with label i if the i-th column of M has 1's in rows r and s. The number of vertices of such an edge-labeled multigraph corresponds to the number of balls which are thrown. These are enumerated by the numbers of vertices and labeled edges in [12] (see also `A098233` in the OEIS [17]). We illustrate this connection in Figure 1.16 where we show the three edge-labeled multigraphs on two edges and the corresponding card sequences which generate the identity permutation.

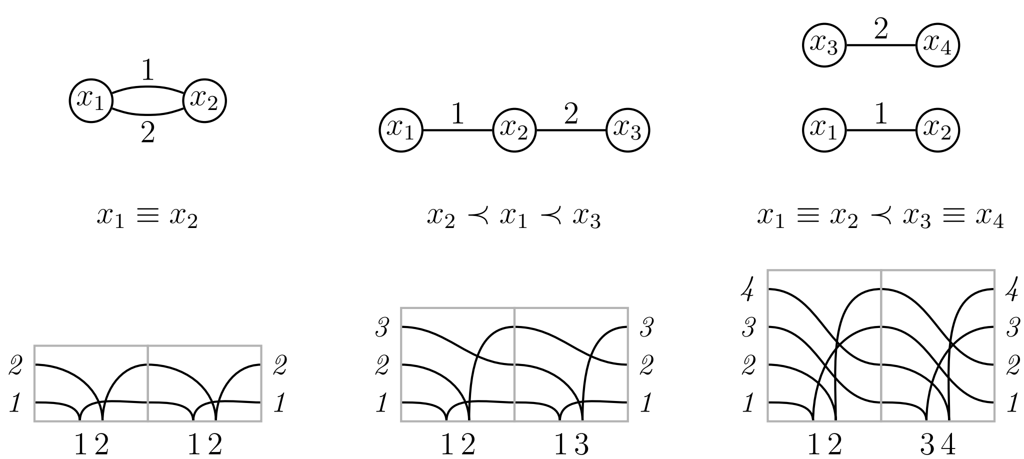


Figure 1.16: Edge-labeled multigraphs with two edges, and the corresponding card sequences

In the special case that the desired permutation $\pi_A = \sigma = \mathrm{id}$, the identity permutation, then *any* 2-cover can generate this permutation. Thus, there is a bijection between 2-covers of $[n]$ and sequences A of n cards with $\pi_A = \mathrm{id}$.

The asymptotic behavior of the number of 2-covers of an n-set, denoted $Cov(n)$, has been studied in [6]. In particular, it is shown there that

$$Cov(n) \sim B_{2n} 2^{-n} \exp\left(-\frac{1}{2}\log\left(\frac{2n}{\log n}\right)\right)$$

where B_{2n} is the well-known Bell number (see [23]).

Counting the number of juggling card sequences which generate permutations other than the identity is more complicated.

In the more general case of throwing $m \geq 3$ balls, we want to consider m-covers of the set $[n]$. An m-cover of $[n]$ is a collection of k (not necessarily distinct) subsets S_i of $[n]$ with the property that each element j of $[n]$ occurs in exactly m of the S_i. As before, we can represent the m-cover by a $k \times n$ matrix M where for $1 \leq i \leq k$, $1 \leq j \leq n$, $M(i,j) = 1$ if $j \in S_i$, and $M(i,j) = 0$ otherwise.

The same analysis holds in this case of general m as in the case of $m = 2$. Namely, for each subset S_i in the m-cover, we can associate a virtual ball x_i. Then we can use the sets B_j corresponding to the columns of M to induce a partial order $\prec$ on the x_i. As before, any permutation σ on $[k]$ which respects the order of equivalent elements can be achieved by a unique sequence of cards. In the case that σ is the identity permutation, then any m-cover of $[n]$ is able to generate this permutation with an appropriate sequence of cards. In this case the number of such juggling card sequences is the number of hyperedge-labeled multi-hypergraphs, (similar to the edge-labeled multigraphs for the case $m = 2$).

1.6 Juggling card sequences with minimal crossings

We now return to throwing a single ball at a time. Any juggling card sequence of n cards will produce a valid siteswap sequence which has period n. However most such siteswap sequences will result in having the balls be non-trivially permuted amongst themselves after n throws. So one natural family to focus on are those which satisfy $\pi_A = \mathrm{id}$, i.e., after n throws the same balls are in the same position and ready to repeat.

Suppose now we follow the balls as they traverse the cards of some sequence A. Then when a card C_k is used, we see that the path of the thrown ball has $k-1$ "crossings" in that card, i.e., locations where the tracks intersect.

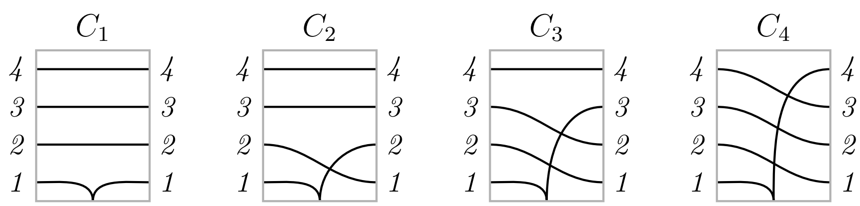


Figure 1.17: Cards with 0, 1, 2 and 3 crossings, respectively

For a sequence $A = C_{i_1}C_{i_2}\ldots C_{i_n}$, the total number of crossings is $X(A) = \sum(i_k - 1)$. In the case when a juggling card sequence has b balls, uses the card C_b, and has $\pi_A = \mathrm{id}$, then the number of crossings satisfies $X(A) \geq b(b-1)$. To see this we note that *every* ball must be thrown. Suppose $I < J$. Then Ball I must at some point be thrown above Ball J to allow Ball J to be thrown, incurring one crossing between this pair. But since Ball J winds up above Ball I at the end of the card sequence, at some point Ball J must be thrown above Ball I, incurring a second crossing. Since each of the $\binom{b}{2}$ pairs crosses at least twice, in total there will be at least $2 \cdot \binom{b}{2} = b(b-1)$ crossings.

We will say a juggling card sequence A is a *minimal crossing juggling card sequence* if the sequence has b balls, uses the card C_b, has $\pi_A = \mathrm{id}$, and $X(A) = b(b-1)$. We will prove that the number of ways is in each case a Narayana number. We will first prove this directly, using recursion, and then then will provide two bijections with classes of Dyck paths, which are also counted by Narayana numbers. In the process we will demonstrate some important structural properties of minimal crossing juggling card sequences.

Theorem 8. *The number of minimal crossing juggling card sequences with b balls and n cards is*

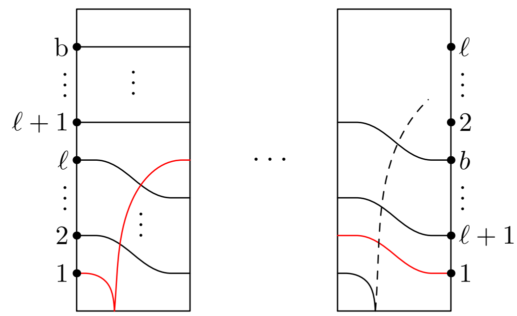
$$f(n,b) = \frac{1}{b}\binom{n-1}{b-1}\binom{n}{b-1} = \frac{1}{n}\binom{n}{b}\binom{n}{b-1},$$

the Narayana numbers.

Proof. Assume that A is a b-valid arrangement of n cards where $\pi(A) = \mathrm{id}$ and $X(A) = b(b-1)$. Let $N_0(n,b)$ be the number of such arrangements. In the previous section we described precisely what these juggling card sequences can look like. Part of that argument quickly gives a nice recurrence. We'll repeat the important points now.

Suppose Ball 1's first throw is to level ℓ. By the next time Ball 1 first returns to level 1, each ball with label less than ℓ will be inactive and hence occupy the top $\ell - 1$ levels of the juggling card, and their order at the top of this card must match their relative order in $\pi(A)$.

Then, unless $\ell = b$, this pattern repeats. Ball 1 is thrown up to some level $\ell' < b - \ell + 1$, balls $\ell + 1$ though ℓ' eventually are all thrown above Ball 1 and they



align themselves properly above balls 2 through ℓ. This process continues until Ball 2 has once again returned to level 2.

For the recurrence, though, we can stop after the first time that Ball 1 returns to level 1. If its first throw is to level ℓ and it takes m more cards for Ball 1 to return to level 1, then the number of ways this can happen is $N_0(m, \ell-1)$. Once these balls have moved to the top of the card (this will include the use of a b-card), we can ignore them and only focus on the remaining $b-(\ell-1)$ balls. Since the first set will never bounce again, and every subsequent ball's bounce is above this whole set or below it, there are simply $N_0\Big(n-(m+1), b-\ell+1\Big)$ ways to conclude the b-valid, length n arrangement. Summing over all choices of ℓ and m gives the recurrence

$$N_0(n,b) = \sum_{\ell=1}^{b}\sum_{m=0}^{n-1} N_0(m,\ell-1)\cdot N_0(n-m-1, b-\ell+1), \tag{1.1}$$

where $N_0(n,n) = 1$ for all $n \in \mathbb{N}_0$ and $N_0(n,b) = 0$ for all $b > n$ except that $N_0(0,1) = 1$.

The solution to this recurrence is the Narayana Numbers $N(n,b) = \frac{1}{n}\binom{n}{b}\binom{n}{b-1}$. We will show this by showing the these number satisfy the same recurrence. To do so, recall that $N(n,b)$ is the number Northeast-Southeast Dyck paths which travel in the first quadrant from the origin to $(2n,0)$ and have b peaks. An example is shown below.

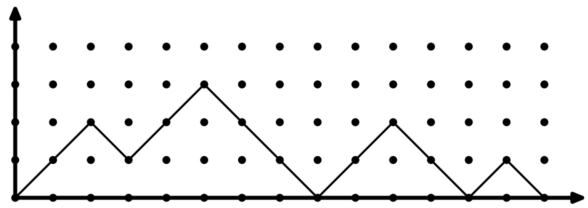
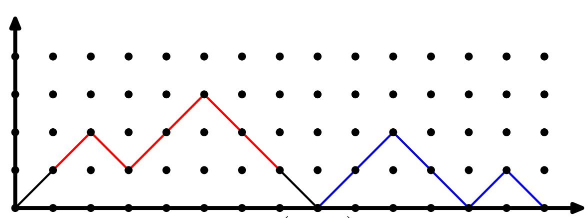


Figure 1.18: One possible path when $(n, b) = (7, 4)$

To create a recurrence, fix n and b with $n > b$ and partition all valid paths by the first time the path returns to the x-axis. If the first return is at the point $(2m, 0)$ and there have been $\ell \in [b]$ peaks thus far, then number of valid conclusions from this point to $(2n, 0)$ is clearly $N(n - m, b - \ell)$.

To count the ways to reach $(2m, 0)$ without touching the x-axis and incurring precisely ℓ peaks, observe that each such path must begin with a northeast step, end with a southeast step, and stay about the line y=1 for all $2(m - 1)$ steps in-between. There are consequently $N(m - 1, \ell)$ options. Here is the previous example where there are $N(m - 1, \ell)$ options for the red path, $N(n - m, b - \ell)$ for the blue, and only one option for the two black steps.



Summing over all possibly values of m and ℓ gives

$$N(n, b) = \sum_{\ell=1}^{b} \sum_{m=1}^{n} N(m - 1, \ell) \cdot N(n - m, b - \ell),$$

which is equivalent to (1). Since the initial conditions also match, $N(n, b)$ is the Narayana number $N_0(n, b)$, as desired.

□

In the following two sections we exhibit a pair of bijections between Dyck paths and and minimal-crossing juggling card sequences.

1.6.1 A reduction bijection with Dyck Paths

Dyck paths are one of many well known combinatorial objects that are connected with the Catalan numbers. In this section and the following we exhibit two bijections between Dyck paths and minimal-crossing juggling card sequences.

We begin by building a list of structural facts, which collectively we call *The Structure Lemma*.

The Structure Lemma

Let A be a b-valid arrangement of n cards for which $X(A) = b(b-1)$ and $\pi(A) = \mathrm{id}$. We break up our argument in the following way.

- For each juggling card in an arrangement A, call a ball *active* if on that card or on some later card that ball will bounce. Otherwise, in the event that all of the ball's bounces were on previous cards, call that ball *inactive*.

- Consider the first ball B_i to bounce for it's final time; i.e. on the, say, k^{th} juggling card B_i bounces and after this card this ball never again bounces, and furthermore on no earlier card did some other ball have this property. Then we can conclude that the k^{th} card must be a b-card; this ball must land above all other balls since otherwise there would be an active ball located above the inactive B_i, an impossibility.

- The next ball B_j to bounce for its final time must again bounce above all active balls, and whether it lands just above B_i or just below B_i is determined by whether in $\sigma = \pi(A)$ we have $\sigma(i) > \sigma(j)$ or $\sigma(i) < \sigma(j)$ (for now we are assuming that $\pi(A) = \text{id}$, so $\sigma(i) = i$). Clearly their relative arrangement at the top of the card must match their relative arrangement in $X(A)$, since inactive balls can never switch their relative arrangement.

- In general we always have that inactive balls occupy the positions at the top of the card and the height of each ball's final throw is determined uniquely by $\pi(A)$.

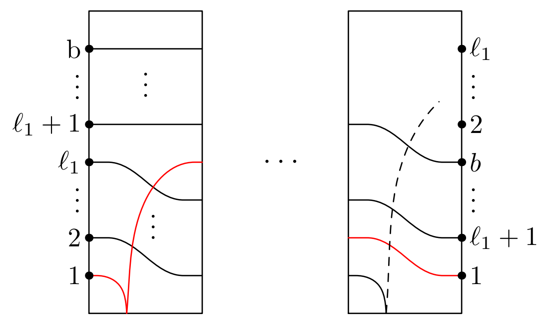
- Now let's incorporate our condition on the number of crossings. This is the minimum number possible, so in particular if $i < j$ then B_i's path will cross B_j's path twice: first during a B_i throw (an upcrossing), while the second during a B_j throw (a downcrossing).

- With this in mind, let's take a macro look at A by focusing on the throw pattern of B_1. Ball 1 will initially be thrown to some level ℓ_1. Let's first suppose that $\ell_1 > 2$. Balls 2 through ℓ_1 have just incurred an upcrossing from Ball 1, and so as soon as they are thrown above Ball 1, incurring a down crossing, they can never cross Ball 1 again.

  Consequently, for any $j \in \{2, 3, \ldots, \ell_1\}$, the bounce in which B_j is thrown above B_1 is also its final throw; moreover, as noted before, the height of that throw is determined by $\pi(A)$. Therefore all of the "action" involving Balls 2 through ℓ_1

involves only each other and occurs entirely below Ball 1.

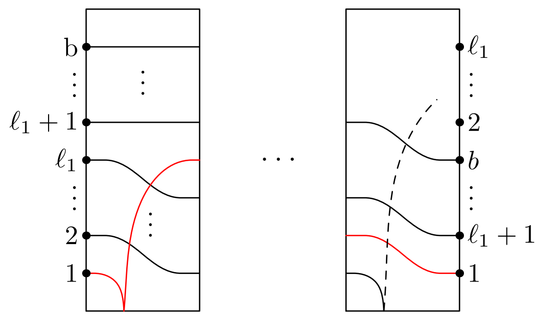
- In fact, it is easy to see that a subproblem is obtained: The set of ℓ_1-valid arrangements A' of m cards are in bijection with the set of possible sequences of $m+1$ cards which appear as the start of some arrangement A of n cards in which A's first card is an ℓ_1-card and the first time that B_1 returns to level 1 is during $(m+1)^{\text{st}}$ card. In the subsequent bullet points we will work out an example.



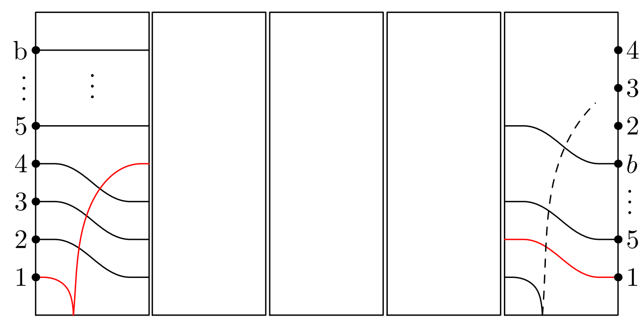
- When Ball 1 returns to level 1 we get a different sort of subproblem: The $\ell-1$ inactive balls can now be ignored. The number of $(b-\ell+1)$-valid arrangements of $n-m-1$ cards with $\pi(A) = \text{id}$ and $X(A) = (b-\ell+1)(b-\ell)$ are in bijection with the number of ways to complete A. The only difference is that the final throw of the former must be altered deterministically so that they land above all active balls. Before continuing with our description, let's look at an example.

In general B_1 is thrown first to some level ℓ_1, and after, say, m cards B_1 returns to level 1.

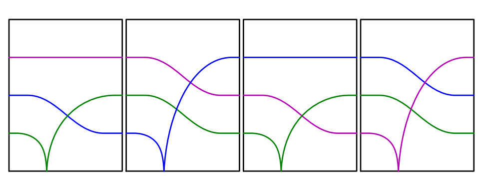
In our example we will assume that $\ell = m = 4$, which then gives the following picture in which three cards are left to be filled in, and the details of the fourth still need to be determined. Notice how, at the end, balls 2 through ℓ are at the



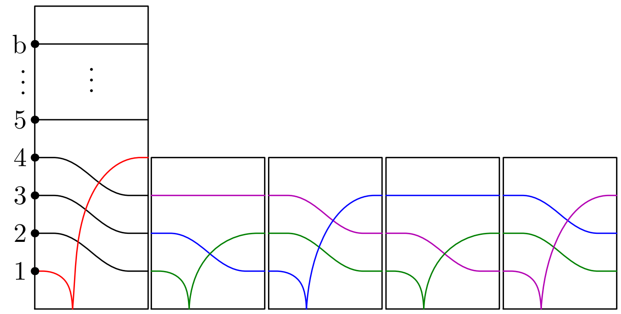
top of the card in the order matching $\pi(A) = \text{id}$. These are the inactive balls, while all others are still active.



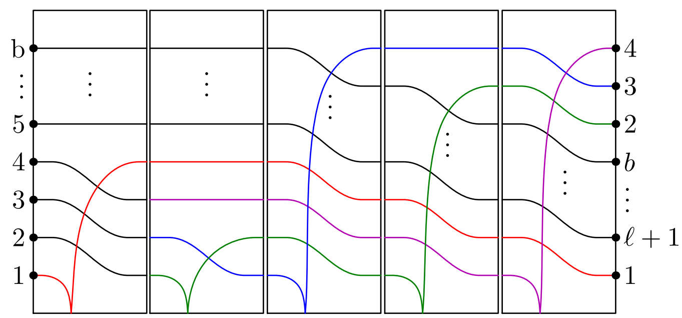
Our claim is that the $\ell - 1 = 3$ balls below B_1's initial throw create a subproblem, and the number of ways to fill in the above cards is precisely the number of ways to have an $(\ell - 1)$-valid sequence of m cards with a minimal number of crossings. Moreover we claim that there is an easy bijection between these two sets. Suppose we have the following 3-valid sequence of 4 cards with the minimal $3(3-1) = 6$ crossings.



We place this next to our first throw. Here we can visualize our previous statement that all of the "action" can be thought of as taking place "below" Ball 1.



Next, we adjust the cards corresponding to each ball's final throw (in this case the final three cards) so that rather than being thrown to the top of these smaller cards, they instead are thrown to the top of height-b cards, all while maintaining their respective ordering.



And now, with Ball 1 returning to level 1 we would next do the same thing again: Ball 1 is thrown to some level $\ell_2 < b - \ell + 1$ and all the action must happen below this second throw. Note that we can now ignore these inactive balls, and instead think about cards of height $b - \ell + 1$.

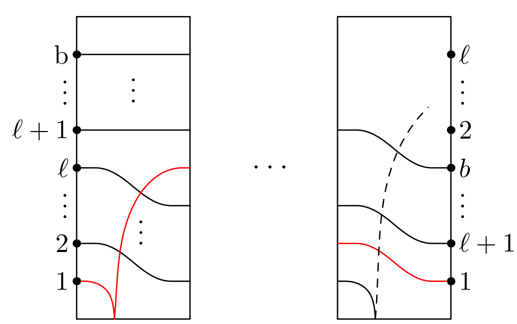
- Let's now return to the first subproblem which occurred "below" Ball 1's initial throw. Here, Ball 2 can do one of two things. The first option is that it can be thrown to the top of the card and become inactive. The second option is that it can now take the role of Ball 1 from before: B_2 is thrown to some level $\ell_2 < \ell_1$ and all of the balls below it now have a sub-subproblem. Continuing in this way, we get a sequence of balls $B_{i_1}, \ldots, B_{i_t}$ and a sequence of throw heights

$\ell_1 > \cdots > \ell_t$ until eventually $\ell_t = 2$. At this point the following ball B_{i_t+1} must bounce to the top of the card, since as we have pointed out before, once it crosses B_{i_t} it must become inactive.

- In general, each ball can either elect to become inactive by being thrown to the top, can be forced to become inactive because the previous throw was a 2-throw, or can be thrown below the previous throw, inducing another subproblem (i.e. another sequence of ℓ_i's).
- We have now completely characterized what A can look like.

Assume that A is a b-valid arrangement of n cards where $\pi(A) = \text{id}$ and $X(A) = b(b-1)$. Let $N_0(n, b)$ be the number of such arrangements. In the previous section we described precisely what these juggling card sequences can look like. Part of that argument quickly gives a nice recurrence. We'll repeat the important points now.

Suppose Ball 1's first throw is to level ℓ. By the next time Ball 1 first returns to level 1, each ball with label less than ℓ will be inactive and hence occupy the top $\ell - 1$ levels of the juggling card, and their order at the top of this card must match their relative order in $\pi(A)$.



Then, unless $\ell = b$, this pattern repeats. Ball 1 is thrown up to some level $\ell' < b - \ell + 1$, balls $\ell + 1$ though ℓ' eventually all are thrown above Ball 1 and they align themselves properly above balls 2 through ℓ. This process continues until Ball 2 has once again returned to level 2.

For the recurrence, though, we can stop after the first time that Ball 1 returns to level 1. If its first throw is to level ℓ and it takes m more cards for Ball 1 to return to level 1, then the number of ways this can happen is $N_0(m, \ell - 1)$. Once these balls have moved to the top of the card (this will include the use of a b-card), we can ignore them and only focus on the remaining $b - (\ell - 1)$ balls. Since the first set will never bounce again, and every subsequent ball's bounce is above this whole set or below it, there are simply $N_0\Big(n - (m + 1), b - \ell + 1\Big)$ ways to conclude the b-valid, length n arrangement. Summing over all choices of ℓ and m gives the recurrence

$$N_0(n, b) = \sum_{\ell=1}^{b} \sum_{m=0}^{n-1} N_0(m, \ell - 1) \cdot N_0(n - m - 1, b - \ell + 1), \tag{1.2}$$

where $N_0(n, n) = 1$ for all $n \in \mathbb{N}_0$ and $N_0(n, b) = 0$ for all $b > n$ except that $N_0(0, 1) = 1$.

The solution to this recurrence is the Narayana Numbers $N(n, b) = \frac{1}{n}\binom{n}{b}\binom{n}{b-1}$. We will show this by showing the these number satisfy the same recurrence. To do so, recall that $N(n, b)$ is the number Northeast-Southeast Dyck paths which travel in the first quadrant from the origin to $(2n, 0)$ and have b peaks. An example is shown below.

To create a recurrence, fix n and b with $n > b$ and partition all valid paths by the first time the path returns to the x-axis. If the first return is at the point $(2m, 0)$

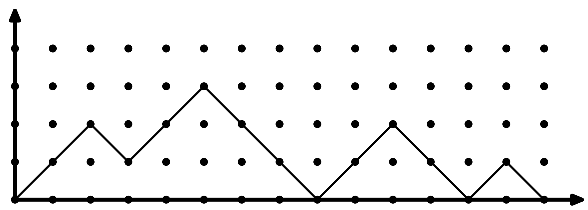
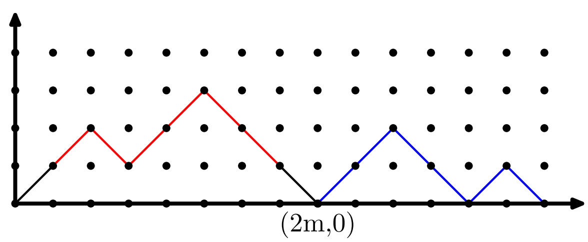


Figure 1.19: One possible path when $(n, b) = (7, 4)$

and there have been $\ell \in [b]$ peaks thus far, then number of valid conclusions from this point to $(2n, 0)$ is clearly $N(n-m, b-\ell)$.

To count the ways to reach $(2m, 0)$ without touching the x-axis and incurring precisely ℓ peaks, observe that each such path must begin with a northeast step, end with a southeast step, and stay about the line y=1 for all $2(m-1)$ steps in-between. There are consequently $N(m-1, \ell)$ options. Here is the previous example where there are $N(m-1, \ell)$ options for the red path, $N(n-m, b-\ell)$ for the blue, and only one option for the two black steps.



Summing over all possibly values of m and ℓ gives

$$N(n, b) = \sum_{\ell=1}^{b} \sum_{m=1}^{n} N(m-1, \ell) \cdot N(n-m, b-\ell),$$

which is equivalent to (1). Since the initial conditions also match, $N(n, b)$ is the Narayana number $N_0(n, b)$, as desired. $\square$

A bijection between Dyck paths and Juggling Sequences

We have shown that the number of b-valid juggling sequences A of length n where $X(A) = b(b-1)$ and $\pi(A) = \mathrm{id}$ is equal to the number of Northeast-Southeast Dyck paths from the origin to $(2n, 0)$ with b peaks. Let's now find a bijection between them. The important ingredients in discovering it were the previous recurrence relation and the prior precise description of the structure of such juggling card sequences.

We record a few properties which will be used to map a valid Dyck path to a juggling card sequence.

- Our bijection will receive a valid Dyck path and output a juggling sequence. The function does this by generating one card at a time, in order. The two Dyck steps that correspond to the next generated card are typically not adjacent; in fact, they are only adjacent if they meet to form a peak. The idea is to "reduce" the problem to a subproblem at each stage. Each successive card will make an analogous reduction.

- If a Dyck path only touches the x-axis at its starting and ending points then the first card will be a 1-card. After removing the first and last steps of the path and shifting the result, this reduces the Dyck path to another which travels from the origin to $(2(n-1), 0)$ and has b peaks. This is called a *length reduction*. It reduces the juggling card sequence by one card while keeping everything else the same; a 1-card accomplishes this.

- If the Dyck path touches the x-axis some time between the starting and ending points, say the first touch is at $(2m, 0)$, then we reduce the problem by partitioning into two parts: The Dyck path from $(1, 1)$ to $(2m-1, 1)$, and the second Dyck path from the $(2m, 0)$ to $(2n, 0)$; when considering each part, for convenience we shift it so that it begins at the origin. This is called a *partition reduction*.

If there are $\ell - 1$ peaks in the first part, then the first card will be an ℓ-card. The soundness of this follows from the Structure Lemma, which describes how such a throw will reduce the juggling card sequence into two subproblems, one of length m where the cards of height $\ell - 1$, and one of length $n - m$ where the cards of height $b - \ell + 1$.

- If we perform a length reduction, we will next look at the reduced Dyck path to generate the next card. If we perform a partition reduction, we will look at the left part first and only move to the right part when the all cards corresponding to the left have been generated. Once we have a reduced Dyck path we return to the previous two bullet points to decide which card to generate next.

- Note that if the Dyck path's first return is at $(2, 0)$ then the first card will be a b-card. In general, if after a reduction we encounter such a length-2 path, the card chosen is the one which makes the corresponding ball inactive. We saw in the Structure Lemma that the card chosen is the one which sends the ball above all active balls, and whose relative order among the inactive balls matches $\pi(A)$.

With these guidelines we can generate all juggling sequences. For example, the path below corresponds to the card sequence $C_3, C_4, C_1, C_4, C_2, C_4, C_1$. In the caption below this figure we have colored this sequence. In the Dyck path, the two steps whose color matches that of a card are the two steps which generated that card.

Notice that the first card generated, colored red, was produced by a partition

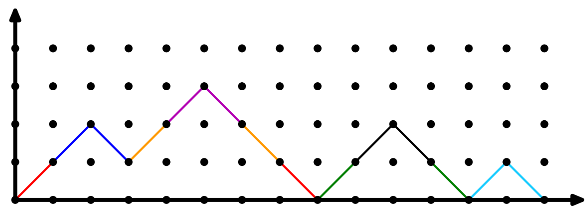


Figure 1.20: C_3, C_4, C_1, C_4, C_2, C_4, C_1

reduction. Then, looking at the subproblem containing the colors blue, orange and purple we see that again we have a partition reduction, generating the blue card; recall that a single peak always corresponds to a ball becoming inactive, so Ball 2 at this point is being sent to the top of the card via a b-card ($b = 4$), becoming the first inactive ball.

Next we are just looking at the orange and purple, so by removing the orange steps we have a length reduction, giving a 1-Card. The purple peak, just like the blue one, corresponds to a ball becoming inactive. Therefore the ball must be sent to the top of the card and its position among the inactive balls must match $\pi(A) = \mathrm{id}$; in this case Ball 3 is bouncing upward and the only other inactive ball is Ball 2, so a 4-Card is needed. The final 3 cards are generated analogously.

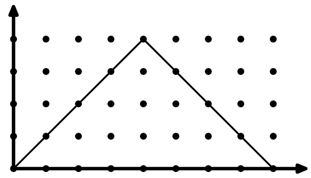
Lastly, observe that through this lens we could also write down an combinatorial interpretation of strong juggling sequences (i.e. sequences which do not use a 1-Card).

This again proves Theorem 8

We now show explicitly all bijection pairings when $n = 4$.

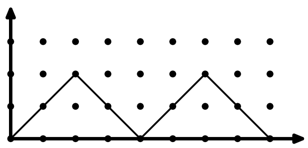
Example 3. In the caption of each Dyck path is the corresponding juggling card sequence.

<u>If $b = 1$ the sole pairing is the following.</u>

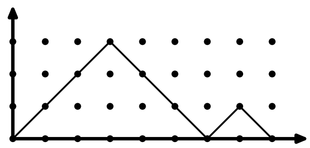


(a) 1111

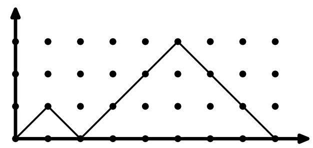
If $b = 2$ the 6 pairings are the following.



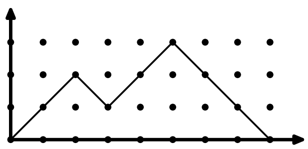
(a) 2211



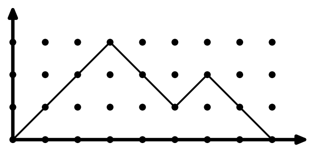
(b) 2121



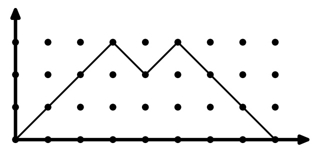
(c) 2112



(d) 1212

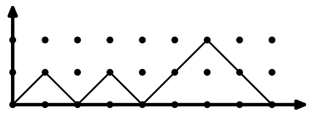


(e) 1221

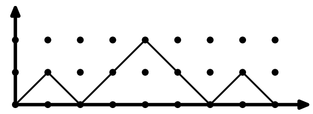


(f) 1122

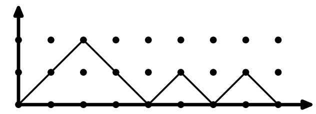
If $b = 3$ the 6 pairings are the following.



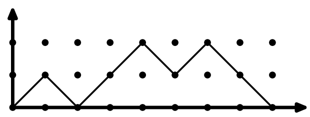
(a) 3313



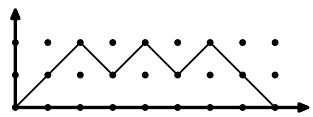
(b) 3232



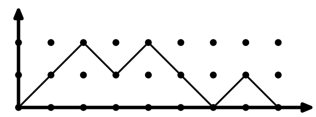
(c) 2323



(d) 3133

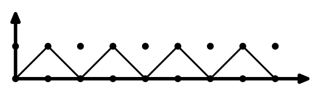


(e) 1333



(f) 3331

If $b = 4$ the sole pairing is the following.



(a) 4444

1.6.2 A parenthesization bijection with Dyck Paths

The second proof also makes use of these decompositions.

Lemma 9. *Given a minimal crossing juggling card sequence A with b balls using n cards, there is a* unique *pair of minimal crossing juggling card sequences (B, C) so that B uses k balls, and m cards and C uses $b - k$ balls and $n - m - 1$ cards (with the possibility that B or C might be empty). Further, given any such pair of minimal crossing juggling card sequences (B, C), the minimal crossing juggling card sequence A can be determined.*

Proof. The first card of A will throw the ball up to some level $k + 1$ and will thus cross paths with balls $2, \ldots, k + 1$. By the time that the first ball is thrown for a second time, the first ball will have had to cross paths with balls $2, \ldots, k + 1$ a second time.

Because each pair of balls can only cross twice it must be that the ball 1 will never again cross with balls $2, \ldots, k+1$. In particular, we will never throw balls $2, \ldots, k+1$ after we throw ball 1 the second time. From this we conclude that all the crossings between balls $2, \ldots, k+1$ will occur between the first two throws of ball 1 and that the relative ordering of balls $2, \ldots, k+1$ will be set when we get to the second throw of ball 1.

So between the first two throws of ball 1, if we ignore balls $1, k+2, \ldots, b$ then we have a juggling card sequence for k balls with $k(k-1)$ crossings with the final arrangement corresponding to the identity.

If we now ignore balls $2, \ldots, k+1$ from the second throw of ball 1 until the end then we must again have all of the $(b-k)(b-k-1)$ crossings among the remaining balls with the final arrangement corresponding to the identity.

We can now conclude that every juggling card sequence that we want to count can be broken into the following three parts:

- The first card which throws ball 1 to height $k+1$.
- The set of cards between the first two occurrences of the throw of ball 1; a juggling card sequence with m cards and k balls having $k(k-1)$ crossings and corresponding to the identity arrangement. We denote this minimal crossing juggling card sequence by B.
- The set of cards from the second time ball 1 is thrown to the end; a juggling card sequence with $n-m-1$ cards and $b-k$ balls having $(b-k)(b-k-1)$ crossings and corresponding the identity arrangement. We denote this minimal crossing juggling card sequence by C.

The first card can be found by knowing the number of balls used in B, so therefore we only need to know B and C. Further, given the above information, we can reconstruct the juggling card sequence for A. Namely, we have the first card. For the next set of cards as determined by B, we initially add balls $1, k+2, \ldots, b$ on top of the balls $2, \ldots, k+1$ and then we continue with the same cards as before *except* for the last time each ball is thrown we increase the height of the throw to move above $1, k+2, \ldots, b$, i.e., the card C_t will be replaced by C_{t+b-k}. For the last set of cards as determined by C, we do the same process where we initially add balls $2, \ldots, k$ on the top and then we continue with the same cards as before *except* for last time each ball $k+2, \ldots, b$ is thrown we increase the height of the throw to move above $2, \ldots, k+1$, i.e., the card C_t will be replaced by C_{t+k}. $\square$

To help illustrate the correspondence used in Lemma 9 in Figure 1.25 we give two juggling card sequences with minimal crossings, one for 2 balls and 3 cards and the other for 3 balls and 4 cards. In Figure 1.26 we give the corresponding juggling card sequence; to help emphasize the structure we shade the portion of the balls which move in unison according to the construction in the lemma in the parts coming from B and C.

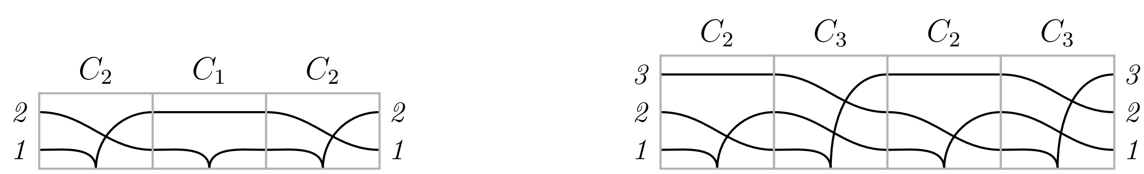


Figure 1.25: Two minimal crossing juggling card sequences

Let us suppose that we indicate the preceding correspondence in the following way, if B and C are the minimal crossing juggling card sequences that generate the minimal crossing juggling card sequence A then we write this as $A = (B)C$. So that

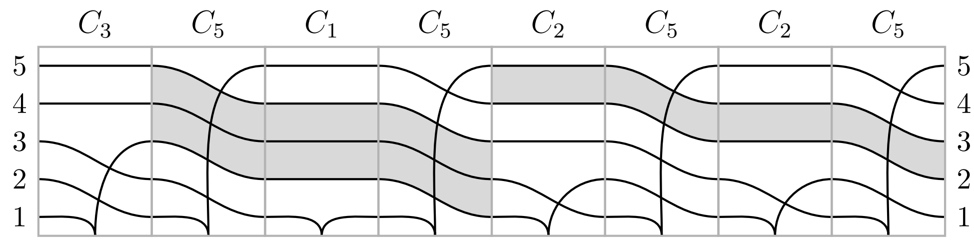


Figure 1.26: The result of combining the two sequences in Figure 1.25

the example from Figures 1.25 and 1.26 would be written as

$$C_3C_5C_1C_5C_2C_5C_2C_5 = (C_2C_1C_2)C_2C_3C_2C_3.$$

Now we simply apply this convention recursively to each minimal crossing juggling card sequence, following the rule that if one part is empty we do not write anything. So $(*)$ would be a juggling card sequence where ball 1 does not return until the last card, $()*$ would be a juggling card sequence where the first card is C_1, and $()$ corresponds to the unique minimal juggling card sequence consisting of a single card, C_1. If we now carry this out on the above example we get the following:

$$\begin{aligned} C_3C_5C_1C_5C_2C_5C_2C_5 &= (C_2C_1C_2)C_2C_3C_2C_3 \\ &= ((C_1C_1))(C_1)C_2C_2 \\ &= ((()C_1))(())(C_1) \\ &= ((()()))(())(()) \end{aligned}$$

This leads naturally to Dyck paths by associating "(" with an up and to the right step and ")" with a down and to the right step, which in our example gives the Dyck path shown in Figure 1.27. This process can be reversed (working from right to left and inside to outside), giving us a bijection between these minimal crossing juggling card sequences and Dyck paths.

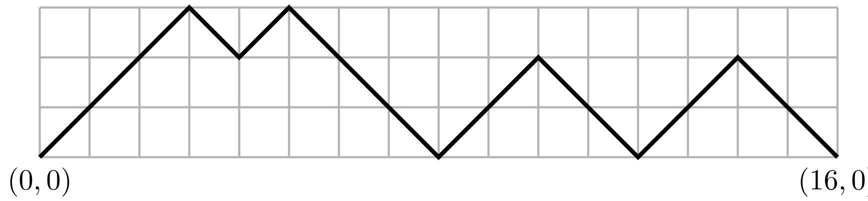


Figure 1.27: The Dyck path for the juggling sequence in Figure 1.26

Careful analysis of the bijection shows that a juggling card sequence with b balls and n cards will produce a Dyck path from $(0, 0)$ to $(2n, 0)$ which has $n + 1 - b$ peaks. This latter statistic on Dyck paths is counted by the Narayana numbers (see `A001263` in [17]). This provides a second proof of Theorem 8. □

We will see yet another proof of this fact in Section 1.6.4, this time using generating functions.

1.6.3 Non-crossing partitions

An alternative way to establish Theorem 8 is to note that the Narayana numbers are the number of ways to partition $[n]$ into b disjoint nonempty sets which are non-crossing, i.e., so that there are no $a < b < c < d$ so that $a, c \in S_i$ and $b, d \in S_j$ (e.g., see [16]). The sets S_i, formed by the locations of when the i-th ball is thrown, form such a non-crossing partition (i.e., if such $a < b < c < d$ exist then balls i and j intersect at least *three* times, which is impossible). One then checks that using the same construction as in Theorem 3 that we can go from a non-crossing partition to one of the juggling card sequences we are counting establishing the bijection.

The important observation to make here, and which we will rely on moving forward, is that if we know the ordering of the balls at the left and right ends *and* we know the order in which the balls are thrown, then we can uniquely determine the cards.

1.6.4 Counting using generating functions

We will now give another proof of Theorem 8 which will employ the use of generating functions. We focus on looking at the *ball throwing patterns* $P = \langle b_1, b_2, \ldots, b_n \rangle$ which list the balls thrown at each step. Given that the minimal crossing juggling card sequences will have each of the b balls thrown we have that P is a partition of $[n]$ into b nonempty sets which are ordered by smallest element.

We will find it convenient to consider a shorthand notation $P^* = \langle d_1, d_2, \ldots, d_r \rangle$ for a pattern P where each d_k denotes a block of d_k's of length at least one, and adjacent d_k's are distinct (note that repeated d_k's correspond to use of the card C_1). Thus, if $P = \langle 1, 1, 1, 2, 2, 2, 1, 3, 3, 3, 3, 2, 2, 4 \rangle$ then the reduced pattern is $P^* = \langle 1, 2, 1, 3, 2, 4 \rangle$. As noted before, in the patterns that we are interested in counting, each pair of balls cross exactly twice and so there cannot be an occurrence of $\langle \ldots, a, \ldots, b, \ldots, a, \ldots, b, \ldots \rangle$ in P^*.

Proof of Theorem 8. We now define the following generating functions:

$$F_b(y) = \textstyle\sum_{n\geq 1} f(b,n)y^n,$$

$$F(x,y) = \textstyle\sum_{b,n\geq 1} f(b,n)x^b y^n = \sum_{b,n\geq 1} F_b(y)x^n.$$

For $b = 1$, we have $f(1,n) = 1$ for all n, since the only possible juggling card sequence consists of n identical cards C_1. Thus,

$$F_1(y) = y + y^2 + y^3 + \cdots = \frac{y}{1-y}.$$

Let us consider the only possible reduced pattern $P^* = \langle 1, 2, \overline{1} \rangle$ of ball throwing patterns for $b = 2$. The notation $\overline{1}$ indicates that this block of 1's may be empty. Thus,

$$F_2(y) = \frac{y}{1-y} F_1(y) \frac{1}{1-y} = \frac{y^2}{(1-y)^3}$$

where the fraction $\frac{1}{1-y}$ allows for the possibility that the second block of 1's may be empty (i.e., this is $1 + F_1(y)$).

For $b = 3$, there are two possibilities for the reduced pattern P^*. The first is that $P^* = \langle 1, C, \overline{1} \rangle$ where C consists of 2's and 3's (and both must occur). The second is that $P^* = \langle 1, 2, 1, 3, \overline{1} \rangle$. Thus, we have

$$F_3(y) = \frac{y}{1-y} F_2(y) \frac{1}{1-y} + \frac{y}{1-y} F_1(y) \frac{y}{1-y} F_1(y) \frac{1}{1-y} = \frac{y^3(y+1)}{(1-y)^5}.$$

Now consider the case for a general $b \geq 3$. Here, we can also partition the possibilities for P^* into two cases. On one hand, we can have $P^* = \langle 1, C \rangle$ where C is a pattern using all $b - 1$ of the balls $\{2, 3, \ldots, b\}$. The number of possible reduced patterns in this case is $\frac{y}{1-y} F_{b-1}(y)$. On the other hand, there may be additional 1's which occur after the first block of 1's. In this case P^* has the form $\langle 1, C_1, C_2 \rangle$ where C_1 uses $i > 0$ balls (not including 1), and C_2 begins with a 1 and uses $j > 0$ balls (including 1). Note this decomposition is the same that was given in Lemma 9. Since $C_1 \cup C_2 = [b]$ then $i + j = b$. In this case the number of possible patterns is given by the following expression:

$$\sum_{\substack{0<i<b\\ i+j=b}} \frac{y}{1-y} F_i(y)F_j(y)$$

Therefore we have,

$$F_b(y) = \frac{y}{1-y} F_{b-1}(y) + \sum_{\substack{0<i<b\\ i+j=b}} \frac{y}{1-y} F_i(y)F_j(y)$$

Multiplying both sides by x^b and summing over $b \geq 2$, we obtain

$$\begin{aligned}
F(x,y) - xF_1(y) &= \sum_{b\geq 2} F_b(y)x^b \\
&= \frac{y}{1-y} \sum_{b\geq 2} x^b F_{b-1}(y) + \frac{y}{1-y} \sum_{b\geq 2} \sum_{\substack{0<i<b,\\ i+j=b}} x^i F_i(y)\, x^j F_j(y) \\
&= \frac{y}{1-y} (xF(x,y) + \big(F(x,y)\big)^2)
\end{aligned}$$

In other words,

$$y\big(F(x,y)\big)^2 = (1-y-xy)F(x,y) - xy. \tag{1.3}$$

Solving this for $F(x,y)$, we get

$$\begin{aligned}
F(x,y) &= \frac{1}{2y}\big(1-y-xy-\sqrt{(1-y-xy)^2-4xy^2}\big) \\
&= \frac{1}{2y}\big(1-y-xy-\sqrt{(1+y-xy)^2-4y}\big) \\
&= \frac{1}{2y}\left(1-y-xy-(1+y-xy)\sqrt{1-\frac{4y}{(1+y-xy)^2}}\right) \\
&= \frac{1}{2y}\Bigg(1-y-xy
\end{aligned}$$

$$
\begin{aligned}
&\qquad - (1+y-xy) + (1+y-xy)\sum_{k\geq 1}\frac{(2k-2)!}{2^{2k-1}k!(k-1)!}\frac{(4y)^k}{(1+y-xy)^{2k}}\Bigg)\\
&= \frac{1}{2y}\left(-2y + 4y\sum_{k\geq 0}\frac{(2k)!}{2^{2k+1}(k+1)!k!}\frac{(4y)^k}{(1+y-xy)^{2k+1}}\right)\\
&= -1 + \sum_{k\geq 0}\frac{1}{k+1}\binom{2k}{k}y^k\sum_{j\geq 0}\binom{2k+j}{j}y^j(x-1)^j.
\end{aligned}
$$

Extracting the coefficient of $x^b y^n$, we obtain

$$
f(b,n) = \sum_{k\geq 0}\frac{1}{k+1}\binom{2k}{k}\binom{n+k}{n-k}\binom{n-k}{b}(-1)^{n-b-k}.
$$

It remains to check that the right-hand side reduces to $\frac{1}{b}\binom{n}{b-1}\binom{n-1}{b-1}$. Rewriting the right hand side, we obtain

$$
f(b,n) = \frac{1}{b}\binom{n-1}{b-1}\sum_{k\geq 0}\binom{n+k}{k+1}\binom{n-b}{k}(-1)^{n-b-k}.
$$

Thus, our proof will be complete if we can show

$$
\sum_{k\geq 0}(-1)^{n-b-k}\binom{n+k}{k+1}\binom{n-b}{k} = \binom{n}{b-1}.
$$

However, this follows at once by identifying the coefficients of x^b in the expressions

$$
\frac{1}{(1-x)^n}(1-x)^{n-b} = (1-x)^{-b}.
$$

$\square$

Knowing that

$$F(x,y) = \sum_{b\geq 1}\sum_{n\geq b} \frac{1}{b}\binom{n}{b-1}\binom{n-1}{b-1}x^b y^n,$$

we can substitute into (1.3) and identify the coefficients of $x^b y^n$ to obtain the following interesting binomial coefficient identity

$$\sum_{\substack{1\leq i\leq b-1\\ 1\leq j\leq n-2}} \frac{1}{i(b-i)}\binom{j}{i-1}\binom{j-1}{i-1}\binom{n-1-j}{b-i-1}\binom{n-2-j}{b-i-1} = \frac{2}{b}\binom{n-1}{b-2}\binom{n-2}{b-1}.$$

1.7 Juggling card sequences with $b(b-1)+2$ crossings

In the preceding section we looked at minimal crossing juggling card sequences. In this section we want to look at the ones which are *almost* minimal, in the sense that we will increase the number of crossings to $b(b-1)+2$. We will focus on the analysis of the ball throwing patterns.

Since each pair of balls cross at least twice and will always cross an even number of times, then it must be the case that there is a special pair of balls, call then a and b with $a < b$, which cross four times. Therefore the ball throwing pattern contains the pattern $\langle \ldots, a, \ldots, b, \ldots, a, \ldots, b, \ldots\rangle$. It is possible that there might be additional copies of the a's and b's so that this problem is not equivalent to counting the number of partitions with one crossing, for which if has been shown (see [1, 3]) that the number of partitions of $[n]$ into b sets which have exactly one crossing is $\binom{n}{b-2}\binom{n-5}{b-3}$. Nevertheless, we will see that the answers are similar and in this section we will establish the following.

Theorem 10. *The number of juggling card sequences A with b balls, using n cards one of which is C_b, having $\pi_A = id$ and $X(A) = b(b-1)+2$ is*

$$g(b,n) = \binom{n}{b+2}\binom{n}{b-2}.$$

1.7.1 Structural result

To help establish Theorem 10 it will be useful to understand the structure of these ball throwing patterns.

Lemma 11. *A ball throwing pattern, P, of length n using b balls with two additional crossings can be decomposed into four ball throwing patterns with no additional crossings, P_0, P_1, P_2, P_3 where P_i has length $m_i \geq 1$ using $c_i \geq 1$ balls, $m_0 + m_2 + m_2 + m_3 = n$, $c_0 + c_1 + c_2 + c_3 = b+2$,* and *a choice of the location of an entry, i_1, in P_0.*

Proof. The crossings between a and b will happen in four of the cards for the juggling card sequence, and using the ball throwing pattern we can determine precisely where this will happen. Namely, we know that since $a < b$ then a must at some first point be thrown higher than b which will occur at the last occurrence of a before the first occurrence of b (i.e., the last time we throw a before we see b); suppose this happens at i_1. Then the next crossing happens at the last occurrence of b before the first occurrence of a after i_1; suppose this happens at i_2. Then the next crossing happens at the last occurrence of a before the first occurrence of b after i_2; suppose this happens at i_3. Finally the last crossing happens at the last occurrence of b before the first occurrence of a after i_3; suppose this happens at i_4. In particular we have the following (where some of the "..." might be empty):

$$\begin{array}{rc} \text{Ball throwing pattern:} & \langle \ldots, a, \ldots, b, \ldots, a, \ldots, b, \ldots \rangle \\ \text{Location of crossings:} & \quad i_1 \qquad i_2 \qquad i_3 \qquad i_4 \end{array}$$

Note that there might be additional occurrences of a and b in the ball throwing pattern, so far we have focused only on the location of the crossings.

We now split the ball throwing pattern into four subpatterns P_i as follows:

- P_1 consists of the entries of P between $i_1 + 1$ and i_2 (inclusive).
- P_2 consists of the entries of P between $i_2 + 1$ and i_3 (inclusive).
- P_3 consists of the entries of P between $i_3 + 1$ and i_4 (inclusive).
- P_0 consists of the remaining entries of P, namely up to i_1 *and* after $i_4 + 1$.

Note that no subpattern contains *both* a and b (by construction), and therefore each one of these subpatterns (by proper relabeling, i.e., so that the first occurrences of the balls in order are $1, 2, \ldots$) give ball throwing patterns with no additional crossings. So we have decomposed the ball throwing pattern into four patterns with no additional crossings, by construction the sum of the lengths of the subpatterns is n. We further have the following which gives information about the number of palls in the subpatterns.

Claim. No ball other than a and b occurs in two of the P_i.

To see this suppose that a ball c occurred both in P_1 and P_2. Then it must be the case that our pattern P contains $\langle \ldots, c \ldots, b, \ldots, c \rangle$. But this is impossible, because between the two occurrences of c in the pattern c had to go above b (one crossing) and then b had to go above c (a second crossing) and so there are no more available crossings for b and c to interact. However we know that the ordering on both ends is the identity and so there must be another crossing at some point either before or after the c's to put them in the correct order at both ends giving us a third crossing which is impossible (since other than the pair a and b, each pair crosses exactly twice). The same argument works for each other pair of intervals.

Therefore we can conclude that a appears in P_0 and P_2, b appears in P_1 and P_3 and each other ball appears in exactly one of the P_i. Letting c_i denote the number of balls in each P_i we can conclude that $c_0+c_1+c_2+c_3=b+2$. Finally we note that the decomposition for P involved splitting the interval for P_0 at some point, for which there are m_0 places we could have chosen (i.e., i_1 is something from $1, 2, \ldots, m_0$).

To finish the bijection we now show how to take four patterns P_0, P_1, P_2, P_3 with no additional crossings with lengths $m_0+m_1+m_2+m_3=n$, number of balls $c_0+c_1+c_2+c_3=b+2$, and a choice $1 \leq i_1 \leq m_0$ to form a pattern P with two additional crossings. We start by first labeling the balls so that they are all distinct among all the P_i and no balls are yet labeled a and b and carry out the following three steps:

1. Whichever ball is thrown in position i_1 in P_0 we relabel that ball a in all its occurrences in P_0. Whichever ball is thrown in position m_1 in P_1 we relabel that ball b in all its occurrences in P_1. Whichever ball is thrown in position m_2 in P_2 we relabel that ball a in all its occurrences in P_2. Whichever ball is thrown in position m_3 in P_3 we relabel that ball b in all its occurrences in P_3. (Note that we now have b different labels in use.)

2. Form a ball throwing pattern by concatenating, in order, the first i_1 entries from P_0, all of P_1, all of P_2, all of P_3, and the remaining $m - i_1$ entries from P_0.

3. Relabel the balls so that the first occurrences of the balls in order are $1, 2, \ldots$.

This produces a ball throwing pattern which has $b(b-1)+2$ crossings (i.e., since a and b will cross four times and no other pair of balls can have more than two crossings). Further, applying the preceding decomposition argument we can precisely recover P_0, P_1, P_2, P_3 and our choice of i_1, establishing the bijection. □

1.7.2 Using generating functions

As in the preceding section, we can define a generating function for what we are trying to count,

$$G(x, y) = \sum_{b \geq 2, n \geq 4} g(b, n) x^b y^n.$$

We are now ready to establish Theorem 10

Proof of Theorem 10. From Lemma 11 we know that the ball throwing patterns we want to count can be decomposed into four ball throwing patterns with no crossings and where there is a choice of where to make a split on the first pattern. Therefore we have

$$g(b, n) = \sum_{\substack{c_i, m_i \geq 1 \\ c_0+c_1+c_2+c_3=b+2 \\ m_0+m_1+m_2+m_3=n}} m_0 f(c_0, m_0) f(c_1, m_1) f(c_2, m_2) f(c_3, m_3). \tag{1.4}$$

We recall the generating function for the ball throwing patterns with no crossings (i.e., for minimal crossing juggling sequences),

$$F(x,y)=\sum_{b,n\geq 1} f(b,n)x^b y^n = \frac{1-y-xy-\sqrt{(1-y-xy)^2-4xy^2}}{2y},$$

and note that

$$y\frac{\partial}{\partial y}\big(F(x,y)\big)=\sum_{b,n\geq 1} nf(b,n)x^b y^n.$$

If we now multiply both sides of (1.4) by $x^b y^n$ and then sum we have the following

$$\begin{aligned}
G(x,y) &= \sum_{b\geq 2,n\geq 4} g(b,n)x^b y^n \\
&= \sum_{b\geq 2,n\geq 4}\Bigg(\sum_{\substack{1\leq c_i,m_i\\ c_0+c_1+c_2+c_3=b+2\\ m_0+m_1+m_2+m_3=n}} m_0 f(c_0,m_0)f(c_1,m_1)f(c_2,m_2)f(c_3,m_3)\Bigg)x^b y^n \\
&= \frac{1}{x^2}\sum_{b\geq 2,n\geq 4}\sum_{\substack{1\leq c_i,m_i\\ c_0+c_1+c_2+c_3=b+2\\ m_0+m_1+m_2+m_3=n}} \begin{matrix}\big(m_0 f(c_0,m_0)x^{c_0}y^{m_0}\times f(c_1,m_1)x^{c_1}y^{m_1}\\ \times f(c_2,m_2)x^{c_2}y^{m_2}\times f(c_3,m_3)x^{c_3}y^{m_3}\big)\end{matrix} \\
&= \frac{1}{x^2}\left(y\frac{\partial}{\partial y}\big(F(x,y)\big)\right)\times F(x,y)\times F(x,y)\times F(x,y) \\
&= y\frac{\partial}{\partial y}\left(\frac{\big(F(x,y)\big)^4}{4x^2}\right).
\end{aligned}$$

Taking the known expression for $F(x,y)$ and letting $z=1-y-xy$ we have

$$\frac{\big(F(x,y)\big)^4}{4x^2}=\frac{8z^4-32xy^2z^2+16x^2y^4-(8z^3-16xy^2z)\sqrt{z^2-4xy^2}}{64x^2y^4},$$

Further we have

$$\sqrt{z^2-4xy^2}=z\sqrt{1-\frac{4xy^2}{z^2}}$$

$$
\begin{aligned}
&= z - z \sum_{k\geq 1} \frac{(2k-2)!}{2^{2k-1}k!(k-1)!} \frac{(4xy^2)^k}{z^{2k}} \\
&= z - \frac{2xy^2}{z} - z \sum_{k\geq 2} \frac{(2k-2)!}{2^{2k-1}k!(k-1)!} \frac{(4xy^2)^k}{z^{2k}} \\
&= z - \frac{2xy^2}{z} - 2 \sum_{k\geq 0} \frac{(2k+2)!}{(k+2)!(k+1)!} \frac{x^{k+2}y^{2k+4}}{z^{2k+3}}.
\end{aligned}
$$

Substituting this in and simplifying we have

$$
\begin{aligned}
\frac{\big(F(x,y)\big)^4}{4x^2} &= \frac{1}{4}(z^2 - 2xy^2) \sum_{k\geq 0} \frac{(2k+2)!}{(k+2)!(k+1)!} \frac{x^k y^{2k}}{z^{2k+2}} \\
&= \frac{1}{4} \sum_{k\geq 0} \frac{(2k+2)!}{(k+2)!(k+1)!} \frac{x^k y^{2k}}{z^{2k}} - \frac{1}{2} \sum_{k\geq 0} \frac{(2k+2)!}{(k+2)!(k+1)!} \frac{x^{k+1}y^{2k+2}}{z^{2k+2}} \\
&= \frac{1}{4} + \frac{1}{2} \sum_{k\geq 2} \frac{(2k)!(k-1)}{k!(k+2)!} \frac{x^k y^{2k}}{z^{2k}},
\end{aligned}
$$

where in going to the last line we pull off the first term on the first summand and shift the second summand and then combine noting we can drop the $k = 1$ case. We also have

$$
\frac{1}{z^{2k}} = \frac{1}{\big(1 - y(x+1)\big)^{2k}} = \sum_{j\geq 0} \binom{2k-1+j}{j} y^j (x+1)^j.
$$

Substituting this we now have

$$
\frac{\big(F(x,y)\big)^4}{4x^2} = \frac{1}{4} + \frac{1}{2} \sum_{\substack{j\geq 0\\ k\geq 2}} \frac{(2k)!(k-1)}{k!(k+2)!} \binom{2k-1+j}{j} x^k (x+1)^j y^{2k+j}.
$$

Finally, we can recover $G(x, y)$ since what remains is to take the derivative with respect to y and then multiply by y, which is equivalent to bringing down the power of y. After simplifying, we can conclude

$$\begin{aligned}
G(x,y) &= \frac{1}{2}\sum_{\substack{j\geq 0\\ k\geq 2}} \frac{(2k)!(k-1)(2k+j)}{k!(k+2)!}\binom{2k-1+j}{j} x^k(x+1)^j y^{2k+j} \\
&= \sum_{\substack{j\geq 0\\ k\geq 2}} \binom{2k+j}{k+2,k-2,j} x^k(x+1)^j y^{2k+j} \\
&= \sum_{\substack{n\geq 4\\ k\geq 2}} \binom{n}{k+2,k-2,n-2k} x^k(x+1)^{n-2k} y^n,
\end{aligned}$$

where $\binom{a}{b,c,d}$ is the multinomial coefficient $\frac{a!}{b!c!d!}$ and in going to the last line we make the substitution $j \to n-2k$.

We can now get the coefficient of $x^b y^n$, which is done by using the binomial theorem and summing over possible k. In particular we can conclude

$$\begin{aligned}
g(b,n) &= \sum_k \binom{n}{k+2,k-2,n-2k}\binom{n-2k}{b-k} \\
&= \sum_k \binom{n}{k+2,k-2,b-k,n-b-k}.
\end{aligned}$$

By the special case $a=2$ of Proposition 12 (given below) this is equal to $\binom{n}{b+2}\binom{n}{b-2}$, finishing the proof. $\square$

Proposition 12. $\displaystyle\sum_k \binom{n}{k+a,k-a,b-k,n-b-k} = \binom{n}{b+a}\binom{n}{b-a}.$

Proof. We count the number of ways to select two sets A and B from n elements, with $|A| = b+a$ and $|B| = b-a$. This is clearly equal to the right hand side, so it remains to show how the left hand side equals this as well.

We begin by noting that we can rewrite the multinomial coefficient as a product of binomial coefficients in the following way,

$$\sum_k \binom{n}{k+a, k-a, b-k, n-b-k} = \sum_k \binom{n}{b+k}\binom{b+k}{2k}\binom{2k}{k+a}.$$

We now choose our sets in the following way: First we pick $b+k$ elements which will correspond to $A \cup B$, then among those $b+k$ elements we choose the $2k$ elements which will belong to precisely one of the sets, finally among the $2k$ elements which will belong to exactly one set we choose $k+a$ of them for A and the remaining $k-a$ go to B. Summing over all possibilities for k now gives the desired count. □

1.8 Juggling card sequences with $b(b-1)+4$ crossings

The case of 4 additional crossings proved to be more difficult. One approach we pursued was to simply the types of juggling card sequences we are counting. Let us call a juggling card sequence A *primitive* if it does not use the "trivial" card C_1, i.e., the card which generates the identity permutation. Such a card does not contribute to the number of crossings $X(A)$ of A, nor does it (nontrivially) permute the balls.

Let us denote by $P_\delta(n, b)$ the number of primitive juggling card sequences A with n cards using the card C_b with $\pi(A) = \text{id}$ and $X(A) = b(b-1)+d$, and let $N_\delta(n, b)$ denote the number of such sequences which are not necessarily primitive. Since crossings occur in pairs, d must be even. Then

$$N_\delta(n, b) = \sum_{k=1}^{n} \binom{n}{k} P_\delta(k, b).$$

$P_4(n,b)$	b=2	b=3	b=4	b=5	b=6	b=7	b=8	b=9
n=6	1	3						
n=7		$2{\cdot}7$	$3{\cdot}7$					
n=8		$2^2{\cdot}3$	$2^4{\cdot}7$	$2^2{\cdot}3{\cdot}7$				
n=9			$2^2{\cdot}3^2{\cdot}5$	$2^2{\cdot}3{\cdot}7^2$	$2^3{\cdot}3^2{\cdot}7$			
n=10			$2{\cdot}3^2{\cdot}5$	$2^5{\cdot}3^2{\cdot}5$	$2{\cdot}3{\cdot}5{\cdot}7{\cdot}11$	$2{\cdot}3^2{\cdot}5{\cdot}7$		
n=11				$3^3{\cdot}5{\cdot}11$	$2^4{\cdot}3^3{\cdot}5{\cdot}11$	$2^5{\cdot}3{\cdot}7{\cdot}11$	$2{\cdot}3^2{\cdot}7{\cdot}11$	
n=12				$2{\cdot}5^2{\cdot}11$	$2{\cdot}3^2{\cdot}5{\cdot}11{\cdot}13$	$2^2{\cdot}3^2{\cdot}5{\cdot}11{\cdot}17$	$2^3{\cdot}3{\cdot}7{\cdot}11^2$	$2^2{\cdot}3^2{\cdot}7{\cdot}11$
n=13					$2{\cdot}5{\cdot}7{\cdot}11{\cdot}13$	$2^2{\cdot}3^3{\cdot}5{\cdot}11{\cdot}13$	$2^2{\cdot}3^2{\cdot}11{\cdot}13{\cdot}23$	$2^2{\cdot}3{\cdot}11{\cdot}13{\cdot}29$
n=14					$3{\cdot}7{\cdot}11{\cdot}13$	$5{\cdot}7{\cdot}11{\cdot}13{\cdot}19$	$2^3{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11{\cdot}13$	$2^3{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11{\cdot}13$

Table 1.1: Data for $P_4(n,b)$

The hope would be that $P_\delta(n,b)$ could be simpler in some sense than $N_\delta(n,b)$ and would therefore be easier to recognize. It turns out that if we write $n = b + t$ then it is not hard to show that

$$P_0(n,b) = \frac{1}{t+1}\binom{b-2}{t}\binom{b+t}{t}$$

and

$$P_2(n,b) = \binom{b+t}{2t}\binom{2t}{t-2}.$$

In Table 1.1 we give data (in factored form) for $P_4(n,b)$ for small values of n and b.

The fact that there are many small factors suggest that $P_4(n,b)$ could be made up of binomial coefficients in some way. However, the presence of occasional "large" prime factors makes it difficult to guess what the expressions might actually be (for example, $P_4(14,10) = 3{\cdot}7{\cdot}11{\cdot}13{\cdot}37$). Nevertheless, computations suggested that

$P_4(n,b)$ is given by the following expression:

$$P_4(n,b) = \frac{(bn-b-8)}{2(b+4)}\binom{n}{b+3}\binom{n}{b-2}.$$

Once we had this guess, we were able to confirm it by finding a complicated expression for the answer and then having a computer verify that the two expressions were in fact equal. We will run through this argument, but first we will build up to it by showing the $\delta = 2$ case, which is easier but still captures most of the ideas.

1.8.1 Juggling diagrams for $\delta = 2$

We begin by reviewing some definitions and introducing some more.

Definition. Let A be a juggling card sequence. A ball B_M is called *active* on a given card if B_M will be thrown on some future card from A; otherwise B_M is called *inactive*. The final time that B_M is thrown is called its *deactivation thrown*. If B_M contributes additional crossings, meaning that there exists some other ball B_N such that the paths of B_M and B_N cross more than twice, then call B_M an *AC ball*.

Let A be a b-valid juggling card sequence in which $X(A) = b(b-1)+2$. Let's describe the structure of A.

Every pair of balls must cross each other's paths at least twice. If every pair of balls crossed each other exactly twice then we would have 0 additional crossings, so there must exist exactly two balls, say B_K and B_L ($K < L$), which cross each other four times. We introduce the notion of a *juggling diagram*, such as seen in Figure 1.28. Juggling diagrams are read from the bottom up, one box at a time. We will see that the boxes partition the arrangement in a nice way; the balls listed in the m^{th} box will correspond to the balls which are thrown during the m^{th} part of the arrangement.

So B_K and B_L are our two AC balls. There are two cases to consider: Either B_K becomes inactive after it's second time being thrown above B_L, or it remains active after this throw.

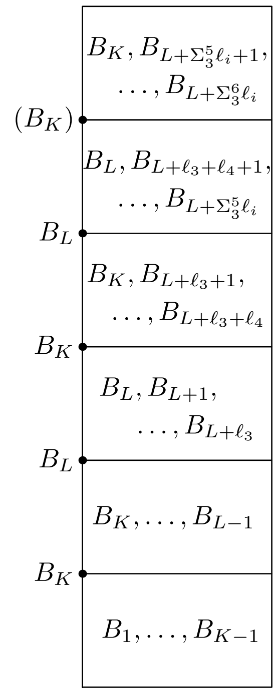


Figure 1.28: $\delta = 2$ juggling diagram

The most important cards in a juggling card arrangement are the cards which have the following two properties. First, the ball being thrown is an AC ball. Second, the ball being thrown has either never been thrown before, or it is the ball's first time being thrown since another AC ball crossed paths with it. Because of their importance, we distinguish the cards in our juggling diagram with horizontal lines. We will show that there are few valid lists of balls for the produced boxes, and that each choice partitions the juggling sequence nicely. Note that after the second time B_K is thrown past B_L, it is possible that B_K is either active or inactive. We represent this in the juggling diagram in Figure 1.28 by putting parentheses around the top B_K. We could have also drawn two separate diagrams, one with B_K there and one without B_K there.

As mentioned, the AC balls are shown on the left of the diagram. Their relative order throughout A is recorded by this list, read starting from the bottom. In particular, in the case with two additional crossings we see that there are exactly two AC balls, B_K and B_L, and if we remove all balls except these two and all juggling cards except the ones in which these balls' paths cross, A necessarily reduces to the ball sequence B_K, B_L, B_K, B_L.

Again observe that it is possible that the B_K could be thrown again after B_L's final throw, as the 2-ball, 5-card juggling arrangement C_2, C_2, C_2, C_2, C_1 shows. Thus, despite using the reduced diagram notation with parentheses, when counting we find it simpler to keep in mind the two possible diagram structures for the $\delta = 2$ case: either B_K, B_L, B_K, B_L if the second time B_K is thrown past B_L is its deactivation throw, or B_K, B_L, B_K, B_L, B_K if that throw is not its deactivation throw.

Next we aim to show that the boxes work "independently" of each other, meaning that the non-AC balls in one box only cross the non-AC balls in another box during a deactivation throw; the one exception to this is that the balls in the bottom-most box and the top-most box may mix, as will soon become evident. For each box, except the bottom and top boxes, we will then be able to read the diagram from the bottom to the top and each ball in that selected box will become inactive before any of the balls in the following box are thrown. The behavior of the AC balls will be slightly different, but will be determined by their position along the left-hand side of the diagram.

To this end, consider a non-AC ball B_C appearing in some box other than the top or bottom box, and a non-AC ball B_D appearing in another box. We now show that B_C appears only its one box. To see this, suppose for a contradiction that B_C crosses B_D during a non-deactivation throw.

Consider one box containing B_C and one containing B_D. Since it is not the case that B_C and B_D are in the bottom and top boxes, respectively, it is easy to see that some AC ball B_M is both inside one of the boxes between B_C and B_D inclusive, and also inside another box. This ball will generate additional crossings with B_C, since by ignoring all balls but these and all cards but the ones in which these balls cross (just as before), we see that the reduced jumping sequence is B_M, B_C, B_M, B_C if $M < C$ or B_C, B_M, B_C, B_M if $M > C$. In either case, this implies that B_C is an AC ball, contrary to assumption.

Observe that the above argument falls apart if B_C is in the bottom box and B_D is in the top box. Indeed, it is straightforward to construct examples in which such a pair do cross during non-deactivation throws.

For the central boxes, the count is now clear in light of the Structure Lemma (see Section 1.6.1). If a box contains ℓ cards and m balls, then there are $N_0(m, \ell)$ ways to choose how these balls can execute their throws.

For the balls in the first and last box, though, the balls may interact. Indeed, consider any ball B_C in the first box and B_D in a box other than the first or the last. If B_C is only in the first box then certainly there are no additional crossings occurring between this pair. Now consider the case where B_C is in both the first and the last boxes. Still, before B_C is thrown from the top box, B_D will only have two crossings with B_C: one from B_C being thrown from the first box to the last box, and one from B_D's deactivation throw. But since $C < D$, $\pi(A) = \mathrm{id}$ and B_D is inactive, B_C's deactivation throw nor any other will cross B_D. So we have shown that no additional crossings are obtained between balls and the outer boxes and balls in the internal boxes by allowing balls in the first box to also appear in the last box.

Now we will show which possible ways of mixing these balls are able to be realized. Assume that there are ℓ balls and m cards used among the top and bottom boxes combined. There are $N_0(m, \ell)$ possible juggling cards arrangements with these quantities, but we must further decide how to take such an arrangement $\tilde{A}$ and split it between the two boxes.

To do this, consider again our original juggling card arrangement (with the additional crossings). Remove all cards which do not correspond to these first and last boxes, and all balls which are only thrown within those cards. As we have seen, we are necessarily left with only the balls $B_1, B_2, \ldots, B_{K-1}, B_M, B_N, \ldots, B_b$ where B_K is the first AC ball, B_M is some AC ball (maybe B_K, maybe not) and N is the first index greater than the indices used in the second-to-top box.

The important observation is that the first ball used in the top box is B_M, which in this reduced arrangement has not been used before. Moreover, given any ball B_P in $\tilde{A}$ other than B_1, it is clear that we can split the arrangement into two pieces, the part before B_P's first thrown, and the rest.

One way to express this is the following. Suppose that the bottom box contains ℓ_1 balls, the top box (the j^{th}, say) contains ℓ_j balls, and m_1 cards are used within the two boxes combined. Then what we have shown is that the total number of ways to execute these throws is $N_0(m_1, \ell_1 + \ell_j)$. Moreover, given a choice of $\ell_1 \geq 0$ and $\ell_j \geq 1$ we can determine which balls are in the first box, which are in the last box, and which ball is the AC ball that appears first in the top box.

From all this we can deduce that the total number of ways this can happen is

$$N_0(m_1, \ell_1 + \ell_j) \cdot \prod_{i=2}^{j-1} N_0(m_i, \ell_i).$$

Summing this over all possible values of the arguments then gives us the total number of ways to realize this juggling diagram.

So we have figured out how to write down a closed-form solution for $N_2(n, b)$ in terms of the Narayana numbers. Note that we could have done the same thing to find $P_2(n,b)$ in terms of $P_0(n,b)$. We will in fact take this approach in the next section.

1.8.2 Juggling diagrams for $\delta = 4$

We can now apply the reasoning from the last section to write down a closed form solution for the case when $\delta = 4$. As soon as we write down all possible juggling diagrams nearly everything is just as before.

Suppose B_K is the smallest-labelled AC ball. Then there is at least one other ball which incurs additional crossings with B_K, so let's suppose B_L is the smallest-labeled AC ball which does this.

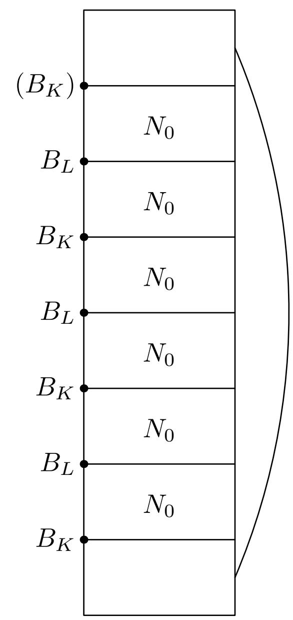


Figure 1.29: Diagram for two AC balls

It is possible that there are 2, 3 or 4 AC balls. The first is case is that B_K and B_L are the only two AC balls. Such a juggling diagram is shown in Figure 1.29. In the later cases it is going to be a little more complicated to determine which boxes can interact in the way described in the previous section. To clarify this we will draw arcs between such boxes.

The next case is that there are exactly three AC balls; call the third ball B_M. This third ball has to fit within the structure $B_K, B_L, B_K, B_L, (B_K)$ as before; we will emphasize this structure by coloring these labels red in the juggling diagrams. Because of their size we will move these diagrams to Appendix blah.

The final case is that there are four AC balls; say B_M and B_N are the second pair. Then B_K And B_L will cross four times and B_M and B_N will cross four times, but there will be no additional crossings between these two pairs. Hence the second pair must fit entirely within the "gaps" of $B_K, B_L, B_K, B_L, (B_K)$. The six possibilities are in the appendix. Keep in mind that minimality conditions in the definition of B_K and B_L.

From this we can write down a closed form solution for each juggling diagram resembling the one in the previous section (but with $P_0(n, b)$ instead). Adding these all together gives a closed form solution for $P_4(n, b)$. We then used the code in Appendix blah to show that the resulting messy sum in fact reduces to

$$P_4(n, b) = \frac{(bn - b - 8)}{2(b + 4)} \binom{n}{b+3} \binom{n}{b-2}.$$

1.9 Higher crossing numbers

The next natural step in our problem is to ask for the enumeration of sequences A with larger values of $X(A)$. A direct proof was only accessible for the $\delta = 0$ case. Generating functions were useful up to $\delta = 2$, but even then became difficult. Juggling diagrams were again a useful tool, but $\delta = 4$ was already messy and without another insight this may be the last case they can reasonable handle.

In search for another approach or simplification, we turn to primitive juggling sequences. Recall that a primitive juggling sequence does not have any 1-throws, and is therefore simpler in some sense. We have already noted the relationship

$$N_\delta(n, b) = \sum_{k=1}^{n} \binom{n}{k} P_\delta(k, b).$$

Fix b and the number of crossings $b(b-1)+\delta$. If n is much larger than $b(b-1)+\delta$

then any length n, b-ball juggling card sequence A for which $X(A) = b(b-1) + \delta$ and $\pi(A) = \text{id}$ is not very interesting – most of the cards are 1-cards. Indeed, moving from n to $n+1$ simply amounts to adding yet another 1-card to one of the juggling card sequences in the length-n case.

The important fact is that, for a fixed b and δ, primitive juggling card sequences are of bounded length. One easy upper bound is $b(b-1) + \delta$, since every card incurs at least one crossing. This bound is not tight for $b > 2$, though, since if each card only incurs exactly one crossing then Ball i will have never been thrown for all $i > 2$.

An easy lower bound is b, since the fastest way to get every ball to be thrown is to have b consecutive b-cards. This bound is also not tight in general, as it fails whenever $\delta > 0$.

We ask for tight bounds on n, in terms of b and δ, for which there exist a positive number of length n juggling card sequences. This is of interest both to learn more about crossings in primitive juggling card sequences, but also because such bounds could help us learn more about the growth rate of $N_\delta(n, b)$, or even specifically about what its values are. We proceed now with our theorem on these bounds.

1.9.1 Bounds on primitive juggling sequences

Theorem 13. *Fix any $\delta \in 2\mathbb{N}_0$ and let $k \in \mathbb{N}_0$ be the smallest possible such that $\delta \leq k(k-1)$. Then there exists a primitive, b-valid ($b > 1$) sequence A of n juggling cards of height b with δ additional crossings if and only if*

$$b + k \leq n \leq 2(b-1) + \delta.$$

Proof. We will break up the proof into four parts. We will prove the upper bound, prove it's tight, prove the lower bound, and finally prove that it is also tight. We begin with the lower bound.

The lower bound.

By removing balls that only bounce once, we find a minimal structure generating all additional crossings. Let A' be the arrangement formed by removing these balls. Suppose this structure contains b' balls and is of length n'. Note that the number of balls removed, $b - b'$, equals the number of cards removed, $n - n'$. Moreover, since each ball will be thrown at least twice we see that $n' \geq 2b'$. Consequently

$$n = (n - n') + n' \geq (b - b') + 2b' = b + b'.$$

Suppose first that the largest card used in A' is at least a k-Card. Since A' has only b' levels and also contains a k-Card, certainly $b' \geq k$. Thus

$$n \geq b + b' \geq b + k.$$

Next suppose that the largest card used is strictly smaller than a k-Card (implying that it has at most $k - 2$ crossings). By the definition of k we know that $\delta > (k-1)(k-2)$. Consequently strictly more than $k - 1$ cards must correspond to additional crossings, implying that $n' \geq b' + k$. This again gives

$$n = (n - n') + n' \geq (b - b') + (b' + k) = b + k,$$

proving the lower bound.

The lower bound is tight.

Fix δ and b and write $n = \ell b + r$ where $r \in [b-1]_0$. We first show that there exists an arrangement A of length $\ell b + k$, where k is as defined in the theorem. First note that if $\delta = k(k-1)$, then the card arrangement

$$A_0^k := \underbrace{C_b, C_b, \ldots, C_b}_{\ell b - k \text{ copies}}, \underbrace{C_k, C_k, \ldots, C_k}_{k \text{ copies}} \underbrace{C_b, C_b, \ldots, C_b}_{k \text{ copies}}$$

works.

Next, if $\delta = k(k-1) - 2$, then the arrangement

$$A_1^k := \underbrace{C_b, C_b, \ldots, C_b}_{\ell b - k \text{ copies}}, \underbrace{C_k, C_k, \ldots, C_k}_{k-1 \text{ copies}} C_{k-1} \underbrace{C_b, C_b, \ldots, C_b}_{k-1 \text{ copies}} C_{b-1}$$

works. Certainly the number of of crossings has decreased by two. To see that the final permeation has not changed, first note that only two cards are different, and in the first of these Ball b is thrown, and in the second ball $b-1$ is thrown. This shows that for any pair of balls B_i and B_j such that $i, j \notin \{b-1, b\}$, the relative arrangement of B_i and B_j in $\pi(A_1^k)$ matches their relative arrangement in $\pi(A_0^k) = \text{id}$. Consequently, provided B_{b-1} returns to level $b-1$ and B_b returns to level b, we may conclude that $\pi(A_1^k) = \text{id}$.

This is easy though. Ball b is the second-to-last to bounce and Ball $b-1$ is the last to bounce, therefore since the last two cards are $C_b C_b - 1$, we are done.

For an arbitrary $m \in [k-1]_0$ define

$$A_m^k := \underbrace{C_b, C_b, \ldots, C_b}_{\ell b-k \text{ copies}}, \underbrace{C_k, C_k, \ldots, C_k}_{k-m \text{ copies}} \underbrace{C_{k-1}, C_{k-1}, \ldots, C_{k-1}}_{m \text{ copies}} \underbrace{C_b, C_b, \ldots, C_b}_{k-1 \text{ copies}} C_{b-m}.$$

Reasoning inductively we get the same conclusion from before. This time the only pair of balls whose relative arrangement has been changed after the final C_{k-1} card are (B_b, B_{b-m}), (B_{b-1}, B_{b-m}), $\ldots$, (B_{b-m+1}, B_{b-m}). Therefore, once again, we immediately see that for all $i, j \neq B_{b-m}$, if $i < j$ then $\pi(i) < \pi(j)$. Consequently, once again, it suffices to show that $\pi(b-m) = b-m$. This holds since, as already noted, Ball $b-m$ is the last ball to bounce, and the last card is C_{b-m}, guaranteeing that $\pi(A_m^k) = \text{id}$.

The upper bound.

For the upper bound, first consider the case when $\delta = 0$. Suppose the first throw sends Ball 1 to level $\ell + 1 \geq 3$. Then, from the Structure Lemma, we get our typical subproblem of some length m. By induction on n the maximum length of this subproblem is $2\ell - 2$. Once these balls have moved to the top of the card we again have a subproblem, this time our cards have height $b - \ell$ and the number of cards is $n - m$. Consequently the maximum length of the juggling card sequence is $2(b-\ell) - 2$. Adding these two together and accounting for the first throw of Ball 1 we have a grand total of at most

$$(2\ell - 2) + (2(b-\ell) - 2) + 1 = 2b - 3$$

juggling cards, which satisfies the desired inequality.

If, however, Ball 1's first throw is to level 2, then the second card must be a

b-card sending ball 2 to the top of the card and returning Ball 1 to level 1. Then by induction on n, the maximum length of the juggling card sequence is

$$2 + (2(b-1) - 2) = 2b - 2,$$

as desired.

It will be helpful to note now that this bound is achievable. Indeed, the following works.

$$\underbrace{C_2, C_b, C_2, C_b, \ldots, C_2, C_b,}_{2(b-1) \text{ total cards}}$$

Now assume that $\delta > 0$ and consider the juggling diagram corresponding to our juggling card sequence A. We can bound the number of cards in A by using what we just proved: If a block has $\ell \geq 2$ two balls in it, then the maximum number of cards corresponding to that block is $2(\ell - 1)$. If block is just the one AC ball inside it, then the maximum number of cards is 1.

Now, if more than one block has multiple balls in it, then this is wasteful. For example, assume one block has $\ell_1 \geq 2$ balls in it, and another has $\ell_2 \geq 2$ balls. Then combined the number of maximum number of cards contributed is

$$2(\ell_1 - 1) + 2(\ell_2 - 1) = 2(\ell_1 + \ell_2) - 4,$$

whereas if you put all $\ell_1 + \ell_2$ of those balls into a single block, then the maximum number of balls is $2(\ell_1 + \ell_2 - 1)$, which is greater.

In general, given a juggling diagram structure, to get as many cards as possible it is optimal to put all non-AC balls into a single block.

Suppose that there are m AC balls. Then the block containing one AC ball and all the non-AC balls can correspond to at most $2((b-m+1)-1)=2b-2m$ cards, while every other block contributes one. Clearly the juggling diagram with the most blocks is the one with exactly two AC balls, and this choice also maximizes $2b-2m$. Therefore it is clear what the best structure is. And this structure is characterized by the $B_K, B_L, B_K, B_L, \ldots, B_K, B_L$ diagram, where $m=2$. This diagram has precisely $\delta+2$ non-empty boxes in the primitive case. This gives a total maximum number of cards of

$$2b-2m+\delta+2=2(b-1)+\delta.$$

The upper bound is tight.

Our work above strongly suggests how to generate an arrangement of maximum length, given b and δ. The below works.

$$\underbrace{C_2, C_b, C_2, C_b, \ldots, C_2, C_b,}_{2(b-1)\ \text{total cards}}, \underbrace{C_2, C_2, \ldots, C_2}_{\delta\ \text{copies}}.$$

$\square$

1.10 Final arrangements consisting of a single cycle

Suppose that we draw cards at random from the set $\{C_1, C_2, \ldots, C_b\}$ with replacement to form a juggling card sequence A. We can then ask for the probability that π_A has some particular property. For example, what is the probability that it is equal to some given permutation, such as the identity, or that the permutation consists of a single cycle. The first question can be answered using Theorem 3. The answer for the second question is especially nice. We state the result as follows.

Theorem 14. *The probability that a random sequence A of n cards taken from the set of juggling sequence cards $\{C_1, C_2, \ldots, C_b\}$ has π_A consisting of a single cycle is $1/b$. In particular, this is independent of n.*

The following proof is due to Richard Stong [18]. We start with the following two basic lemmas.

Lemma 15. *The probability that a random permutation σ of $[b]$ has $L(\sigma) \geq k$ is $1/k!$ for $1 \leq k \leq b$.*

Proof. Select a k-element subset $\{a_0 > a_1 > \cdots > a_{k-1}\}$ from $[b]$. Define the permutation ρ by first setting $\rho(b-i) = a_i$ for $0 \leq i \leq k-1$. There are exactly $(b-k)!$ ways to complete ρ so that it is a permutation of $[b]$. Clearly, $L(\sigma) \geq k$ and there are $\binom{b}{k}(b-k) = b!/k!$ choices for ρ (and any ρ with $L(\rho) = k$ must be formed this way). Thus, the probability that a random ρ has $L(\rho) \geq k$ is $1/k!$ as claimed. $\square$

We note here that the number of permutations of $[b]$ that consist of a single cycle is $(b-1)!$.

Lemma 16. *The probability that a random permutation σ of $[b]$ which consists of a single cycle has $L(\sigma) \geq k$ is $1/k!$ for $1 \leq k \leq b-1$.*

Proof. The proof is similar to that of Lemma 15. In this case we choose k elements $\{a_0 > a_1 > \cdots > a_{k-1}\}$ from $[b-1]$ and map $\rho(b-i)$ to a_i for $0 \leq i \leq k-1$ as before. The reason that we don't allow $a_0 = b$ is that if $\rho(b) = a_0 = b$ then ρ would have a fixed point and so, could not be a single cycle. Now the question is how to complete the definition of ρ so that it becomes a single cycle. This is actually quite easy. We have the beginning of $b-k$ chains, namely, $b \to a_0$, $b-1 \to a_1$, $\ldots$, $b-k+1 \to a_{k-1}$, together with the remaining single points not included in the points listed so far. It is just a matter of piecing these fragments together to form a single cycle. The fact that some of the a_i might be equal to some of the $b-j$ causes no problem. It is easy to see that there are just $(b-k-1)!$ ways to complete the definition of ρ so that it becomes a single cycle with $L(\rho) \geq k$, and furthermore all such ρ can be constructed this way. Since $\binom{b-1}{k}(b-k-1)! = (b-1)!/k!$, and there are $(b-1)!$ permutations of $[b]$ that are cycles of length b, this completes the proof of Lemma 16. $\square$

Proof of Theorem 14. Partition the set of $b!$ permutations of $[b]$ into b disjoint classes X_k, for $1 \leq k \leq b$. Namely, $\sigma \in X_k$ if and only if $L(\sigma) = k$. By Lemma 15, $|X_k| = b!\big(\frac{1}{k!} - \frac{1}{(k+1)!}\big)$ for $1 \leq k \leq b-1$, while $|X_b| = 1$. Similarly, we can partition the set of $(b-1)!$ permutations which are b-cycles into disjoint sets Y_k, for $1 \leq k \leq b-1$, where $\sigma \in Y_k$ if and only if $L(\sigma) = k$. By Lemma 16, $|Y_k| = (b-1)!\big(\frac{1}{k!} - \frac{1}{(k+1)!}\big)$ for $1 \leq k \leq b-2$, while $|Y_{b-1}| = 1$. Note that $L(\sigma) \geq b-1$ if and only if $\sigma \in X_{b-1} \cup X_b$.

Now by Theorem 3, each $\sigma \in X_k$ accounts for exactly $\sum_{k=b-L(\sigma)}^{b} \left\{ n \atop k \right\}$ different card sequences A with $\pi_A = \sigma$, and the same is true for each $\sigma \in Y_k$, where $1 \leq k \leq b-2$. Furthermore, $|X_k| = b|Y_k|$ for these k. In addition, each $\sigma \in X_{b-1} \cup X_b$ and each $\sigma \in Y_{b-1}$ accounts for exactly $\sum_{k=1}^{b} \left\{ n \atop k \right\}$ different card sequences A with $\pi_A = \sigma$. Thus, since $|X_{b-1} \cup X_b| = \frac{b!}{(b-1)!} = b = b|Y_{b-1}|$ then it follows that the number of card sequences accounted for by all σ (which is b^n) is exactly b times the number accounted for by the σ which are b-cycles. In other words, the probability that a random sequence of n cards generates a permutation which is a b-cycle is just $1/b$, independent of n. $\square$

It turns out that the analog of Theorem 14 holds for cards where m balls are thrown.

Theorem 17. *The probability that a random sequence A of length n using cards where m balls are thrown at a time has π_A equal to a b-cycle is $1/b$. In particular, this is independent of n.*

The proof follows the same lines as the proof of Theorem 14 and will be omitted. The basic point is that in this case each σ with $L(\sigma) = k$ accounts for exactly $\sum_{k=b-L(\sigma)}^{b} \left\{ {n \atop k} \right\}_m$ sequences of m-cards with $\pi_A = \sigma$. Note that it is not obvious that Theorem 17 even holds for $n = 1$.

The surprising thing is that these results apply for all n and is not tied to a limiting process. Indeed, in the limit this is a special case of a much more general group theoretic principle that we prove now.

Theorem 18. *Let G be a group, let $\mathcal{S} = \{g_1, \ldots, g_k\}$ be a generating set of G, and let $\mathcal{P} = \{p_1, \ldots, p_k\}$ be a corresponding set of non-zero probabilities summing to 1. Consider the Markov chain on G where at each stage the current element is multiplied by a random $g \in \mathcal{S}$ chosen with probability given by $\mathcal{P}$. Then the stationary distribution of this process is the uniform distribution, independent of the group structure or $\mathcal{P}$.*

Proof. For simplicity we will assume that our walk begins at the identity element. Consider the formal sum $\mathcal{D} = \sum_{g_i \in S} p_i g_i$. The probability distribution of the random walk after n steps is then given by the formal sum $\mathcal{D}^n$. Let $\mathcal{F} = \sum_{g \in G} q_g g$ be the stationary distribution of this Markov chain. Then we have that $\mathcal{F}$ acts as a fixed point, i.e., $\mathcal{D}\mathcal{F} = \mathcal{F}$.

Let h be a group element whose probability q_h in the stationary distribution is maximum, i.e., $q_h \geq q_g$ for all $g \in G$. Applying this after equating the h coefficients on each side of $\mathcal{DF} = \mathcal{F}$ gives

$$q_h = \sum_{i=1}^{k} p_i q_{g_i^{-1}h} \leq \sum_{i=1}^{k} p_i q_h = q_h,$$

which can only hold if each $q_{g_i^{-1}h} = q_h$. Now, for each i, apply this same argument by choosing $g_i^{-1}h$ as the maximum element instead of h. Since $\{g_i^{-1} : i \in [k]\}$ is also a generating set of G, by continuing in this way we see that $q_g = q_h$ for all $g \in G$, completing the proof. $\square$

Thus in the case of S_n, the probability of having ℓ distinct cycles after choosing n random juggling cards tends to $\left[{b \atop \ell}\right]/b!$ as n tends to infinity, where $\left[{b \atop \ell}\right]$ denotes the Stirling number of the first kind, i.e. the number of ways to decompose $\{1, 2, \ldots, b\}$ into ℓ disjoint cycles. Indeed, we note without proof that it converges to this quite rapidly. By following the lines of the proof of Theorem 14, only in the "end cases" where $L(\sigma)$ is within ℓ of b does the proportion not equal precisely $\left\{{b \atop \ell}\right\}/b!$.

Chapter 2

Edge Flipping

2.1 Introduction

In a previous paper, Chung and Graham [20] considered the following "edge flipping" process on a connected graph G (originally suggested to them by Persi Diaconis, see also [19]). Beginning with the graph in some arbitrary coloring, repeatedly select an edge (with replacement) at random and color both of its vertices blue with probability p and red with probability $q := 1 - p$. This creates a random walk on all possible red/blue colorings of the graph and has a unique stationary distribution. Chung and Graham were able to determine the stationary distributions for paths and cycles as well as obtain some asymptotic results related to these graphs.

We remark that finding the stationary distribution is difficult since the state space of this random walk generally is exponential in the number of vertices in the graph. As a result, computations can only be carried out for small graphs. The goal of this chapter is to show how to find the stationary distribution of this process for the complete graph K_n. In Figure 2.1 we illustrate the state space of this process for K_3 (where for simplicity we use symmetry to reduce from eight possible colorings down to the four (light) blue/(dark) red colorings shown).

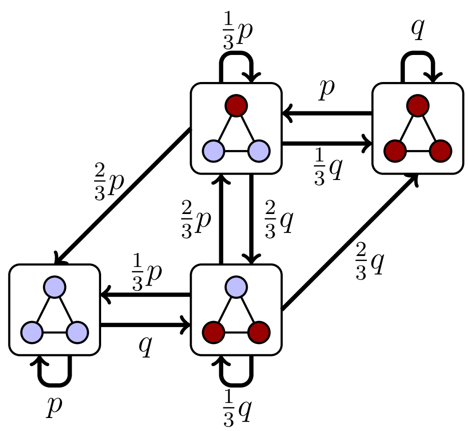


Figure 2.1: The edge flipping process on K_3

Using the figure it is a straightforward exercise to verify that the stationary distribution for the edge flipping process on K_3 satisfies the following:

$$\mathbb{P}(3 \text{ blue; } 0 \text{ red}) = p^2$$

$$\mathbb{P}(2 \text{ blue; } 1 \text{ red}) = pq$$

$$\mathbb{P}(1 \text{ blue; } 2 \text{ red}) = pq$$

$$\mathbb{P}(0 \text{ blue; } 3 \text{ red}) = q^2$$

The primary goal of chapter is to establish the following general result.

Theorem 19. *Let $b + r = n$. The stationary distribution for the edge flipping process on the complete graph K_n satisfies the following:*

$$\mathbb{P}(b \textit{ blue; } r \textit{ red}) = \frac{2^n p^b q^r}{\binom{2n-2}{n-2}} \sum_j \sum_k \binom{n-1}{b-2j, r-2k, j+k-1, j, k} (4p)^{-j} (4q)^{-k}.$$

To find the probability that b specific vertices are blue and the remaining r are red, divide by $\binom{n}{b} = \binom{n}{r}$.

Here we use $\binom{n}{i_1, i_2, \ldots, i_k}$ with $n = i_1 + i_2 + \cdots + i_k$ to represent the multinomial coefficient, i.e., we have

$$\binom{n}{i_1, i_2, \ldots, i_k} = \frac{n!}{i_1! i_2! \cdots i_k!},$$

with the convention that if any of the terms are negative then the value is 0.

Since flipping an edge can only change the color of at most two vertices, a direct proof of Theorem 19 can be carried out by verifying

$$\sum_b \mathbb{P}(b \text{ blue}; n - b \text{ red}) = 1$$

and that for each $b + r = n$

$$\begin{aligned}\binom{n}{2}\mathbb{P}(b \text{ blue}; r \text{ red}) &= p\binom{r}{2}\mathbb{P}(b-2 \text{ blue}; r+2 \text{ red}) \\ &\quad + pbr\mathbb{P}(b-1 \text{ blue}; r+1 \text{ red}) + \left(p\binom{b}{2} + q\binom{r}{2}\right)\mathbb{P}(b \text{ blue}; r \text{ red}) \\ &\quad + qbr\mathbb{P}(b+1 \text{ blue}; r-1 \text{ red}) + q\binom{b}{2}\mathbb{P}(b+2 \text{ blue}; r-2 \text{ red}).\end{aligned}$$

We will take a different approach which will give more insight into the process and introduce more tools that might be useful for working with other graphs.

We will proceed by first going into more details on the edge flipping process in Section 2.2. Then in Section 2.3 we will focus on establishing the probability that the first k (labeled) vertices are blue and use this to find the probability that all the vertices are blue. In Section 2.4 we use what is known about the all-blue probability to establish Theorem 19. In Section 2.6 we look at some of the asymptotics as n grows without bound, and in Section 2.7 we will take a look more closely at graph builds. Then in Section 2.8 we will look at the generalization to hypergraphs and finally in Section 2.9 we give some concluding remarks including further directions of research.

Throughout the chapter we will always use p to denote the probability a selected edge will have its vertices be colored blue, and q to denote the probability a selected edge will have its vertices be colored red.

2.2 Equivalent interpretations to edge flipping

The goal of this section is to take a closer look at the edge flipping process, and in particular look at different ways to analyze what is going on. We will assume that the graphs we are working with are connected. We start with the original interpretation of edge flipping.

> **Random walk interpretation of edge flipping:** Take a graph G with edges $\{e_1, e_2, \ldots, e_m\}$, and some initial random red/blue vertex coloring of G. Randomly choose edges (with replacement) and change the color of the vertices of the selected edge to blue with probability p and to red with probability q. Continue this process indefinitely.

Our first observation is that the edge flipping process is memoryless, i.e., a vertex v is only affected by the last edge drawn that was incident to v. This suggests that we should focus only on the last time that a particular edge was selected, and leads us to the following interpretation.

> **Reduced interpretation of edge flipping:** Take a graph G and a deck of $|E(G)|$ cards, i.e., one card for each edge. Randomly shuffle the deck and then deal out the cards one at a time. For each card, change the

vertices of the indicated edge to blue with probability p and to red with probability q.

Note that since each edge of G will be considered, the initial red/blue vertex coloring of G is not important.

Proposition 20. *Given a fixed vertex coloring c of a graph G, the probability of being at c in the stationary distribution for the random walk interpretation is the same as the probability of realizing c by the reduced interpretation.*

Proof. The probability of being at c in the random walk can be found by looking at the probability that we are at c after N steps as $N \to \infty$. For large N almost every list of N edges will contain each edge once (i.e., the contribution from lists not containing all edges $\to 0$).

So now let us look at a fixed list $\mathcal{L}$ of N edges from G in which each edge appears at least once in the list. Given any permutation π of $\{e_1, \dots, e_m\}$, define $\pi(\mathcal{L})$ to be the list that applies π to each member of $\mathcal{L}$. Since at each stage we randomly chose an edge, the probability that a random list of length N is $\mathcal{L}$ is the same as the probability that it is $\pi(\mathcal{L})$; indeed, both probabilities are $1/m^N$.

Moreover, the locations of the final edge appearances in $\mathcal{L}$ are the same as in $\pi(\mathcal{L})$, i.e., if e_i's final appearance is in position j of $\mathcal{L}$, then $\pi(e_i)$'s final appearance in $\pi(\mathcal{L})$ is in position j. This shows that, given two orderings $\mathcal{O}_1$ and $\mathcal{O}_2$ of $E(G)$, the number of length-N edge sequences in which the final appearances of each edge occurs in the order $\mathcal{O}_1$ is the same as the number that appear in the order $\mathcal{O}_2$. And hence, the probability of the ordering $\mathcal{O}_1$ is the same as the probability of the ordering $\mathcal{O}_2$.

Since these orderings are the only thing that determine the final coloring, and each ordering is equally likely, it suffices to look over all possible orderings $\mathcal{O}$ of the

m edges and determine the resulting final coloring, i.e., the reduced interpretation of edge flipping. $\square$

One immediate consequence of this is that there are only finitely many possible orderings of the edges and each occurs with probability some monomial in p and q. Therefore the probability of realizing a particular coloring c is a polynomial function of p and q.

In the reduced interpretation we only considered each edge once. In fact, in the process of coloring the edges even fewer of the cards have an impact on the final coloring. This is because cards that occurred near the start of the deck are likely to have both of its vertices recolored by some other edges later in the process. This suggests we focus only on edges which will impact the final coloring, and leads to our next interpretation.

> **Reversed interpretation of edge flipping:** Take a graph G and a deck of $|E(G)|$ cards, i.e., one card for each edge, and start with no coloring on the vertices of G. Randomly shuffle the deck and then deal out the cards one at a time. For each card, if one (or both) of the vertices of the corresponding edge is uncolored, then with probability p color the uncolored vertex (or vertices) blue, and with probability q color the uncolored vertex (or vertices) red. If a vertex is colored already, do not recolor it.

Proposition 21. *Given a vertex coloring c of G, the probability that we end with coloring c is the same for both the reduced interpretation and the reversed interpretation.*

Proof. We could instead color in the reverse order, as follows. Given some ordering $\mathcal{O}$, let $\mathcal{O}'$ be the reverse ordering. Run through the reversed order, coloring edges as before, but now when presented with a vertex that is already colored, instead of recoloring it just leave it as is. This gives the same coloring as before.

We can now view this as taking a deck of m cards, one card for each edge of G, randomly shuffling this deck, and dealing the cards out one at a time. When e_i's card is dealt, locate e_i in G and, if any of its end vertices are uncolored, color them blue with probability p and red with probability q. $\square$

The coloring on the reversed interpretation grows in bits and pieces, i.e., a card for an edge either colors two vertices, one vertex, or no vertex. We can approach the process through understanding this evolving coloring, which gives our last interpretation.

> **Constructive interpretation of edge flipping:** Take a graph G and a deck of $|E(G)|$ cards, i.e., one card for each edge. Start with an uncolored graph on the vertices of G but with no edges. Randomly shuffle the deck and deal out the cards one at a time. Each time an edge comes up, insert the edge into the graph. If at the time of insertion of the edge, one (or both) of the vertices of the edge is uncolored, then with probability p color the uncolored vertex (or vertices) blue and with probability q color the uncolored vertex (or vertices) red.

Since this works in the same manner as the reversed interpretation we have the following result.

Proposition 22. *The probability that we end with coloring c for the reversed interpretation is*

$$\frac{1}{m!}\sum_{\mathcal{O}} p^s q^t$$

where the sum is taken over all orderings, $\mathcal{O}$, of $E(G)$ which yield the final coloring of c in the constructive interpretation, and s and t are the number of cards which colored at least one vertex blue and red, respectively for the given ordering.

When applying the constructive process on the graph G, if we disregard the edges which do not color a vertex then the growing sequence of graphs induced by the resulting collection of edges will form a forest with no isolated vertices. In particular, each tree will have one edge which colored both its vertices and the remaining edges on the tree colored precisely one vertex. We can conclude that k is the number of trees in the forest if and only if there were exactly $n-k$ edges which contributed to the final coloring.

We now consider the special case of the all-blue coloring of the graph (so each card which colors at least one vertex was chosen to color blue with probability p). We have that the coefficient of p^{n-k} is the proportion of all orderings where $n-k$ edges contributed to the final coloring, i.e., the final graph when ignoring edges which did not color has k trees.

Let $F_G(k)$ denote the number of orderings of the edges of G so that the final graph when ignoring edges which did not color has k trees. When $G = K_n$ we simply write $F_n(k)$. Then we have

$$\mathbb{P}(c \text{ is all blue}) = \sum_k \frac{F_G(k)}{m!} p^{n-k}.$$

A similar analysis can be done when we are not using the all-blue coloring, and we will return to this in a later section.

2.3 Probability the first k vertices are colored blue

In this section we will look at the probability that the first k (labeled) vertices are colored blue in the stationary distribution. While this is an interesting question in its own right, we will mainly use this to establish the all-blue case for Theorem 19.

Before we begin, we will need to introduce the idea of a *restricted* coloring. Let G be a graph and c be a coloring of a subgraph H of G. Define $\mathbb{P}_G(c)$ to be the probability of realizing the coloring c on G, where the vertices $V(G) \setminus V(H)$ are allowed to be any color.

Lemma 23. *Let G be a graph and c be a coloring of a subgraph H of G. Let G' be the graph obtained by removing all the edges from G which have neither endpoint in H. If G' has no isolated vertices then*

$$\mathbb{P}_G(c) = \mathbb{P}_{G'}(c).$$

Proof. We will use the reverse interpretation of edge flipping. Let T be the deleted edges with $m = |E(G)|$ and $t = |T|$. Let L be any list of the edges of G which gives the coloring c. Notice that the edges from T do not color any of the vertices of $V(H)$ whose colors we demand match c. Therefore removing them from L still leaves a list L' of the edges of G' which color H exactly as before (note that every vertex of G still gets a color since G' has no isolated vertices).

Moreover, for a fixed list L' of the edges of G' giving the coloring c, it is easy to see that the number of lists of $E(G)$ which reduce to L' is precisely $\binom{m}{t}$. Note that this quantity is independent of our choice of L'. Likewise, given a list of $E(G')$ that gives a coloring different than c, there are again precisely $\binom{m}{t}$ lists of $E(G)$ which reduce to the chosen list.

Since every list of G can be reduced we conclude that the proportion of lists of $E(G)$ giving the coloring c is the same as the proportion of lists of $E(G')$ giving the coloring c. $\square$

So to determine the probability that the first k are blue we can work on a "simpler" graph. In particular we get the following recurrence.

Theorem 24. *Let $G = K_n$ and let $2 \le k \le n$. Let $Q_n(k)$ denote the probability that k* specified *vertices are blue (regardless of the coloring on the remaining $n - k$ vertices). Then*

$$Q_n(k) = \frac{(2n-2k)p}{2n-k-1} Q_n(k-1) + \frac{(k-1)p}{2n-k-1} Q_n(k-2)$$

with $Q_n(0) = 1$ and $Q_n(1) = p$.

Proof. We will use the reversed interpretation of the problem. Observe that the initial condition $Q_n(0) = 1$ holds since any coloring works (i.e., there is no restriction). Also $Q_n(1) = p$ since in any ordering of the edges, the first edge to contain the specified vertex will determine its color, and with probability p that card will color the vertex blue.

By Lemma 23, we may instead consider the graph G' obtained by removing all edges disjoint from our specified vertices, $\{v_1, \dots, v_k\}$. Note that G' is the lexicographic graph $K_k \vee \big((n-k)K_1\big)$. We now use this to establish the recurrence.

Consider a list L of $E(G')$, and let e_1 be the first edge in L. The first case is that e_1 has exactly one vertex in the clique on $\{v_1, v_2, \dots, v_k\}$. Say $e_1 = \{v_i, u\}$ where $u \notin \{v_1, \dots, v_k\}$. This case occurs $k(n-k)/\big(\binom{k}{2} + k(n-k)\big) = (2n-2k)/(2n-k-1)$ proportion of the time, and in order for v_i to be colored blue we must have that the vertices of e_1 were chosen to be blue, which happens with probability p.

Assuming the above occurs, we claim that the probability that the remainder of the process produces a legal coloring is $Q_n(k-1)$. Consider the current state of v_i. Vertex v_i is now the correct color and so for the rest of the list it does not matter what colors its incident edges are given; this is exactly the coloring property of the vertices $V(G) \setminus \{v_1, \dots, v_k\}$. Moreover, all edges from v_i to $V(G) \setminus \{v_1, \dots, v_k\}$ will not affect whether the coloring c occurs, and so (essentially by another application of Lemma 23) we may disregard these edges. Therefore by moving v_i out of the clique (shrinking k by 1), we see that we have an analogous problem with probability $Q_n(k-1)$.

The second case is that e_1 has both vertices in the clique on $\{v_1, \ldots, v_k\}$. This occurs $\binom{k}{2}/\left(\binom{k}{2} + k(n-k)\right) = (k-1)/(2n-k-1)$ proportion of the time and with probability p the vertices of the edge will be colored blue, as required. Given this, the only edges containing either of these vertices that matter are the ones between one of them and another vertex in $\{v_1, \ldots, v_k\}$. Therefore, as before, we may move these vertices out of the clique and with probability $Q_n(k-2)$ the remaining process will produce a legal coloring. This gives the recurrence. $\square$

We work out an example below, for when $n = 4$.

Example 4. We are given that $Q_4(0) = 1$ and $Q_4(1) = p$. We now compute the rest of the polynomials using the recurrence.

$$\begin{aligned} Q_4(2) &= \sum_{i=1}^{2} \frac{\binom{2}{i}\binom{4-2}{2-i}p}{\binom{4}{2} - \binom{4-2}{2}} Q_4(2-i) \\ &= \frac{4p}{5} Q_4(1) + \frac{p}{5} Q_4(0) \\ &= \frac{4}{5}p^2 + \frac{1}{5}p \end{aligned}$$

$$\begin{aligned} Q_4(3) &= \sum_{i=1}^{2} \frac{\binom{3}{i}\binom{4-3}{2-i}p}{\binom{4}{2} - \binom{4-3}{2}} Q_4(3-i) \\ &= \frac{p}{2} Q_4(2) + \frac{p}{2} Q_4 Q_4(1) \\ &= \frac{3}{5}p^2 + \frac{2}{5}p^3 \end{aligned}$$

$$\begin{aligned} Q_4(4) &= \sum_{i=1}^{2} \frac{\binom{4}{i}\binom{4-4}{2-i}p}{\binom{4}{2} - \binom{4-4}{2}} Q_4(4-i) \\ &= 0 \cdot Q_4(3) + p \cdot Q_4(2) \\ &= \frac{4}{5}p^3 + \frac{1}{5}p^2 \end{aligned}$$

By examining small cases like the above, an explicit solution to this recurrence was found.

Theorem 25. *Let $Q_n(k)$ denote the probability that k* specified *vertices are blue (regardless of the coloring on the remaining $n-k$ vertices). Then*

$$Q_n(k) = \frac{\sum_j (n-k+j)^{\overline{\lfloor k/2 \rfloor - j}} 2^{\lfloor k/2 \rfloor - 2j} p^{k-j} \binom{k}{2j} \frac{(2j)!}{j!}}{\prod_{\ell=1}^{\lfloor k/2 \rfloor} (2n - 2\ell - 1)},$$

where $m^{\overline{k}} = m(m+1)\cdots\big(m+(k-1)\big)$ denotes the rising factorial.

Proof. For $k=0$ and 1, the denominator will be an empty product which by convention is 1 and the sum in the numerator will be nonzero only for $j=0$, which gives 1 and p respectively. This establishes the base cases.

It now suffices to verify the recurrence, and for this we will find it useful to treat these expressions as polynomials in p, i.e.,

$$Q_n(k) = \sum_j c_{n,k}(j) p^{k-j}$$

where

$$c_{n,k}(j) = \frac{(n-k+j)^{\overline{\lfloor k/2 \rfloor - j}} 2^{\lfloor k/2 \rfloor - 2j} \binom{k}{2j} \frac{(2j)!}{j!}}{\prod_{\ell=1}^{\lfloor k/2 \rfloor} (2n - 2\ell - 1)}.$$

Note that the $\binom{k}{2j}$ term indicates this will only be nonzero when $0 \le j \le \lfloor k/2 \rfloor$.

The recurrence from Theorem 24 translates into the following recurrence on the coefficients:

$$c_{n,k}(j) = \frac{2n-2k}{2n-k-1}c_{n,k-1}(j) + \frac{k-1}{2n-k-1}c_{n,k-2}(j-1). \tag{2.1}$$

So it suffices to verify (2.1). Since the expression for the coefficient involves the term $\lfloor k/2 \rfloor$ we will find it useful to separate the verification of (2.1) into two cases depending on the parity of k.

First suppose that k is odd. Then we have the following.

$$\begin{aligned}
&\frac{2n-2k}{2n-k-1}c_{n,k-1}(j) + \frac{k-1}{2n-k-1}c_{n,k-2}(j-1) \\
&= \frac{2n-2k}{2n-k-1}\left[\frac{(n-k+j+1)^{\overline{\lfloor k/2\rfloor - j}}2^{\lfloor k/2\rfloor-2j}\binom{k-1}{2j}\frac{(2j)!}{j!}}{\prod_{\ell=1}^{\lfloor k/2\rfloor}(2n-2\ell-1)}\right] \\
&\qquad\qquad + \frac{k-1}{2n-k-1}\left[\frac{(n-k+j+1)^{\overline{\lfloor k/2\rfloor - j}}2^{\lfloor k/2\rfloor-2j+1}\binom{k-2}{2j-2}\frac{(2j-2)!}{(j-1)!}}{\prod_{\ell=1}^{\lfloor k/2\rfloor-1}(2n-2\ell-1)}\right] \\
&= \frac{(n-k+j+1)^{\overline{\lfloor k/2\rfloor - j}}2^{\lfloor k/2\rfloor-2j}\binom{k}{2j}\frac{(2j)!}{j!}}{(2n-k-1)\prod_{\ell=1}^{\lfloor k/2\rfloor}(2n-2\ell-1)} \times \left[(2n-2k)\frac{k-2j}{k} + (k-1)\frac{(2n-k)2j}{(k-1)k}\right] \\
&= \frac{(n-k+j+1)^{\overline{\lfloor k/2\rfloor - j}}2^{\lfloor k/2\rfloor-2j}\binom{k}{2j}\frac{(2j)!}{j!}}{(2n-k-1)\prod_{\ell=1}^{\lfloor k/2\rfloor}(2n-2\ell-1)} \times \left[2(n-k+j)\right] \\
&= \frac{(n-k+j)^{\overline{\lfloor k/2\rfloor - j}}2^{\lfloor k/2\rfloor-2j}\binom{k}{2j}\frac{(2j)!}{j!}}{\prod_{\ell=1}^{\lfloor k/2\rfloor}(2n-2\ell-1)} = c_{n,k}(j)
\end{aligned}$$

The first step is substitution (using $\lfloor (k-1)/2\rfloor = \lfloor k/2\rfloor$, $\lfloor (k-2)/2\rfloor = \lfloor k/2\rfloor - 1$). We then pulled out the common terms from both parts (in the second term the $(2n-k) = (2n-2\lfloor k/2\rfloor - 1)$ comes from compensating for pulling too much out of the denominator). For the third step we then simplify the last term. Finally for the fourth step we see that we have $(n-k+j+1)^{\overline{\lfloor k/2\rfloor - j}}$ while we want $(n-k+j)^{\overline{\lfloor k/2\rfloor - j}}$ so we need to have an $(n-k+j)$ to insert (which comes from the end), and we also need to eliminate the last term in the product which is $\big(n-k+j+1+(\lfloor k/2\rfloor - j - 1)\big) = (n-k+\lfloor k/2\rfloor)$. Now at this point we note that we also have an extra 2 in the numerator and that $2(n-k+\lfloor k/2\rfloor) = 2n-2k+(k-1) = 2n-k-1$, and this cancels with what we have in the denominator. Therefore when we simplify we end up with $c_{n,k}(j)$ as desired.

Now suppose that k is even. Then we have the following.

$$\begin{aligned}
&\frac{2n-2k}{2n-k-1}c_{n,k-1}(j) + \frac{k-1}{2n-k-1}c_{n,k-2}(j-1)\\
&= \frac{2n-2k}{2n-k-1}\left[\frac{(n-k+j+1)^{\overline{\lfloor k/2\rfloor-j-1}}2^{\lfloor k/2\rfloor-1-2j}\binom{k-1}{2j}\frac{(2j)!}{j!}}{\prod_{\ell=1}^{\lfloor k/2\rfloor-1}(2n-2\ell-1)}\right]\\
&\quad + \frac{k-1}{2n-k-1}\left[\frac{(n-k+j+1)^{\overline{\lfloor k/2\rfloor-j}}2^{\lfloor k/2\rfloor-2j+1}\binom{k-2}{2j-2}\frac{(2j-2)!}{(j-1)!}}{\prod_{\ell=1}^{\lfloor k/2\rfloor-1}(2n-2\ell-1)}\right]\\
&= \frac{(n-k+j+1)^{\overline{\lfloor k/2\rfloor-j-1}}2^{\lfloor k/2\rfloor-2j}\binom{k}{2j}\frac{(2j)!}{j!}}{(2n-k-1)\prod_{\ell=1}^{\lfloor k/2\rfloor-1}(2n-2\ell-1)} \times \left[(2n-2k)\frac{k-2j}{2k} + (k-1)\frac{(n-\frac{k}{2})2j}{(k-1)k}\right]\\
&= \frac{(n-k+j+1)^{\overline{\lfloor k/2\rfloor-j-1}}2^{\lfloor k/2\rfloor-2j}\binom{k}{2j}\frac{(2j)!}{j!}}{(2n-k-1)\prod_{\ell=1}^{\lfloor k/2\rfloor-1}(2n-2\ell-1)} \times \big[(n-k+j)\big]\\
&= \frac{(n-k+j)^{\overline{\lfloor k/2\rfloor-j}}2^{\lfloor k/2\rfloor-2j}\binom{k}{2j}\frac{(2j)!}{j!}}{\prod_{\ell=1}^{\lfloor k/2\rfloor}(2n-2\ell-1)} = c_{n,k}(j)
\end{aligned}$$

The first few steps are as before (except now $\lfloor (k-1)/2\rfloor = \lfloor k/2\rfloor - 1$). For the final step we see that we have $(n-k+j+1)^{\overline{\lfloor k/2\rfloor-j-1}}$ while we want $(n-k+j)^{\overline{\lfloor k/2\rfloor-j}}$ so we need to have an $(n-k+j)$ to insert (which comes from the end), in the meantime the term $(2n-k-1)$ in the denominator is equal to $(2n-2\lfloor k/2\rfloor-1)$ which is the next term that would be in the product so we move it into the product. Therefore when we simplify we end up with $c_{n,k}(j)$ as desired.

This establishes (2.1) and the desired result. $\square$

The important case for us is when $k=n$, which is equivalent to having all of the vertices colored blue. In particular by evaluating $Q_n(k)$ at $k = n$ we have the following.

Corollary 26. *The probability that the complete graph is colored all-blue under the edge flipping process is given by*

$$\sum_{j=1}^{\lfloor n/2\rfloor} \frac{\binom{2j-1}{j}\binom{n-1}{2j-1}2^{n-2j}}{\binom{2n-2}{n}} p^{n-j}.$$

Proof. We have

$$Q_n(n) = \frac{\sum_j j^{\overline{\lfloor n/2\rfloor - j}} 2^{\lfloor n/2\rfloor - 2j}\binom{n}{2j}\frac{(2j)!}{j!}p^{n-j}}{\prod_{\ell=1}^{\lfloor n/2\rfloor}(2n-2\ell-1)}.$$

We only need to sum from $0 \le j \le \lfloor n/2\rfloor$ since the $\binom{n}{2j}$ term is zero otherwise. But when $j = 0$ the rising factorial term starts at 0 and so drops out, hence we can start our sum at 1. Further we have that $j^{\overline{\lfloor n/2\rfloor - j}} = \big(\lfloor n/2\rfloor - 1\big)!/(j-1)!$.

The other thing to notice is that the product of odd terms in the denominator can be turned into a ratio of factorials and a power of 2 by inserting in extra terms and then simplifying. In particular it is easy to check that

$$\prod_{\ell=1}^{\lfloor n/2\rfloor}(2n-2\ell-1) = \frac{(2n-2)!(n-\lfloor n/2\rfloor-1)!}{2^{\lfloor n/2\rfloor}(n-1)!(2n-2\lfloor n/2\rfloor-2)!}.$$

Combining these we can now write the coefficient in terms of factorials and powers of 2. In particular the coefficient of p^{n-j} is

$$\frac{\frac{(\lfloor\frac{n}{2}\rfloor-1)!}{(j-1)!}2^{\lfloor\frac{n}{2}\rfloor-2j}\frac{n!}{(2j)!(n-2j)!}\frac{(2j)!}{j!}}{\frac{(2n-2)!(n-\lfloor\frac{n}{2}\rfloor-1)!}{2^{\lfloor\frac{n}{2}\rfloor}(n-1)!(2n-2\lfloor\frac{n}{2}\rfloor-2)!}} = \frac{n!(n-1)!(2n-2\lfloor\frac{n}{2}\rfloor-2)!(\lfloor\frac{n}{2}\rfloor-1)!2^{2\lfloor\frac{n}{2}\rfloor-2j}}{(j-1)!(n-2j)!j!(2n-2)!(n-\lfloor\frac{n}{2}\rfloor-1)!}.$$

When n is even then $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$ and this simplifies to the following:

$$\frac{n!(n-1)!(n-2)!(\frac{n}{2}-1)!2^{n-2j}}{(j-1)!(n-2j)!j!(2n-2)!(\frac{n}{2}-1)!} = \frac{n!(n-1)!(n-2)!2^{n-2j}}{(j-1)!(n-2j)!j!(2n-2)!}.$$

When n is odd then $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ and this simplifies to the following:

$$\frac{n!(n-1)!(n-1)!(\frac{n}{2}-\frac{3}{2})!2^{n-2j-1}}{(j-1)!(n-2j)!j!(2n-2)!(\frac{n}{2}-\frac{1}{2})!} = \frac{n!(n-1)!(n-2)!2^{n-2j}}{(j-1)!(n-2j)!j!(2n-2)!}.$$

Here we used $(n-1)!2^{-1} = (n-2)!(\frac{n}{2}-\frac{1}{2})$ and $(\frac{n}{2}-\frac{1}{2})! = (\frac{n}{2}-\frac{1}{2})(\frac{n}{2}-\frac{3}{2})!$ to simplify.

In both cases we ended up with

$$\frac{n!(n-1)!(n-2)!2^{n-2j}}{(j-1)!(n-2j)!j!(2n-2)!} = \frac{\binom{2j-1}{j}\binom{n-1}{2j-1}2^{n-2j}}{\binom{2n-2}{n}}.$$

This establishes the result and concludes the proof. $\square$

Recall that $F_n(k)$ denotes the number of orderings of the edges of K_n in the constructive interpretation of edge flipping so that the final graph (when ignoring edges which did not color) has k trees. We also have

$$\mathbb{P}(c \text{ is all blue}) = \sum_k \frac{F_n(k)}{m!} p^{n-k}.$$

Combining this with the previous corollary now gives us the following.

Corollary 27. $F_n(k) = \dfrac{\binom{2k-1}{k}\binom{n-1}{2k-1}2^{n-2k}\binom{n}{2}!}{\binom{2n-2}{n}}.$

2.4 Probability of an arbitrary coloring

Once we have the probability that b specified vertices are colored blue, then by use of an inclusion-exclusion argument we can find the probability that b specified vertices are blue and the remainder are red. However, this will result in an alternating sum, and be entirely in terms of p, when we want a sum with non-negative terms which is in terms of p and q. Therefore we will take a slightly different approach by using the constructive interpretation of edge flipping.

Theorem 28. *For the edge flipping process on the complete graph $G = K_n$, let c be a coloring which assigns red to r* specified *vertices with the remainder being blue. Then the probability of having the coloring c is given by*

$$\frac{1}{\binom{2n-2}{n-2,n-r,r}} \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{s=0}^{\lfloor r/2 \rfloor} \binom{n-1}{n-2k-r+2s, r-2s, k-1, k-s, s} 2^{n-2k} p^{n-k-r+s} q^{r-s}.$$

We note the result does not depend on which vertices have been specified red and blue, and so if we only care that r of the vertices are red and the remainder are blue then we multiply the probability in Theorem 28 by $\binom{n}{r}$. Further, making the substitutions $n = b + r$, $-k + s = -j'$ and $s = k'$ and then simplifying shows that this result is equivalent to Theorem 19.

Before we begin the proof of Theorem 28 we point out that when $r = 0$ then s is forced to be 0 and this reduces to the expression to the one given in Corollary 26. So the result is true in this case.

Proof. We use the constructive interpretation of edge flipping. Let $\mathcal{F}_n(k)$ be the set of ways to build K_n (ignoring the coloring for now), one edge at a time, so that the edges which actually colored forms a forest with k trees ($1 \le k \le \lfloor n/2 \rfloor$). Note that

$F_n(k) = |\mathcal{F}_n(k)|$. For each tree there will be a unique edge which colors both of its vertices, and the remaining edges in the tree color one of its vertices. In particular there will be k edges which color 2 vertices, $n-2k$ edges which color one vertex, and the remaining edges will not color.

Consider an arbitrary $S \in \mathcal{F}_n(k)$ and let $\mathcal{T}_S$ be the set of k edges which color two vertices. Clearly these edges color an even number of vertices red, so suppose $2s$ is that number ($0 \leq s \leq \lfloor r/2 \rfloor$). We will now count the number of such sequences produce such a coloring c.

Define an r-*set* to be a set of r vertices from G. We first count the total number of r-sets for which exactly $2s$ of its members are paired up by s edges from $\mathcal{T}_S$, and the remaining $r-2s$ vertices are disjoint from the edges of $\mathcal{T}_S$; call this property of an r-set *property-s*. This is easy, as there are $\binom{k}{s}$ ways to pick the s edges from $\mathcal{T}_S$ containing our first $2s$ vertices, and $\binom{n-2k}{r-2s}$ ways to choose the remaining. Summing over all $S \in F_n(k)$ gives a total of $\binom{n-2k}{r-2s}\binom{k}{s}|F_n(k)|$ distinct occurrences of property-s, among all builds in $\mathcal{F}_n(k)$.

Since G is symmetric between r-sets, and the set $\mathcal{F}_n(k)$ is symmetric within $V(G)$ in the sense that any build from this collection can be translated to another by simply permuting the vertex set, it is clear that any two r-sets will have property-s the same number of times among the entire collection. Therefore this total must be evenly distributed among all $\binom{n}{r}$ of the r-sets. In particular, the unique r-set of red vertices must occur precisely

$$\frac{\binom{n-2k}{r-2s}\binom{k}{s}|F_n(k)|}{\binom{n}{r}}$$

times.

Given such a such a sequence S, there is a $1/\binom{n}{2}!$ probability that it will appear, and if it does there is a $p^{n-k-r+s}q^{r-s}$ probability that it will be colored in the unique way giving the coloring c. In particular, the s edges from $\mathcal{T}_S$ that pair up $2s$ red vertices must be red, and the remaining $r-s$ red vertices will each be colored by whatever is the first edge in S to contain it. Since these vertices are not part of $\mathcal{T}_S$, precisely one edge is needed for each. Therefore there are $r-s$ edges which will color the red vertices, and since precisely $n-k$ edges contribute to the coloring, there must be $n-k-(r-s)$ edges corresponding to coloring the vertices of an edge blue. Summing over all choices of s and k gives a probability of

$$P_G(c) = \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{s=0}^{\lfloor r/2 \rfloor} \frac{\binom{n-2k}{r-2s}\binom{k}{s}|F_n(k)|}{\binom{n}{r}\binom{n}{2}!} p^{n-k-r+s}q^{r-s}.$$

of obtaining the coloring c. Applying Corollary 27 and collapsing the resulting binomial coefficients to multinomial coefficients gives the result. □

2.5 Other graph families

We hope that in the future more classes of graphs are studied. In this section we initiate that endeavor by finding the probability polynomials for the all-blue coloring of some additional, non-trivial classes of graphs.

2.5.1 Friendly flips

Our first infinite family of graph are the friendship graphs. The m^{th} friendship graph is obtained by taking m distant copies of K_3, choosing one vertex from each, and identifying those vertices. Theorem 29 below gives the probability polynomial for the all-blue coloring of any friendship graph, but our proof actually proves something more general. Define a *half-edge* to be a an edge with only one end-vertex, and let $f(m,n)$ be the m^{th} friendship graph with n half-edges attached to the central vertex.

Proposition 29. *The probability polynomial for the friendship graph $f(m,0)$ is*

$$\sum_{i=0}^{m}\left(\prod_{j=0}^{i-1}\frac{(m-j)p}{3m-j}\right)\cdot\frac{\left(\frac{p}{3}+\frac{2p^2}{3}\right)^{m-i-1}\left(\frac{6m-4i}{3}p^2+\frac{4i}{3}p^3\right)}{3m-i}.$$

Proof. Let $\tilde{P}_n$ be the probability polynomial for the graph obtained by replacing the two pendant edges in P_n with half edges. Below we will be concerned with $\tilde{P}_3$ (which has one vertex, zero edges, and two half edges) and $\tilde{P}_4$ (which has two vertices, one edge, and two half edges).

We will use the reversed interpretation of edge flipping. If the first edge or half edge chosen is a half edge, then this half edge will color the central vertex, and with probability p it will color it blue. Once this occurs, observe that all the half edges are guaranteed to not affect the final coloring and thus may be ignored by the subgraph lemma. Furthermore, each triangle no has one vertex colored. Thus for the remainder of the coloring, the set of triangle behave precisely like m disjoint copies of $\tilde{P}_4$.

Since a half edge is chosen first with probability $n/(n+3m)$, the probability that $\tilde{P}_4$ is all-blue is $p/3+2p^2/3$, and the probability polynomial of a union of disjoint graphs is just the product of each's probability polynomial, we see that this case occurs and gives the all-blue coloring with probability

$$\frac{np}{n+3m}\left(\frac{p}{3}+\frac{2p^2}{3}\right)^m.$$

Similarly, the probability of first choosing a central edge and then realizing the all-blue coloring is

$$\frac{2mp^2}{n+3m}\left(\frac{p}{3}+\frac{2p^2}{3}\right)^{m-1}.$$

Now there is a $2m/(n+3m)$ chance of being in this case and with probability p this first edge is blue. Once this edge is chosen, again the half edges do not matter and the triangles break apart, as far as the coloring is concerned. The difference, though, is that the edge chosen was now part of one of these triangles. Whichever triangle it is apart of can now be viewed as a copy of $\tilde{P}_3$, which is blue with probability p, while the remaining $m-1$ triangles can still be viewed as $m-1$ disjoint copies of $\tilde{P}_4$.

Lastly, there is a $m/(n+3m)$ chance that the first edge chosen was a non-central edge. Such an edge must still be chosen to be blue, but after this the remaining two edges in its triangle can be viewed as half-edges. Thus the corresponding polynomial is $f(m-1, n+2)$.

With the initial conditions $f(0, n) = p$ for all n, we have obtained a recurrence relation. Explicitly:

$$f(m,n) = \frac{np}{n+3m}\left(\frac{p}{3}+\frac{2p^2}{3}\right)^{m} + \frac{2mp^2}{n+3m}\left(\frac{p}{3}+\frac{2p^2}{3}\right)^{m-1} + \frac{mp}{n+3m}f(m-1,n+2).$$

From this we can deduce a closed-form expression for the friendship graph polynomial $f(m, 0)$ by applying the above m times. The answer for general n is

$$f(m,n) = \sum_{i=0}^{m-1}\left(\prod_{j=1}^{i}\frac{(m-j+1)p}{(n+2j-2)+3(m-j+1)}\right)\cdot\left(\frac{(n+2i)p}{(n+2i)+3(m-i)}\left(\frac{p}{3}+\frac{2p^2}{3}\right)^{m-i}\right.$$

$$+ \frac{2(m-i)p^2}{(n+2i)+3(m-i)} \left(\frac{p}{3}+\frac{2p^2}{3}\right)^{m-i-1} \Bigg) + p \cdot \prod_{j=0}^{m-1} \frac{(m-j)p}{(n+2j)+(3(m-j))}$$

$$= \sum_{i=0}^{m-1} \left(\prod_{j=0}^{i-1} \frac{(m-j)p}{n+3m-j} \right) \cdot \Bigg(\frac{(n+2i)p}{n+3m-i} \left(\frac{p}{3}+\frac{2p^2}{3}\right)^{m-i}$$

$$+ \frac{2(m-i)p^2}{n+3m-i} \left(\frac{p}{3}+\frac{2p^2}{3}\right)^{m-i-1} \Bigg) + p^{m+1} \cdot \prod_{j=0}^{m-1} \frac{(m-j)}{n+3m-j}$$

$$= \sum_{i=0}^{m-1} \left(\prod_{j=0}^{i-1} \frac{(m-j)p}{n+3m-j} \right) \cdot \Bigg(\frac{p\left(\frac{p}{3}+\frac{2p^2}{3}\right)^{m-i-1}}{n+3m-i}$$

$$\cdot \left(\left(\left(\frac{p}{3}+\frac{2p^2}{3}\right) + 2p^2/3)(n+2i) + 2(m-i)p \right) \right) + p^{m+1} \cdot \prod_{j=0}^{m-1} \frac{(m-j)}{n+3m-j}$$

$$= \sum_{i=0}^{m-1} \left(\prod_{j=0}^{i-1} \frac{m-j}{n+3m-j} \right) \cdot \Bigg(\frac{p^{i+1}\left(\frac{p}{3}+\frac{2p^2}{3}\right)^{m-i-1}}{n+3m-i}$$

$$\cdot \left(\left(\frac{2n+4i}{3} \right) p^2 + \left(\frac{n+6m-4i}{3} \right) p \right) \Bigg) + p^{m+1} \cdot \prod_{j=0}^{m-1} \frac{(m-j)}{n+3m-j}$$

To get $f(m,0)$ we simply set $n=0$.

$$\sum_{i=0}^{m-1} \left(\prod_{j=0}^{i-1} \frac{m-j}{3m-j} \right) \cdot \left(\frac{\left(\frac{p}{3}+\frac{2p^2}{3}\right)^{m-i-1} \left(\frac{4i}{3}p^{3+i} + \frac{6m-4i}{3}p^{2+i}\right)}{3m-i} \right)$$

$$+ p^{m+1} \cdot \prod_{j=0}^{m-1} \frac{m-j}{3m-j}$$

$$= \sum_{i=0}^{m} \left(\prod_{j=0}^{i-1} \frac{m-j}{3m-j} \right) \cdot \left(\frac{\left(\frac{p}{3}+\frac{2p^2}{3}\right)^{m-i-1} \left(\frac{4i}{3}p^{3+i} + \frac{6m-4i}{3}p^{2+i}\right)}{3m-i} \right)$$

$\square$

2.5.2 Bright stars

Now consider the class of *bright stars*, i.e., paths of length two, all joined at an endpoint. Like above, let $g(m,n)$ be the probability polynomial for the all-blue coloring of the graph which is a bright star with m paths of length 2 emanating from the central vertex, along with n half edges attached to the central vertex.

Proposition 30. *The probability polynomial for the bright star $g(m,0)$ is*

$$\sum_{i=0}^{m}\left(\prod_{j=0}^{i-1}\frac{(m-j)p}{2m-j}\right)\cdot\frac{\left(\frac{p}{2}+\frac{p^2}{2}\right)^{m-i-1}\left(\frac{2m-i}{2}p^2+\frac{i}{2}p^3\right)}{2m-i}.$$

Proof. Again we use the reverse interpretation of edge flipping. There are m edges not incident with the central vertex. If one of these is the first edge chosen then the the remaining consequential edges form a graph of the same form only with one fewer path of length two and one more half edge. Thus the probability that we are in this case, the first edge is chosen to be blue, and the rest of the graph is also colored blue is

$$\frac{mp}{2m+n}g(m-1,n+1).$$

Let Q_1 be the graph consisting of one edge attached to one half edge. Let Q_2 be a single half edge. If the first edge chosen is one of the (full) edges incident with the central vertex, then after being chosen to be blue with probability p, the remaining consequential edges form a disjoint union of $m-1$ copies of Q_1 and one copy of Q_2. It's easy to see that the probability of being in this case and realizing the all-blue coloring is then

$$\frac{mp}{2m+n}p\left(\frac{p}{2}+\frac{p^2}{2}\right)^{m-1}.$$

Finally, if a half-edge is chosen first then what remains is m disjoint copies of Q_1. Adding in this final term gives the recurrence

$$\begin{aligned} g(m,n) &= \frac{mp}{2m+n} g(m-1,n+1) + \frac{mp}{2m+n} p \left(\frac{p}{2} + \frac{p^2}{2}\right)^{m-1} + \frac{np}{2m+n} \left(\frac{p}{2} + \frac{p^2}{2}\right)^{m} \\ &= \frac{mp}{2m+n} g(m-1,n+1) + \frac{\left(\frac{p}{2} + \frac{p^2}{2}\right)^{m-1}}{2m+n} \left(mp^2 + np\left(\frac{p}{2} + \frac{p^2}{2}\right)\right) \end{aligned}$$

with initial sequence $g(0,n) = p$ for all n.

Then by inspection we can deduce that $g(m,n)$ is

$$\begin{aligned} &\sum_{i=0}^{m-1} \left(\prod_{j=0}^{i-1} \frac{(m-j)p}{2(m-j)+(n+j)}\right) \left(\frac{\left(\frac{p}{2} + \frac{p^2}{2}\right)^{m-i-1} \left((m-i)p^2 + \left(\frac{p}{2} + \frac{p^2}{2}\right)(n+i)p\right)}{2(m-i)+(n+i)}\right) \\ &\qquad + p \cdot \prod_{j=0}^{m-1} \frac{(m-j)p}{2(m-j)+(n+j)} \\ &\quad = \sum_{i=0}^{m} \left(\prod_{j=0}^{i-1} \frac{(m-j)p}{2m+n-j}\right) \cdot \frac{\left(\frac{p}{2} + \frac{p^2}{2}\right)^{m-i-1} \left(\frac{2m+n-i}{2}p^2 + \frac{n+i}{2}p^3\right)}{2m+n-i}. \end{aligned}$$

By setting $n = 0$ we obtain the all-blue probability polynomial $g(m,0)$ for the two-edge star:

$$\sum_{i=0}^{m} \left(\prod_{j=0}^{i-1} \frac{(m-j)p}{2m-j}\right) \cdot \frac{\left(\frac{p}{2} + \frac{p^2}{2}\right)^{m-i-1} \left(\frac{2m-i}{2}p^2 + \frac{i}{2}p^3\right)}{2m-i}.$$

$\square$

2.5.3 Other families of graphs

This analysis can be carried out for any graph or families of graphs. Another natural family to consider would be complete bipartite graphs. For the special case $K_{1,n}$ the analysis is again straightforward to carry out. Namely the probability that the "center" vertex is blue and b of the remaining are blue and $r = n - b$ are red is

$$\binom{b+r}{b}\frac{b}{b+r}p^b q^r.$$

Similarly the probability that the center vertex is red and b of the remaining are blue and $r = n - b$ are red is

$$\binom{b+r}{b}\frac{r}{b+r}p^b q^r.$$

We note without proof that the probability polynomial for the all-blue coloring of any complete bipartite graph is known, and was discovered through the study of graph builds, c.f. Section 2.7.

In addition to cycles, paths, and complete graphs it is possible to implement a program to directly determine the stationary distribution for any *small* graph. The study of these small graphs might offer some insight into what is happening for larger graphs and warrant further exploration. For example, one can show that for the edge flipping process on the Petersen graph, P, that

$$\mathbb{P}(P \text{ is all blue}) = \frac{326p^9 + 4352p^8 + 10923p^7 + 4744p^6 + 130p^5}{20475}.$$

2.6 Asymptotic results

In this section we will consider some asymptotic results related to Theorem 28, in particular focusing on how likely a particular coloring will occur. The approach we will give is to first express results in terms of a multivariate generating function and then apply known asymptotic tools to estimate the coefficients of these functions.

2.6.1 Generating functions

We begin with the following expression for the probability that b vertices are blue and r vertices are red:

$$\frac{1}{\binom{2n-2}{n-2}}\sum_{k\geq 1}\sum_{s\geq 0}\binom{n-1}{n-2k-r+2s,\, r-2s,\, k-1,\, k-s,\, s}2^{n-2k}p^{n-k-r+s}q^{r-s}, \tag{2.2}$$

where $n=b+r$ and we have multiplied by $\binom{n}{r}$ and simplified the term in front since we don't care which r vertices are red. Define

$$\begin{aligned} g(n,r) &= \sum_{k\geq 1,s}\binom{n-1}{n-2k-r+2s,\, r-2s,\, k-1,\, k-s,\, s}2^{n-2k}p^{n-k-r+s}q^{r-s} \qquad (2.3)\\ &= \sum_{k\geq 1,s}\binom{n-1}{n-2k-r+2s,\, r-2s,\, k-1,\, k-s,\, s}(2p)^n\left(\frac{1}{4p}\right)^k\left(\frac{q}{p}\right)^r\left(\frac{p}{q}\right)^s. \end{aligned}$$

Next, we define the generating function

$$\begin{aligned} F(x,y,z,w) &= \sum_{n,k\geq 1,r,s}\binom{n-1}{n-2k-r+2s,\, r-2s,\, k-1,\, k-s,\, s}x^n y^k z^r w^s\\ &= \sum_{n,k\geq 1,s}\binom{n-1}{n-2k,\, k-1,\, k-s,\, s}(1+z)^{n-2k}x^n y^k w^s z^{2s} \end{aligned}$$

$$= \sum_{n,k\geq 1,s} \binom{n-1}{n-2k, k-1, k-s, s} (x(1+z))^n \left(\frac{y}{(1+z)^2}\right)^k (z^2w)^s$$

$$= \sum_{n,k\geq 1} \binom{n-1}{n-2k, k-1, k} (1+z^2w)^k (x(1+z))^n \left(\frac{y}{(1+z)^2}\right)^k$$

$$= \sum_{n,k\geq 1} \binom{n-1}{n-2k, k-1, k} (x(1+z))^n \left(\frac{y(1+z^2w)}{(1+z)^2}\right)^k$$

$$= \sum_{n,k\geq 1} \frac{1}{2}\binom{n-1}{2k-1}\binom{2k}{k} (x(1+z))^n \left(\frac{y(1+z^2w)}{(1+z)^2}\right)^k$$

$$= \sum_{k\geq 1} \frac{1}{2}\binom{2k}{k} (x(1+z)) \left(\frac{y(1+z^2w)}{(1+z)^2}\right)^k \sum_{N} \binom{N}{2k-1} (x(1+z))^N$$

(where we set $N = n-1$)

$$= \sum_{k\geq 1} \frac{1}{2}\binom{2k}{k} (x(1+z)) \left(\frac{y(1+z^2w)}{(1+z)^2}\right)^k \frac{(x(1+z))^{2k-1}}{(1-x(1+z))^{2k}}$$

$$= \frac{1}{2}\sum_{k\geq 1} \binom{2k}{k} \left(\frac{x^2y(1+z^2w)}{(1-x(1+z))^2}\right)^k$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{1-\frac{4x^2y(1+z^2w)}{(1-x(1+z))^2}}} - 1\right)$$

$$= \frac{1}{2}\left(\frac{(1-x(1+z))}{\sqrt{(1-x(1+z))^2 - 4x^2y(1+z^2w)}} - 1\right)$$

Finally, we consider the generating function:

$$G(X,Y) = \sum_{n\geq 1,r} g(n,r)X^nY^r. \qquad (2.4)$$

From (2.2) and (2.3), we have

$$G(X,Y) = F\left(2pX, \frac{1}{4p}, \frac{q}{p}Y, \frac{p}{q}\right)$$
$$= \frac{1}{2}\left(\frac{1-2pX(1+\frac{q}{p}Y)}{\sqrt{(1-2pX(1+\frac{q}{p}Y))^2 - 4(2pX)^2\frac{1}{4p}(1+\frac{q^2}{p^2}Y^2\frac{p}{q})}} - 1\right)$$

$$= \frac{1}{2}\left(\frac{1-2X(p+qY)}{\sqrt{(1-2X(p+qY))^2-4X^2(p+qY^2)}}-1\right) \tag{2.5}$$

2.6.2 Asymptotics for the all blue coloring

We consider the special case that $r=0$, i.e., all the vertices of K_n are blue. The corresponding generating function is given by substituting $Y=0$ in (2.5).

$$G(X,0)=\sum_n g(n,0)X^n = \frac{1}{2}\left(\frac{1-2pX}{\sqrt{(1-2pX)^2-4pX^2}}-1\right) \tag{2.6}$$

To determine the asymptotic behavior of $g(n,0)$ as $n\to\infty$, we use the following result (see [23]).

Theorem 31 (Darboux [21])**.** *Suppose that $f(z)$ is analytic for $|z|<r$, $r>0$, and has only algebraic singularities on $|z|=r$. Let a be the minimum of $\mathbf{Re}(\alpha)$ for the terms of the form $(1-z/w)^\alpha h(z)$ at the singularities of $f(z)$ on $|z|=r$, and let w_j, α_j and $h_j(z)$ be the w, α, and $h(z)$ for those terms of the form $(1-z/w)^\alpha h(z)$ for which $\mathbf{Re}(\alpha)=a$. Then, as $n\to\infty$,*

$$[z^n]f(z)=\sum_j \frac{h_j(w_j)n^{-\alpha_j-1}}{\Gamma(-\alpha_j)w_j^n}. \tag{2.7}$$

Here, $[z^n]f(z)$ denotes the coefficient of z^n in the series expansion for $f(z)$. To apply this to our situation, we take

$$f(X)=\frac{1}{\sqrt{1-2(p+\sqrt{p})X}},$$
$$h(X)=\frac{1-2pX}{\sqrt{1-2(p-\sqrt{p})X}},$$

$$w = \frac{1}{2(p+\sqrt{p})},$$
$$\alpha = a = -\frac{1}{2}.$$

Plugging these values into (2.7), using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and simplifying gives

$$g(n,0) = \frac{1}{2}[X^n]f(X) = \frac{1}{2\sqrt{2\pi n}\sqrt{1+\sqrt{p}}}(2(p+\sqrt{p}))^n + o\left(\frac{(2(p+\sqrt{p}))^n}{\sqrt{n}}\right) \quad (2.8)$$

where the extra factor of $\frac{1}{2}$ comes from the (easy-to-forget) factor of $\frac{1}{2}$ in (2.6).

Now, to get the asymptotic value of the probability $\mathbb{P}(n \text{ blue}; 0 \text{ red})$, we must divide by $\binom{2n-2}{n-2}$ which by Stirling's formula is asymptotic to $\frac{2^{2n-2}}{\sqrt{\pi n}}$. Putting this together with (2.8) gives the final result.

Theorem 32. *For $0 \leq p \leq 1$, the probability that all the vertices of K_n are blue is*

$$\mathbb{P}(n \textit{ blue; } 0 \textit{ red}) = \sqrt{\frac{2}{1+\sqrt{p}}}\left(\frac{p+\sqrt{p}}{2}\right)^n (1+o(1)) \quad \textit{as } n \to \infty. \quad (2.9)$$

Setting $p = \frac{1}{2}$ we obtain,

Corollary 33. *With $p = \frac{1}{2}$, the probability that all the vertices of K_n are blue is*

$$\mathbb{P}(n \textit{ blue; } 0 \textit{ red}) = (4-2\sqrt{2})\left(\frac{1+\sqrt{2}}{4}\right)^n (1+o(1)) \quad \textit{as } n \to \infty. \quad (2.10)$$

2.6.3 Asymptotics for a fixed proportion of the vertices blue

We now look at the problem of estimating $\mathbb{P}(b \text{ blue}; r \text{ red})$ along a ray where the ratios $\frac{b}{r}$ and $\frac{r}{b}$ are both bounded away from 0. For this case we will rely on some relatively recent tools which obtain asymptotic estimates for multivariate generating functions (e.g., see [24]), as opposed to the univariate generating functions which we had in the previous section.

Consider a general generating function G of the form

$$G(x,y) = \frac{F(x,y)}{(H(x,y))^{\beta}} = \sum_{n,r} c_{n,r} x^n y^r$$

where H is analytic and β is positive. The growth rates of the coefficients $c_{n,r}$ are determined by the solutions of $H(x,y) = 0$. For the "directional" asymptotics $r/n \sim \lambda$, with $0 < \lambda < 1$, the growth rate is determined by solving the following system of two equations:

$$\begin{aligned} H(x,y) &= 0, \\ \frac{yH_y}{xH_x} &= \lambda. \end{aligned}$$

The solutions for these equations are called *critical points*.

We need the following result, which is a special care of a more general result.

Theorem 34 (Greenwood [22]). *Let H be an analytic function with a single smooth strictly minimal critical point (x_0, y_0), where x_0 and y_0 are real and positive. Suppose H has only real coefficients in its power series expansion about the origin. Assume $H(0,0) > 0$, and consider $H^{-\beta}$ for β a real positive number with the standard branch chosen along the negative real axis, so that $H(0,0)^{\beta} > 0$. Let $\lambda = \frac{r+O(1)}{n}$ be fixed, $0 < \lambda < 1$, as $n, r \to \infty$. Define the following quantities:*

$$\theta_1 = \frac{H_y(x_0,y_0)}{H_x(x_0,y_0)} = \frac{\lambda x_0}{y_0},$$

$$\theta_2 = \frac{1}{2H_x}(\theta_1^2 H_{xx} - 2\theta_1 H_{xy} + H_{yy}) \text{ evaluated at the point } (x_0, y_0)$$
$$\theta_3 = \frac{1}{\sqrt{\frac{2\theta_2}{x_0} + \frac{\theta_1^2}{x_0^2} + \frac{\lambda}{y_0^2}}},$$

In the definition of θ_3, the term underneath the square root is always positive, and the positive square root should be taken. Assume that $H_x(x_0, y_0)$ and $\frac{2\theta_2}{x_0} + \frac{\theta_1^2}{x_0^2} + \frac{\lambda}{y_0^2}$ are nonzero. Then the following expression holds as $n, r \to \infty$ with $\lambda = \frac{r+O(1)}{n}$:

$$c_{n,r} \sim \frac{\theta_3 (H_x(x_0, y_0)x_0)^{-\beta} F(x_0, y_0) n^{\beta - 3/2}}{\Gamma(\beta)\sqrt{2\pi}}.$$

Using the above theorem, we have the following result for general p.

Theorem 35. *For $0 < p < 1$, with $p + q = 1$, we have*

$$\mathbb{P}(pn \text{ blue; } qn \text{ red}) = \frac{1}{\sqrt{3pq\pi n}} + o\left(\frac{1}{\sqrt{n}}\right). \tag{2.11}$$

Proof. We start with the generating function in (2.5). Our goal is to estimate $g(n, \lambda n)$ for $\lambda = 1 - p$. (As one would expect, when $\lambda \neq 1 - p$, then the probability that there are just λn red vertices goes to 0 exponentially rapidly in n; see the example after the proof).

The generating function G can be written (replacing X by x and Y by y) as

$$G(x, y) = \frac{F(x, y)}{(H(x, y))^{1/2}} - \frac{1}{2}$$

where

$$H(x, y) = (1 - 2x(p + qy))^2 - 4x^2(p + qy^2),$$

$$F(x, y) = \frac{1 - 2x(p + qy)}{2}.$$

For the directional asymptotics $r/n \sim \lambda$, the growth rate is determined by solutions (x_0, y_0) of the following system of two equations:

$$\begin{aligned} H(x, y) &= (1 - 2x(p + qy))^2 - 4x^2(p + qy^2) = 0, \\ \frac{yH_y}{xH_x} &= \lambda. \end{aligned}$$

The unique solution satisfying $0 < x_0 \leq 1/2$ and y_0 positive is $x_0 = 1/4$ and $y_0 = 1$. Then in our case (where $\beta = \frac{1}{2}$ and $\lambda = 1 - p$),

$$g(n, \lambda n) \sim C(n) x_0^{-n} y_0^{-\lambda n} = C(n)\, 4^n$$

where $C(n)$ is determined by the following values:

$$\begin{aligned} \theta_1 &= \frac{\lambda x_0}{y_0} = \frac{1 - p}{4}, \\ \theta_2 &= \frac{1}{2H_x}(\theta_1^2 H_{xx} - 2\theta_1 H_{xy} + H_{yy}) \text{ evaluated at the point } (x_0, y_0) = (1/4, 1) \\ &= \frac{(5p - 4)(1 - p)}{16}, \\ \theta_3 &= \frac{1}{\sqrt{\frac{2\theta_2}{x_0} + \frac{\theta_1^2}{x_0^2} + \frac{\lambda}{y_0^2}}} = \sqrt{\frac{2}{3p(1 - p)}}, \\ C(n) &= \frac{\theta_3 (H_x(x_0, y_0) x_0)^{-\beta} F(x_0, y_0) n^{\beta - 3/2}}{\Gamma(\beta)\sqrt{2\pi}} = \frac{1}{4\pi n \sqrt{3pq}} 4^n. \end{aligned}$$

Therefore we can estimate the probability of pn blue nodes and qn red nodes by

$$\mathbb{P}(pn \text{ blue}; qn \text{ red}) = \frac{g(n,(1-p)n)}{\binom{2n-2}{n}} \sim \frac{C(n)4^n}{\frac{4^n}{4\sqrt{\pi n}}} = \frac{1}{\sqrt{3pq\pi n}}$$

as desired. $\square$

We note that when λ differs from $1-p$, the solution of the two equations has x_0' strictly greater than $1/4$ and consequently the probability $\mathbb{P}((1-\lambda)n \text{ blue}; \lambda n \text{ red})$ will be $O((\frac{1}{4x_0'})^n)$, i.e., it will go to 0 exponentially rapidly in n. For example, with $p = \frac{1}{2} = q$ and $\lambda = \frac{1}{3}$, we find that the two equations have a unique solution with $y_0' = 0.62741\ldots$ being the positive root of $4y^3 + 8y^2 - 5y - 1$ and $x_0' = \frac{10}{9}y_0'^2 + 2y_0' - \frac{25}{18} = 0.303313\ldots$.

2.7 Graph Builds

In this section we take an independent look at the function $F_G(k)$, which was introduced in Section 2.2.

Fix a positive integer k and a graph G with vertex set $V(G)$ and edge set $E(G)$. Let $|V(G)| = n$ and $|E(G)| = m$. Consider the following graph procedure. Begin with the empty graph on $V(G)$ and pick an edge from G to place onto $V(G)$. The result is a graph H_1 with a single edge. Next pick another edge from G and place it onto $V(G)$. Continuing in this way, for each $\ell \in [m]$ we obtain a graph H_ℓ which is a subgraph of G containing ℓ edges and n vertices. Note that each H_ℓ is a subgraph of $H_{\ell+1}$.

Let $\tilde{H}_\ell$ be the graph obtained from H_ℓ by removing the lone vertices. Let $F_G(k)$ be the number of sequences $\tilde{H}_1, \tilde{H}_2, \ldots, \tilde{H}_m$ where exactly k times is $\tilde{H}_{i+1}$ obtained from $\tilde{H}_i$ by adding an edge disjoint from all others thus far.

The case $k = 1$ corresponds to the number of ways to "build up" a graph G, one edge at a time, so that at each point the so-far created graph is connected. Said differently, after the first edge has been chosen each subsequent chosen edge must share at least one vertex with a previously chosen edge. Thus $\left|V\left(\tilde{H}_{\ell+1}\right)\right| \leq \left|V\left(\tilde{H}_{\ell}\right)\right| + 1$ for all $\ell \geq 1$.

Example 5. If $G = K_4$ and $k = 1$, one valid sequence is

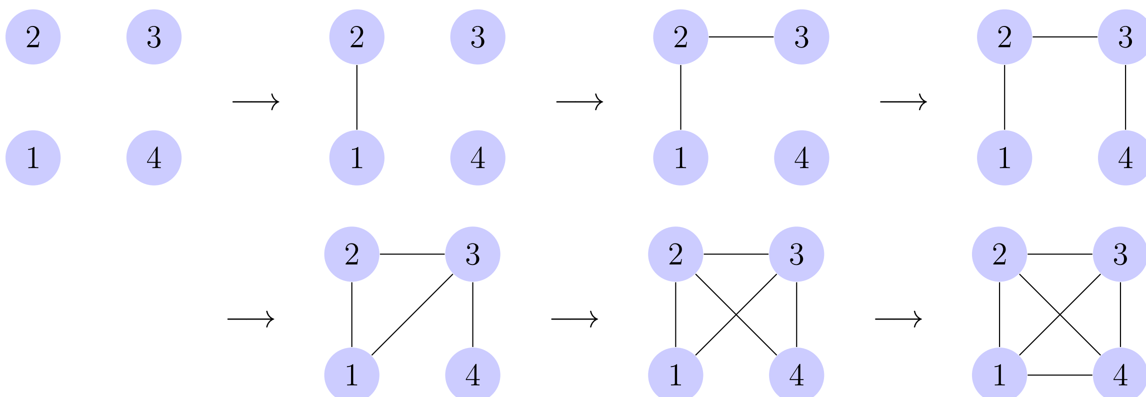


Figure 2.2: One build of K_4 in the case $k = 1$

while the following is not a valid sequence (but it is valid when $k = 2$)

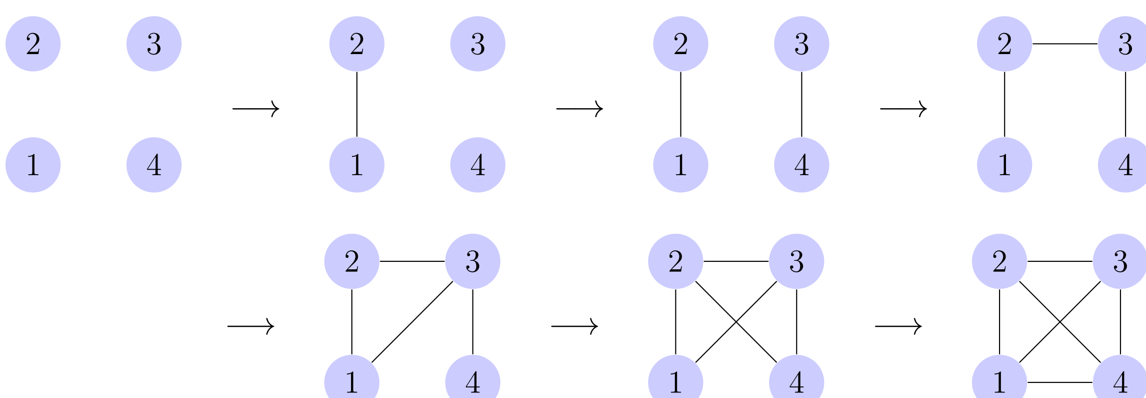


Figure 2.3: One build of K_4 in the case $k = 2$.

As we already saw, this structural graph theory question arose from edge-flipping on graphs, in which the following theorem was found. To properly state it, let's phrase the question slightly differently. Let $P(G,k)$ denote the probability that, if the edges of G are randomly ordered, that the resulting build has the property that exactly k of these edges are disjoint from all previous edges. Then we have the following theorem.

Theorem 36.

$$P\left(K_n, 1\right) = \frac{2^{n-2}}{C_{n-1}},$$

where C_n is the n^{th} Catalan number.

Proof. Perform the above random sequence and let S_i be the set of vertices that have been seen after the i^{th} edge that exposes a new vertex was placed. Our random sequence satisfies the desired condition exactly when $|S_{i+1}| \leq |S_i| + 1$ for all $i \geq 1$.

Assume that $|S_i| = t$. When the edge that creates $|S_{i+1}|$ is placed, one of two things happens. Either that edge is one of the $t(n-t)$ edges which adds one vertex to the set S_i, or that edge is one of the $\binom{n-t}{2}$ which adds two vertices to S_i. The desirable case occurs with probability

$$\frac{t(n-t)}{t(n-t) + \binom{n-t}{2}} = \frac{2t}{2t+n-t-1}.$$

Our final probability is then simply the product of all these probabilities, namely

$$\prod_{t=2}^{n-1} \frac{2t}{2t+n-t-1} = 2^{n-2} \prod_{t=2}^{n-1} \frac{t}{2t+n-t-1} = \frac{2^{n-2}}{C_{n-1}},$$

where the last equality is easily obtained by writing C_{n-1} in factorial form and canceling where you can. □

At the 2015 Graduate Research Workshop in Combinatorics I lead a research group to study graph builds. During that time we discovered many more theorems, but for various reasons I will not state those here. I will note, though, that at least for the $k = 1$ case much is known for paths, cycles, complete bipartite graphs, and spiders. More is also known for random graphs and edge-transitive graphs.

2.8 Hyperedge flipping

It is natural to ask which results discussed can be generalized to complete t-uniform hypergraphs. In particular, we uniformly at random pick t vertices and color them all blue with probability p and red with probability $q = 1 - p$. If $t = 1$, then we are simply flipping each vertex at random, and so the stationary distribution is easily found. In particular,

$$\mathbb{P}(b \text{ blue}; r \text{ red}) = \binom{b+r}{b} p^b q^r.$$

The next case, $t = 2$, is the graph case studied above. In this section we begin the study for $t > 2$. Some of the results generalize easily, but previously we still maintained our focus above on the graph case both for clarity and because our most important theorems do not yet generalize.

2.8.1 Equivalent interpretations to hyperedge flipping

The goal of this section is to take a closer look at the hyperedge flipping process and to demonstrate that the problem reformulations in Section 2.2 still do generalize in the hypergraph case. We begin with the original interpretation of hyperedge flipping.

> **Random walk interpretation of hyperedge flipping:** Take a hypergraph $\mathcal{H}$ with hyperedges $\{e_1, e_2, \ldots, e_m\}$, and some initial, arbitrary red/blue vertex coloring of $\mathcal{H}$. Randomly choose hyperedges (with replacement) and change the color of the vertices of the selected hyperedge to blue with probability p and to red with probability q. Continue this process indefinitely.

Our first observation is that the hyperedge flipping process is memoryless, i.e., a vertex is only affected by the last hyperedge drawn that was incident to v. This suggests that we should focus only on the last time that a particular hyperedge was selected, and leads us to the following interpretation. (In fact, even fewer hyperedges will affect the final coloring.)

> **Reduced interpretation of hyperedge flipping:** Take a hypergraph $\mathcal{H}$ and a deck of $|E(\mathcal{H})|$ cards, i.e., one card for each hyperedge. Randomly shuffle the deck and then deal out the cards one card at a time. For each card, change the vertices of the indicated hyperedge to blue with probability p and to red with probability q.

Note that since each hyperedge of $\mathcal{H}$ will eventually be chosen, any original red/blue vertex coloring of $\mathcal{H}$ is not important.

Proposition 37. *Given a fixed vertex coloring c of $\mathcal{H}$, the probability of being at c in the stationary distribution for the random walk interpretation is the same as the probability of realizing c by the reduced interpretation.*

Proof. The probability of being at c in the random walk is found by looking at the probability that we are at c after N steps where $N \to \infty$. We start by looking at any fixed list $\mathcal{L}$ of N hyperedges from $\mathcal{H}$ in which each hyperedge appears at least once in the list. Given any permutation π of $\{e_1, \ldots, e_m\}$, define $\pi(\mathcal{L})$ to be the list that applies π to each member of $\mathcal{L}$. Since at each stage we randomly chose a hyperedge, the probability that a random list of length N is $\mathcal{L}$ is the same as the probability that it is $\pi(\mathcal{L})$; indeed, both probabilities are $1/m^N$.

Moreover, the locations of the final hyperedge appearances in $\mathcal{L}$ are the same as in $\pi(\mathcal{L})$, i.e., if e_i's final appearance is in position j in $\mathcal{L}$, then $\pi(e_i)$'s final appearance in $\pi(\mathcal{L})$ is in position j. This shows that, given two orderings $\mathcal{O}_1$ and $\mathcal{O}_2$ of $E(\mathcal{H})$, the number of length-N hyperedge sequences in which the final appearances of each hyperedge occurs in the order $\mathcal{O}_1$ is the same as the number that appear in the order $\mathcal{O}_2$. And hence, the probability of the ordering $\mathcal{O}_1$ is the same the probability of the ordering $\mathcal{O}_2$.

Since these orderings are the only thing that determine they final coloring, and each ordering is equally likely, it suffices to take a random ordering $\mathcal{O}$ of the m hyperedges and simply run the procedure on this list.

Finally, we observe that as $N \to \infty$ each list of randomly chosen hyperedges will contain each possible hyperedge with probability 1. In particular the probability of being at coloring c after N steps in the random walk interpretation converges to the probability of realizing c by the reduced interpretation. □

One immediate consequence of this is that there are only finitely many possible orderings of the hyperedges and each occurs with probability some monomial in p and q. Therefore the probability of realizing a particular coloring c is a polynomial function of p.

In looking at the reduced interpretation we only considered each hyperedge once. However in the process of coloring the hyperedges only a few of the cards have an impact on the final coloring. This is because cards that occurred near the start of the deck are likely to have all of their vertices recolored by some other hyperedge later in the process. This suggests we focus only on hyperedges which will impact the final coloring, and leads to our next interpretation.

> **Reversed interpretation of hyperedge flipping:** Take a hypergraph $\mathcal{H}$ and a deck of $|E(\mathcal{H})|$ cards, i.e., one card for each hyperedge, and start with no coloring on the vertices of $\mathcal{H}$. Randomly shuffle the deck and then deal out the cards one at a time. For each card, if any number of the vertices of the corresponding hyperedge are uncolored, then with probability p color the uncolored vertices blue and with probability q color them red. If a vertex is colored already, do not recolor it.

Proposition 38. *Given a vertex coloring c of $\mathcal{H}$, the probability that we end with coloring c is the same for both the reduced interpretation and the reversed interpretation.*

Proof. Since the process is memoryless, we could instead color in the reverse order, as follows. Given some ordering $\mathcal{O}$, let $\mathcal{O}'$ be the reverse ordering. Run through the reversed order, coloring hyperedges as before, but now when presented with a vertex that is already colored, instead of recoloring it just leave it as is. This clearly gives the same coloring as before.

We can now view this as taking a deck of m cards, one card for each hyperedge of $\mathcal{H}$, randomly shuffling this deck, and dealing the cards out one at a time. When e_i's card is dealt, locate e_i in $\mathcal{H}$ and, if any of its vertices are uncolored, color them blue with probability p and red with probability q. □

Next is a very similar interpretation which is clearly equivalent to the previous.

> **Reversed-with-replacement interpretation of hyperedge flipping:** Take a hypergraph $\mathcal{H}$, initially uncolored. Randomly choose hyperedges (with replacement) from $\mathcal{H}$ and for each chosen edge color the uncolored vertices of e blue with probability p and red with probability q. If a vertex is colored already, do not recolor it. Continue this process until every vertex of $\mathcal{H}$ has been colored.

The coloring on the reversed interpretation grows in bits and pieces, i.e., a card for a hyperedge will color ℓ vertices for some $\ell \in \{0, 1, \ldots, t\}$. We can approach the process through understanding this evolving coloring, which gives our last interpretation.

> **Constructive interpretation of hyperedge flipping:** Take a hypergraph $\mathcal{H}$ and a deck of $|E(\mathcal{H})|$ cards, i.e., one card for each hyperedge. Start with a hypergraph on the vertices of $\mathcal{H}$ but with no hyperedges. Randomly shuffle the deck and deal out the cards one at a time. Each time a hyperedge comes up, insert the hyperedge into the hypergraph. If at the time of insertion of the hyperedge, some (or all) of the vertices of the hyperedge are uncolored, then with probability p color the uncolored vertices blue and with probability q color the uncolored vertices red. Otherwise disregard the card.

Since this works in the same manner as the reversed interpretation we have the following result.

Proposition 39. *The probability that we end with coloring c for the reversed interpretation is*

$$\frac{1}{m!}\sum_{\mathcal{O}} p^s q^t$$

where the sum is taken over all orderings of $E(\mathcal{H})$, and s and t are the number cards which colored at least one vertex blue and red, respectively, in the constructive interpretation.

When applying the constructive process on the t-uniform hypergraph $\mathcal{H}$, if we disregard the hyperedges which do not color a vertex then the growing sequence of hypergraphs induced by the resulting collection of hyperedges will form a forest with no isolated vertices. In particular, each hyper-tree (i.e. the graph is not disconnected, but there is no hyperedge which is a subset of the union of the others) will have one hyperedge which colored all t of its vertices and each other hyperedges on the tree colored a smaller positive number of vertices.

We now consider the special case when we are considering the all-blue coloring of the hypergraph (so each card which colors at least one vertex was chosen to color blue with probability p). We have that the coefficient of p^ℓ is the proportion of all orderings where ℓ hyperedges contributed to the final coloring.

Let $F_{\mathcal{H}}(a_1, a_2, \ldots, a_t)$ denote the number of orderings of the hyperedges of $\mathcal{H}$ in which, for each i, there are a_i hyperedges which color i vertices. Observe that $\sum_{i=1}^{t} i \cdot a_i = n$. When $\mathcal{H}$ is the complete t-uniform hypergraph on n vertices we simply write $F_n(a_1, a_2, \ldots, a_t)$. Note that earlier we used $F_n(k)$ in place of $F_n(n-2k, k)$, as defined above.

We have

$$\mathbb{P}(c \text{ is all blue}) = \sum_{\substack{[a_1, \ldots, a_t], \\ \sum i \cdot a_i = n}} \frac{F_{\mathcal{H}}(a_1, a_2, \ldots, a_t)}{m!} p^{\sum_i^t a_i}.$$

A similar analysis can be done when we are not using the all-blue coloring, and we will return to this in a later section.

2.8.2 Probability the first k vertices are colored blue

In this section we will look at the probability that the first k (labeled) vertices are colored blue in the stationary distribution.

Before we begin, we will need to introduce the idea of a *restricted* coloring, which is defined just as before. Let $\mathcal{H}$ be a t-uniform hypergraph and c be a coloring of a subhypergraph H of $\mathcal{H}$. Define $\mathbb{P}_{\mathcal{H}}(c)$ to be the probability of realizing the coloring

c on $\mathcal{H}$, where the vertices $V(\mathcal{H}) \setminus V(H)$ are allowed to be any color.

Lemma 40. *Let $\mathcal{H}$ be a t-uniform hypergraph and c be a coloring of a subhypergraph H of $\mathcal{H}$. Let $\mathcal{H}'$ be the hypergraph obtained by removing all the hyperedges from $\mathcal{H}$ which do not intersect $V(H)$. If $\mathcal{H}'$ has no isolated vertices then*

$$\mathbb{P}_{\mathcal{H}}(c) = \mathbb{P}_{\mathcal{H}'}(c).$$

Proof. We will use the reverse interpretation of hyperedge flipping. Let U be the deleted hyperedges with $m = |E(\mathcal{H})|$ and $u = |U|$. Let L be any list of the hyperedges of $\mathcal{H}$ which gives the coloring c. Notice that the hyperedges from U do not color any of the vertices $V(H)$ whose colors we demand match c. Therefore removing them from L still leaves a list L' of the hyperedges of $\mathcal{H}'$ which color $\mathcal{H}'$ exactly as before (note that every vertex of $\mathcal{H}$ still gets a color since $\mathcal{H}'$ has no isolated vertices).

Moreover, for a fixed list L' of the hyperedges of $\mathcal{H}'$ giving the coloring c, it is easy to see that the number of lists of $E(\mathcal{H})$ which reduce to L' is precisely $\binom{m}{u}$. Note that this quantity is independent of our choice of L'. Likewise, given a list of $E(\mathcal{H}')$ that gives a coloring different than c, there are again precisely $\binom{m}{u}$ lists of $E(\mathcal{H})$ which reduce to the chosen list.

Since every list of $\mathcal{H}$ can be reduced we conclude that the proportion of lists of $E(\mathcal{H})$ giving the coloring c is the same as the proportion of lists of $E(\mathcal{H}')$ giving the coloring c. $\square$

So to determine the probability that the first k are blue we can work on a "simpler" hypergraph. Let $\mathcal{H}_n^{(t)}$ be the complete t-uniform hypergraph on n vertices. Since the specific selected vertices are not important, only the number of them, we define $Q_n^{(t)}(k)$ to be the probability that k *specified* vertices of $\mathcal{H}_n^{(t)}$ are blue (regardless of the coloring on the remaining $n-k$ vertices). We now get the following recurrence.

Proposition 41. *Suppose* $2 \leq k \leq n$. *Then*

$$Q_n^{(t)}(k) = \sum_{i=1}^{t} \frac{\binom{k}{i}\binom{n-k}{t-i}p}{\binom{n}{t} - \binom{n-k}{t}} Q_n^{(t)}(k-i)$$

with $Q_n^{(t)}(0) = 1$.

Proof. We will use the reversed interpretation of the problem. Observe that the initial condition $Q_n^{(t)}(0) = 1$ holds since any coloring works (i.e., there is no restriction).

We now prove the recurrence. By Lemma 40, we may instead consider the hypergraph $\mathcal{H}'$ obtained by removing all hyperedges disjoint from our specified vertices, $\{v_1, \ldots, v_k\}$. Note that $\mathcal{H}'$ is the lexicographic hypergraph.We now use this to establish the recurrence.

Consider a list L of $E(\mathcal{H}')$, and let e_1 be the first hyperedge in L. Then e_1 has some number $i \in [t]$ of vertices that intersect the set $\{v_1, \ldots, v_k\}$. Among the $\sum_{i=1}^{t} \binom{k}{i}\binom{n-k}{t-i} = \binom{n}{t} - \binom{n-k}{t}$ hyperedges in $\mathcal{H}'$, the probability of randomly choosing one which intersects $\{v_1, \ldots, v_k\}$ in exactly i vertices is $\binom{k}{i}\binom{n-k}{t-i} \Big/ \left[\binom{n}{t} - \binom{n-k}{t}\right]$.

Moreover, given such an edge, we must have that edge chosen to be blue if we are to end with the all-blue coloring on $\{v_1, \ldots, v_k\}$.

Finally, with all of the above satisfied, we then realize a reduced problem. Indeed, assume that the i vertices chosen to be blue are $\{v_{k-i+1}, \ldots, v_k\}$ and consider the state of some vertex v from this set. Vertex v is colored blue and since we are working from the reversed interpretation of the problem it's color will not change. Indeed, the only vertices whose future colorings matter are those in $\{v_1, \ldots, v_{k-i}\}$.

Moreover any edge contained in the set $\{v_{k-i+1},\ldots,v_n\}$ will not affect the final coloring for $\mathcal{H}'$, regardless of its position (after e_1) in L.

Therefore by Lemma 40 the probability of completing the all-blue coloring on $\mathcal{H}'$ is precisely $Q_n^{(t)}(k-i)$. Summing over all possible i then gives the recurrence. $\square$

Proposition 42. *The hyperedge flipping probability polynomial for the complete, t-uniform hypergraph corresponding to the event that the first k vertices are blue (the other vertices can be any color) is* $\sum_{\ell=\lceil k/t\rceil}^{n-t+1} c_\ell\cdot p^\ell$ *where*

$$c_\ell=\sum_{\substack{[a_1,\ldots,a_\ell],\\ \sum a_i=k,\\ a_i\geq 1}}\frac{\binom{k}{a_1,a_2,\ldots,a_\ell}\cdot\prod_{i=1}^{\ell}\binom{n-\sum_{j=i}^{\ell}a_j}{t-a_i}}{\prod_{i=1}^{\ell}\left[\binom{n}{t}-\binom{n-\sum_{j=i}^{\ell}a_j}{t}\right]}.$$

Proof. We will show that this satisfies the recurrence in Theorem 41. Clearly the base case is satisfied. Now assume that $k>0$. Then

$$\begin{aligned}
[p^\ell]Q_n^{(t)}(k) &= \sum_{\substack{[a_1,\ldots,a_\ell],\\ \sum a_i=k,\\ a_i\geq 1}}\frac{\binom{k}{a_1,a_2,\ldots,a_\ell}\cdot\prod_{i=1}^{\ell}\binom{n-\sum_{j=i}^{\ell}a_j}{t-a_i}}{\prod_{i=1}^{\ell}\left[\binom{n}{t}-\binom{n-\sum_{j=i}^{\ell}a_j}{t}\right]}\\
&= \sum_{m=1}^{t}\sum_{\substack{[a_1,\ldots,a_{\ell-1},m],\\ \sum a_i=k-m,\\ a_i\geq 1}}\frac{\binom{k}{a_1,a_2,\ldots,a_{\ell-1},m}\cdot\prod_{i=1}^{\ell}\binom{(n-m)-\sum_{j=i}^{\ell-1}a_j}{t-a_i}}{\prod_{i=1}^{\ell}\left[\binom{n}{t}-\binom{(n-m)-\sum_{j=i}^{\ell-1}a_j}{t}\right]}\\
&= \sum_{m=1}^{t}\sum_{\substack{[a_1,\ldots,a_{\ell-1},m],\\ \sum a_i=k-m,\\ a_i\geq 1}}\frac{\binom{k}{a_1,a_2,\ldots,a_{\ell-1},m}\cdot\prod_{i=1}^{\ell}\binom{(n-m)-\sum_{j=i}^{\ell-1}a_j}{t-a_i}}{\prod_{i=1}^{\ell}\left[\binom{n}{t}-\binom{(n-m)-\sum_{j=i}^{\ell-1}a_j}{t}\right]}\\
&= \sum_{m=1}^{t}\frac{\binom{k}{m}\binom{n-m}{t-m}}{\binom{n}{t}-\binom{n-m}{t}}\sum_{\substack{[a_1,\ldots,a_{\ell-1}],\\ \sum a_i=k-m,\\ a_i\geq 1}}\frac{\binom{k-m}{a_1,a_2,\ldots,a_{\ell-1}}\cdot\prod_{i=1}^{\ell-1}\binom{(n-m)-\sum_{j=i}^{\ell-1}a_j}{t-a_i}}{\prod_{i=1}^{\ell-1}\left[\binom{n}{t}-\binom{(n-m)-\sum_{j=i}^{\ell-1}a_j}{t}\right]}\\
&= \sum_{m=1}^{t}\frac{\binom{k}{m}\binom{n-m}{t-m}}{\binom{n}{t}-\binom{n-m}{t}}[p^\ell]Q_n^{(t)}(n-m).
\end{aligned}$$

□

We also note that this formula can be computed directly. We sketch this argument now. By an application of Lemma 40 we may again disregard all edges which do not contribute to the final coloring. To compute c_ℓ we consider all ways in which precisely ℓ hyperedges could contribute to the coloring of the specified k vertices, $\{v_1, \ldots, v_k\}$.

Each of these ℓ hyperedges will color some positive number of the vertices. Suppose these numbers are $a_1, \ldots, a_\ell$. Given such a list, we can compute the probability of obtaining this list. After this we simply sum over all possible such lists.

To compute this, observe that there are $\binom{k}{a_1, a_2, \ldots, a_\ell}$ choices for which vertices each hyperedge colors. Given such a choice, the first hyperege has an $\binom{n-k}{t-a_1} / \left[\binom{n}{t} - \binom{n-k}{t}\right]$ chance of being properly realized. Then next edge has a $\binom{n-k+a_1}{t-a_2} / \left[\binom{n}{t} - \binom{n-k}{t}\right]$ chance, and so forth. Continuing in this way gives the asserted formula.

Proposition 43. *Let $H_n^{(t)}$ be the complete t-uniform hypergraph on n vertices and let c be the all-blue coloring of $H_n^{(t)}$. Then for any $p \in [0,1]$ and $t \in [n-1]$ we have*

$$\mathbb{P}_{\mathcal{H}_n^{(t)}}(c) \leq \mathbb{P}_{\mathcal{H}_n^{(t+1)}}(c),$$

and equality holds if and only if $p \in \{0, 1\}$.

Proof. We use the reversed-with-replacement interpretation of hyperedge flipping. Choosing a random hyperedge from $\mathcal{H}_n^{(t+1)}$ is equivalent to choosing a random set of t vertices and and then choosing a new, additional vertex. After this hyperedge is chosen, a color is then associated to it – blue with probability p and red with probability q. We will define a single list L to record all of this data. In particular, the i^{th} member of this list is an ordered triple (S, a, c) where S is the chosen set of t vertices, a is the additional vertex giving the hyperedge $S \cup \{a\}$, and c is the color which this hyperedge was assigned.

Let $\tilde{L}$ be a list of ordered pairs where if (S, a, c) is the i^{th} element in L then (S, c) is the i^{th} element in $\tilde{L}$. It's possible that at this point the union of the sets S is a proper subset of $V\left(\mathcal{H}_n^{(t)}\right)$. If this is the case, continue picking random hyperedges and colors for them as before, and append each new selection on to $\tilde{L}$. Do this until the sets S cover $V\left(\mathcal{H}_n^{(t)}\right)$.

It is clear that since L was generated randomly, mimicking the hyperedge flipping process on $\mathcal{H}_n^{(t+1)}$, that the corresponding lists $\tilde{L}$ then do the same on $\mathcal{H}_n^{(t)}$; precisely – the lists $\tilde{L}$ give the same distribution of colorings on $\mathcal{H}_n^{(t)}$ as does (the reversed-with-replacement interpretation of) hyperedge flipping.

With this we can then examine the probability of getting the all-blue colorings on $\mathcal{H}_n^{(t+1)}$ and on $\mathcal{H}_n^{(t)}$ by generating a random list L and coloring $\mathcal{H}_n^{(t+1)}$ as prescribed by this list and coloring $\mathcal{H}_n^{(t)}$ by the list $\tilde{L}$. It is clear that whenever $\tilde{L}$ colors $\mathcal{H}_n^{(t)}$ all-blue that L also colored $\mathcal{H}_n^{(t+1)}$ all-blue, which proves that $\mathbb{P}_{\mathcal{H}_n^{(t)}}(c) \leq \mathbb{P}_{\mathcal{H}_n^{(t+1)}}(c)$. But it's also clear that as long as $p \in (0, 1)$ there is at least one list which colors $\mathcal{H}_n^{(t+1)}$ all-blue while $\mathcal{H}_n^{(t)}$ gets a different coloring. This gives the asserted strict inequality and concludes the proof.

□

Once we have the probability that b specified vertices are colored blue, then by use of an inclusion-exclusion argument we can find the probability that b specified vertices are blue and the remainder are red. However, this will result in an alternating sum, and be entirely in terms of p, when we want a sum which is in terms of p

and q. Therefore we will take a slightly different approach by using the constructive interpretation of hyperedge flipping.

Theorem 44. *For the hyperedge flipping process on the complete t-uniform hypergraph $\mathcal{H} = H_n^{(t)}$, let c be a coloring which assigns red to r* specified *vertices and blue to the remainder. Then the probability of having the coloring c is given by*

$$P_G(c) = \sum_{\sum_1^t i\cdot a_i = n} \sum_{\sum_1^t i\cdot r_i = r} \frac{|F_n(a_1,\ldots,a_t)| \cdot \prod_1^t \binom{a_i}{r_i}}{\binom{n}{t}!\binom{n}{r}} p^{\sum(a_i - r_i)} q^{\sum r_i}.$$

We note the result does not depend on which vertices have been specified red and blue, and so if we only care that r of the vertices are red and the remainder are blue then we multiply the probability in Theorem 44 by $\binom{n}{r} = \binom{b+r}{b,r}$.

Proof. We use the constructive interpretation of hyperedge flipping. Let $\mathcal{F}_n^{(t)}(a_1,\ldots,a_t)$ be the set of ways to build $\mathcal{H}$ one hyperedge at a time so that a_i hyperedges have the property that, when placed, they color precisely i vertices. Note that $F_n(a_1,\ldots,a_t) = |\mathcal{F}_n(a_1,\ldots,a_t)|$.

Consider an arbitrary $S \in \mathcal{F}_n(a_1,\ldots,a_t)$ and let $\mathcal{T}_i$ be the set of a_i hyperedges which color i vertices. Clearly the number of vertices that are colored red by these hyperedges is a multiple of i, so suppose $i \cdot r_i$ is that number. Clearly $\sum i \cdot r_i = r$. We will now count the number of such sequences which give the coloring c.

Define an *r-set* to be a set of r vertices from $\mathcal{H}$. We first count the total number of r-sets for which, for each i, exactly $i \cdot r_i$ of its members are grouped together by r_i hyperedges from $\mathcal{T}_i$, and the remaining $r - i\cdot r_i$ vertices are disjoint from the hyperedges of $\mathcal{T}_i$; call this property of an r-set *property $\mathcal{T}$*. This is easy, as for each i there are $\binom{a_i}{r_i}$

ways to pick the these r_i hyperedges from $\mathcal{T}_i$. Summing over all $S \in F_n(a_1, \dots, a_t)$ gives a total of $|F_n(a_1, \dots, a_t)| \cdot \prod_1^t \binom{a_i}{r_i}$ distinct occurrences of property $\mathcal{T}$, among all builds in $\mathcal{F}_n(a_1, \dots, a_t)$.

Since $\mathcal{H}$ is symmetric between r-sets, and the set $\mathcal{F}_n(a_1, \dots, a_t)$ is symmetric within $V(G)$ in the sense that any build from this collection can be translated to another by simply permuting the vertex set, it is clear that any two r-sets will have property $\mathcal{T}$ the same number of times among the entire collection. Therefore this total must be evenly distributed among all $\binom{n}{r}$ of the r-sets. In particular, the unique r-set of red vertices must occur precisely

$$\frac{|F_n(a_1, \dots, a_t)| \cdot \prod_1^t \binom{a_i}{r_i}}{\binom{n}{r}}$$

times.

Given such a sequence S, there is a $1/\binom{n}{t}!$ probability that it will appear, and if it does there is a $p^{\sum(a_i - r_i)} q^{\sum r_i}$ probability that it will be colored in the unique way giving the coloring c. In particular, for each i, the r_i hyperedges from $\mathcal{T}_i$ that group together $i \cdot r_i$ red vertices must be red. Then, since there is a total of $\sum a_i$ hyperedges which color at least one vertex, the remaining $\sum a_i - \sum r_i$ must correspond to blue hyperedges. Summing over all choices gives a probability of

$$P_G(c) = \sum_{\sum_1^t i \cdot a_i = n} \sum_{\sum_1^t i \cdot r_i = r} \frac{|F_n(a_1, \dots, a_t)| \cdot \prod_1^t \binom{a_i}{r_i}}{\binom{n}{t}! \binom{n}{r}} p^{\sum(a_i - r_i)} q^{\sum r_i}.$$

of obtaining the coloring c. $\square$

2.9 Future Work

Below we mention some possible future directions.

- We have studied only a few classes of graphs, and of t-uniform hypergraphs. We look for new work on specific classes of (hyper)graphs.

- One natural question is to ask the rate of convergence of the random walk interpretation of the edge flipping process, i.e., how quickly do we converge to the stationary distribution. One way to determine this is to use the spectrum of the probability transition matrix on the state graph. The spectrum, and in particular how closely the non-trivial eigenvalues cluster around 0, give a bound on the rate of convergence to the stationary distribution. Our work does not show how to establish the spectrum. We remark that the spectrum for the path and cycle were determined in the analysis carried out in the earlier paper of Chung and Graham [20].

- We can also increase the number of colors, for instance blue, red, and yellow. So that an edge changes to blue with probability p, to red with probability q and to yellow with probability $r = 1 - p - q$. We point out that the results of coloring the first k vertices blue given in Section 2.3 still hold since we can combine the two other colors together. So that the work in establishing the general case comes in the bootstrapping given in Section 2.4.

- There is a lot more work to be done in the study of graph builds. This effort has begun but we look forward to much more.

Appendix A

Code

A.1 Generating card sequences

The following is MATLAB code.

```
function [ Final_output ] = CrossingNumSeqs( n,b,type,
    FinalPermutation, AdditionalCrossings )
%           type = 1 means the sequences are valid
%           type = 2 means the sequences are strong
%   valid means that at least one jump is a b-jump
%   strong means that the sequence is valid and also there
     are no 1-jumps



```

```
AllFinalComps = IntegerCompositions(b*(b-1) +
    AdditionalCrossings + n,n,n,type,b);
AllFinalComps = transpose(AllFinalComps);

if type == 1
    temp = [];
    NumComps = size(AllFinalComps,1);
    for i = 1 : NumComps;

        rowi = AllFinalComps{i};
        if 0 == isempty(find(rowi==b,1))
                temp = [temp ; rowi];
        end
    end
    ValidComps = temp;
    temp3 = [];
    NumComps3 = size(ValidComps,1);
    for i = 1 : NumComps3;
        rowi = ValidComps(i,:);
        temp = 1:b;
        for j = 1 : n;
            temp = Shift2(temp,rowi(j));
        end
        if b == sum(temp == FinalPermutation);
```

```
            temp3 = [temp3 ; rowi];
        end
    end
    GoodValidComps = temp3;
    Final_output = GoodValidComps;

else
    temp = [];
    NumComps = size(AllFinalComps,1);
    for i = 1 : NumComps;
        rowi = AllFinalComps{i};
        if 0 == isempty(find(rowi==b,1))
                temp = [temp ; rowi];
        end
    end
    ValidComps = temp;
    temp2 = [];
    NumComps2 = size(ValidComps,1);
    for i = 1 : NumComps2;
        rowi = ValidComps(i,:);
        if 1 == isempty(find(rowi==1,1))
                temp2 = [temp2 ; rowi];
        end
    end
```

```
    StrongComps = temp2;
    temp4 = [];
    NumComps4 = size(StrongComps,1);
    for i = 1 : NumComps4;
        rowi = StrongComps(i,:);
        temp = 1:b ;
        for j = 1 : n;
            temp = Shift2(temp,rowi(j));
        end
        if b == sum(temp == FinalPermutation);
            temp4 = [temp4 ; rowi];
        end

    end
    GoodStrongComps = temp4;
    Final_output = GoodStrongComps;
end

Final_output = unique(Final_output,'rows');

disp(size(Final_output,1));

end

```

```
function L = Shift2(L,m)
%Takes in a list L of ball orders, and outputs the list
    after an m-bounce occurs
L1=L(1);
L(1:m-1)=L(2:m);
L(m)=L1;
end







function cell_out=IntegerCompositions( n,mink,maxk,a,b )
%RIC :
%   Generates restricted and unrestricted integer
    compositions
%   n = integer to be partitioned
```

```
%    kmin = min. no. of summands
%    kmax = max. no. of summands
%    a = min. value of summands
%    b = max.value of summands
cell = [];
rowdec = 0;
cell_out = {};
for i=mink:maxk
    in = n/i;
    if a>1 rowdec = i;
    end
    if a<=in && in <= b
        cell_out = N2N(n,i,a,b,n-1-rowdec,i-1,0,0,cell,
            cell_out);
    end
end
end




function cell_out = N2N(n,i,a,b,row,col,level,cumsum,cell,
    cell_out)
```

```

if col~=0
    if col==1
        jmax = max(a,n-cumsum-b);
        jmin = min(b,n-cumsum-a);
        for j=jmax : jmin
            cell(i-1) = j; cell(i) = n-cumsum-j;
            csize = size(cell_out,2);
            cnext = csize+1;
            cell_out{cnext} = cell;
        end
    else
        cell(level+1) = a;
        tmp = cumsum + a;
        ntmp = round((n - tmp)/(i-level-1));
        if a <= ntmp && ntmp <= b && cell(level+1)
            cell_out = N2N(n,i,a,b,row-a+1,col-1,level+1,
                tmp,cell,cell_out);
        else
            for q=1:min((b-a),(row-a)-(col-1))
                cell(level+1) = cell(level+1)+1;
                tmp = tmp + 1;
                ntmp = round((n - tmp)/(i-level-1));
                if a <= ntmp && ntmp <= b && cell(level+1)
```

```
                    q2 = q; q = min((b-a),(row-a)-(col-1));
                    cell_out = N2N(n,i,a,b,row-a-q2+1,col-1,level+1,tmp,cell,cell_out);
                end
            end
        end
    end
end
if level>0 && row>1
    cell(level) = cell(level)+1;
    cumsum = cumsum + 1;
    if cell(level)<a
        cumsum = cumsum - cell(level) + a;
        cell(level) = a;
        row = row + cell(level) - a;
    end
    toploop = min(b-cell(level),row-col-1);
    npart = round((n-cumsum)/(i-level));
    if a<=npart && npart<=b && cell(level)<=b
        cell_out = N2N(n,i,a,b,row-1,col,level,cumsum,cell,cell_out);
    else
        for p=1:toploop
```

```
                cell(level) = cell(level)+1;
                cumsum = cumsum + 1;
                npart = round((n-cumsum)/(i-level));
                if a<=npart && npart<=b && cell(level)<=b
                    p2 = p; p = toploop;
                    cell_out = N2N(n,i,a,b,row-p2,col,level,
                       cumsum,cell,cell_out);
                end
            end
        end
end
end
```

A.2 $\delta = 4$ formula reduction

The following is Maple code.

```
> with(combinat):

> Narayana := proc (n, b)
  local temp;
  if n < 1 or b < 1 then
  temp := 0
  else
  temp := binomial(n, b)*binomial(n, b-1)/n
  end if;
  end proc:

> NPrimitiveDeltaFour := proc (n, b)
  local temp;
  temp := (b*n-b-8)*binomial(n, b+3)*binomial(n, b-2)/(2*b+8)
  end proc
```

```
> CompSixCombinedPlus := proc (n, m, z)
local a, b, c, d, e, f, g, i, j, k, l, o, p, temp;
temp := 0;
for a in [seq(i, i = 1 ..  n-4)] do
for c in [seq(i, i = 1 ..  n-3-a)] do
for d in [seq(i, i = 1 ..  n-2-a-c)] do
for e in [seq(i, i = 1 ..  n-1-a-c-d)] do
for f in [seq(i, i = 1 ..  n-a-c-d-e)] do
for i in [seq(i, i = 0 ..  m+z-4)] do
for j in [seq(i, i = 1 ..  m+z-4)] do
for k in [seq(i, i = 1 ..  m+z-3-i-j)] do
for l in [seq(i, i = 1 ..  m+z-2-i-j-k)] do
for o in [seq(i, i = 1 ..  m+z-1-i-j-k-l)] do
for p in [seq(i, i = 1 ..  m+z-i-j-k-l-o)] do
if 0 < n-a-c-d-e-f and 0 < m+z-i-j-k-l-o-p then
temp := temp+Narayana(a, i+j)*Narayana(c, k)*Narayana(d, l)*
Narayana(e, o)*Narayana(f, p)*
Narayana(n-a-c-d-e-f, m+z-i-j-k-l-o-p)
end if end do end do end do end do end do end do end do end do
end do  end do end do;
temp;
end proc:
```

```
> CompSevenCombinedPlus := proc (n, m, z)
  local a, b, c, d, e, f, g, i, j, k, l, o, p, q, temp;
  temp := 0;
  for a in [seq(i, i = 1 ..  n-5)] do
  for c in [seq(i, i = 1 ..  n-4-a)] do
  for d in [seq(i, i = 1 ..  n-3-a-c)] do
  for e in [seq(i, i = 1 ..  n-2-a-c-d)] do
  for f in [seq(i, i = 1 ..  n-1-a-c-d-e)] do
  for g in [seq(i, i = 1 ..  n-a-c-d-e-f)] do
  for i in [seq(i, i = 0 ..  m+z-5)] do
  for j in [seq(i, i = 1 ..  m+z-5-i)] do
  for k in [seq(i, i = 1 ..  m+z-4-i-j)] do
  for l in [seq(i, i = 1 ..  m+z-3-i-j-k)] do
  for o in [seq(i, i = 1 ..  m+z-2-i-j-k-l)] do
  for p in [seq(i, i = 1 ..  m+z-1-i-j-k-l-o)] do
  for q in [seq(i, i = 1 ..  m+z-i-j-k-l-o-p)] do
  if 0 < n-a-c-d-e-f-g and 0 < m+z-i-j-k-l-o-p-q then
  temp := temp+Narayana(a, i+j)*Narayana(c, k)*Narayana(d, l)*
  Narayana(e, o)*Narayana(f, p)*Narayana(g, q)*
  Narayana(n-a-c-d-e-f-g, m+z-i-j-k-l-o-p-q)
  end if end do end do end do end do end do end do end do end do
  end do end do end do end do end do;
  temp;
  end proc:
```

```
> CompEightCombinedPlus := proc (n, m, z)
local a, b, c, d, e, f, g, h, i, j, k, l, o, p, q, r, temp;
temp := 0;
for a in [seq(i, i = 1 ..  n-6)] do
for c in [seq(i, i = 1 ..  n-5-a)] do
for d in [seq(i, i = 1 ..  n-4-a-c)] do
for e in [seq(i, i = 1 ..  n-3-a-c-d)] do
for f in [seq(i, i = 1 ..  n-2-a-c-d-e)] do
for g in [seq(i, i = 1 ..  n-1-a-c-d-e-f)] do
for h in [seq(i, i = 1 ..  n-a-c-d-e-f-g)] do
for i in [seq(i, i = 0 ..  m+z-6)] do
for j in [seq(i, i = 1 ..  m+z-6-i)] do
for k in [seq(i, i = 1 ..  m+z-5-i-j)] do
for l in [seq(i, i = 1 ..  m+z-4-i-j-k)] do
for o in [seq(i, i = 1 ..  m+z-3-i-j-k-l)] do
for p in [seq(i, i = 1 ..  m+z-2-i-j-k-l-o)] do
for q in [seq(i, i = 1 ..  m+z-1-i-j-k-l-o-p)] do
for r in [seq(i, i = 1 ..  m+z-i-j-k-l-o-p-q)] do
if 0 < n-a-c-d-e-f-g-h and 0 < m+z-i-j-k-l-o-p-q-r then
temp := temp+Narayana(a, i+j)*Narayana(c, k)*Narayana(d, l)*
Narayana(e, o)*Narayana(f, p)*Narayana(g, q)*Narayana(h, r)*
Narayana(n-a-c-d-e-f-g-h, m+z-i-j-k-l-o-p-q-r)
end if end do end do end do end do end do end do end do end do
end do end do end do end do end do end do end do;
temp;
end proc:
```

```
> CompSevenCombinedThricePlus := proc (n, m, z)
local a, b, c, d, e, f, g, h, i, j, k, l, o, p, q, r, temp;
temp := 0;
for a in [seq(i, i = 1 .. n-5)] do
for d in [seq(i, i = 1 .. n-4-a)] do
for e in [seq(i, i = 1 .. n-3-a-d)] do
for f in [seq(i, i = 1 .. n-2-a-d-e)] do
for g in [seq(i, i = 1 .. n-1-a-d-e-f)] do
for h in [seq(i, i = 1 .. n-a-d-e-f-g)] do
for i in [seq(i, i = 0 .. m+z-6)] do
for j in [seq(i, i = 1 .. m+z-6-i)] do
for k in [seq(i, i = 1 .. m+z-5-i-j)] do
for l in [seq(i, i = 1 .. m+z-4-i-j-k)] do
for o in [seq(i, i = 1 .. m+z-3-i-j-k-l)] do
for p in [seq(i, i = 1 .. m+z-2-i-j-k-l-o)] do
for q in [seq(i, i = 1 .. m+z-1-i-j-k-l-o-p)] do
for r in [seq(i, i = 1 .. m+z-i-j-k-l-o-p-q)] do
if 0 < n-a-d-e-f-g-h and 0 < m+z-i-j-k-l-o-p-q-r then
temp := temp+Narayana(a, i+j+k)*Narayana(d, l)*Narayana(e, o)*
Narayana(f, p)*Narayana(g, q)*Narayana(h, r)*
Narayana(n-a-d-e-f-g-h, m+z-i-j-k-l-o-p-q-r)
end if end do end do end do end do end do end do end do end do
end do end do end do end do end do end do;
temp;
end proc:
```

```
> CompSixCombinedThricePlus := proc (n, m, z)
local a, b, c, d, e, f, g, h, i, j, k, l, o, p, q, r, temp;
temp := 0;
for a in [seq(i, i = 1 .. n-4)] do
for d in [seq(i, i = 1 .. n-3-a)] do
for e in [seq(i, i = 1 .. n-2-a-d)] do
for f in [seq(i, i = 1 .. n-1-a-d-e)] do
for g in [seq(i, i = 1 .. n-a-d-e-f)] do
for i in [seq(i, i = 0 .. m+z-5)] do
for j in [seq(i, i = 1 .. m+z-5-i)] do
for k in [seq(i, i = 1 .. m+z-4-i-j)] do
for l in [seq(i, i = 1 .. m+z-3-i-j-k)] do
for o in [seq(i, i = 1 .. m+z-2-i-j-k-l)] do
for p in [seq(i, i = 1 .. m+z-1-i-j-k-l-o)] do
for q in [seq(i, i = 1 .. m+z-i-j-k-l-o-p)] do
if 0 < n-a-d-e-f-g and 0 < m+z-i-j-k-l-o-p-q then
temp := temp+Narayana(a, i+j+k)*Narayana(d, l)*
Narayana(e, o)*Narayana(f, p)*Narayana(g, q)*
Narayana(n-a-d-e-f-g, m+z-i-j-k-l-o-p-q)
end if end do end do end do end do end do end do end do
end do end do end do end do end do;
temp;
end proc:
```

```
> CompSevenCombinedTwicePlus := proc (n, m, z)
local a, b, c, d, e, f, g, h, i, j, k, l, o, p, q, r, temp;
temp := 0;
for a in [seq(i, i = 1 .. n-5)] do
for c in [seq(i, i = 1 .. n-4-a)] do
for e in [seq(i, i = 1 .. n-3-a-c)] do
for f in [seq(i, i = 1 .. n-2-a-c-e)] do
for g in [seq(i, i = 1 .. n-1-a-c-e-f)] do
for h in [seq(i, i = 1 .. n-a-c-e-f-g)] do
for i in [seq(i, i = 0 .. m+z-6)] do
for j in [seq(i, i = 1 .. m+z-6-i)] do
for k in [seq(i, i = 1 .. m+z-5-i-j)] do
for l in [seq(i, i = 1 .. m+z-4-i-j-k)] do
for o in [seq(i, i = 1 .. m+z-3-i-j-k-l)] do
for p in [seq(i, i = 1 .. m+z-2-i-j-k-l-o)] do
for q in [seq(i, i = 1 .. m+z-1-i-j-k-l-o-p)] do
for r in [seq(i, i = 1 .. m+z-i-j-k-l-o-p-q)] do
if 0 < n-a-c-e-f-g-h and 0 < m+z-i-j-k-l-o-p-q-r then
temp := temp+Narayana(a, i+j)*Narayana(c, k+l)*
Narayana(e, o)*Narayana(f, p)*Narayana(g, q)*Narayana(h, r)*
Narayana(n-a-c-e-f-g-h, m+z-i-j-k-l-o-p-q-r)
end if end do end do end do end do end do end do end do
end do end do end do end do end do end do end do;
temp;
end proc:
```

```
> CompNineCombinedPlus := proc (n, m, z)
  local a, b, c, d, e, f, g, h, x, i, j, k, l, o, p, q, r,
  s, temp;
  temp := 0;
  for a in [seq(i, i = 1 ..  n-7)] do
  for c in [seq(i, i = 1 ..  n-6-a)] do
  for d in [seq(i, i = 1 ..  n-5-a-c)] do
  for e in [seq(i, i = 1 ..  n-4-a-c-d)] do
  for f in [seq(i, i = 1 ..  n-3-a-c-d-e)] do
  for g in [seq(i, i = 1 ..  n-2-a-c-d-e-f)] do
  for h in [seq(i, i = 1 ..  n-1-a-c-d-e-f-g)] do
  for x in [seq(i, i = 1 ..  n-a-c-d-e-f-g-h)] do
  for i in [seq(i, i = 0 ..  m+z-8)] do
  for j in [seq(i, i = 1 ..  m+z-8-i)] do
  for k in [seq(i, i = 1 ..  m+z-6-i-j)] do
  for l in [seq(i, i = 1 ..  m+z-5-i-j-k)] do
  for o in [seq(i, i = 1 ..  m+z-4-i-j-k-l)] do
  for p in [seq(i, i = 1 ..  m+z-3-i-j-k-l-o)] do
  for q in [seq(i, i = 1 ..  m+z-2-i-j-k-l-o-p)] do
  for r in [seq(i, i = 1 ..  m+z-1-i-j-k-l-o-p-q)] do
  for s in [seq(i, i = 1 ..  m+z-i-j-k-l-o-p-q-r)] do
  if 0 < n-a-c-d-e-f-g-h-x and 0 < m+z-i-j-k-l-o-p-q-r-s then
  temp := temp+Narayana(a, i+j)*Narayana(c, k)*Narayana(d, l)*
  Narayana(e, o)*Narayana(f, p)*Narayana(g, q)*Narayana(h, r)*
  Narayana(x, s)*
  Narayana(n-a-c-d-e-f-g-h-x, m+z-i-j-k-l-o-p-q-r-s)
  end if end do end do end do end do end do end do end do end do
  end do end do end do end do end do end do end do end do end do;
temp; end proc:
```

```
> CompEightCombinedTwicePlus := proc (n, m, z)
  local a, b, c, d, e, f, g, h, x, i, j, k, l, o, p, q, r, s, temp;
  temp := 0;
  for a in [seq(i, i = 1 ..  n-6)] do
  for c in [seq(i, i = 1 ..  n-5-a)] do
  for e in [seq(i, i = 1 ..  n-4-a-c)] do
  for f in [seq(i, i = 1 ..  n-3-a-c-e)] do
  for g in [seq(i, i = 1 ..  n-2-a-c-e-f)] do
  for h in [seq(i, i = 1 ..  n-1-a-c-e-f-g)] do
  for x in [seq(i, i = 1 ..  n-a-c-e-f-g-h)] do
  for i in [seq(i, i = 0 ..  m+z-8)] do
  for j in [seq(i, i = 1 ..  m+z-8-i)] do
  for k in [seq(i, i = 1 ..  m+z-6-i-j)] do
  for l in [seq(i, i = 1 ..  m+z-5-i-j-k)] do
  for o in [seq(i, i = 1 ..  m+z-4-i-j-k-l)] do
  for p in [seq(i, i = 1 ..  m+z-3-i-j-k-l-o)] do
  for q in [seq(i, i = 1 ..  m+z-2-i-j-k-l-o-p)] do
  for r in [seq(i, i = 1 ..  m+z-1-i-j-k-l-o-p-q)] do
  for s in [seq(i, i = 1 ..  m+z-i-j-k-l-o-p-q-r)] do
  if 0 < n-a-c-e-f-g-h-x and 0 < m+z-i-j-k-l-o-p-q-r-s then
  temp := temp+Narayana(a, i+j)*Narayana(c, k+l)*Narayana(e, o)*
  Narayana(f, p)*Narayana(g, q)*Narayana(h, r)*
  Narayana(x, s)*Narayana(n-a-c-e-f-g-h-x, m+z-i-j-k-l-o-p-q-r-s)
  end if end do end do end do end do end do end do end do end do
  end do end do end do end do end do end do end do end do;
  temp;
  end proc:
```

```
> CompEightCombinedThricePlus := proc (n, m, z)
  local a, b, c, d, e, f, g, h, x, i, j, k, l, o, p, q, r, s, temp;
  temp := 0;
  for a in [seq(i, i = 1 ..  n-6)] do
  for d in [seq(i, i = 1 ..  n-5-a)] do
  for e in [seq(i, i = 1 ..  n-4-a-d)] do
  for f in [seq(i, i = 1 ..  n-3-a-d-e)] do
  for g in [seq(i, i = 1 ..  n-2-a-d-e-f)] do
  for h in [seq(i, i = 1 ..  n-1-a-d-e-f-g)] do
  for x in [seq(i, i = 1 ..  n-a-d-e-f-g-h)] do
  for i in [seq(i, i = 0 ..  m+z-8)] do
  for j in [seq(i, i = 1 ..  m+z-8-i)] do
  for k in [seq(i, i = 1 ..  m+z-6-i-j)] do
  for l in [seq(i, i = 1 ..  m+z-5-i-j-k)] do
  for o in [seq(i, i = 1 ..  m+z-4-i-j-k-l)] do
  for p in [seq(i, i = 1 ..  m+z-3-i-j-k-l-o)] do
  for q in [seq(i, i = 1 ..  m+z-2-i-j-k-l-o-p)] do
  for r in [seq(i, i = 1 ..  m+z-1-i-j-k-l-o-p-q)] do
  for s in [seq(i, i = 1 ..  m+z-i-j-k-l-o-p-q-r)] do
  if 0 < n-a-d-e-f-g-h-x and 0 < m+z-i-j-k-l-o-p-q-r-s then
  temp := temp+Narayana(a, i+j+k)*Narayana(d, l)*Narayana(e, o)*
  Narayana(f, p)*Narayana(g, q)*Narayana(h, r)*Narayana(x, s)*
  Narayana(n-a-d-e-f-g-h-x, m+z-i-j-k-l-o-p-q-r-s)
  end if end do end do end do end do end do end do end do end do
  end do end do end do end do end do end do end do end do;
  temp;
  end proc:
```

```
> CompTenCombinedPlus := proc (n, m, z)
local a,b,c,d,e,f,g,h,x,y,i,j,k,l,o,p,q,r,s,t,temp;temp := 0;
for a in [seq(i, i = 1 ..  n-8)] do
for c in [seq(i, i = 1 ..  n-7-a)] do
for d in [seq(i, i = 1 ..  n-6-a-c)] do
for e in [seq(i, i = 1 ..  n-5-a-c-d)] do
for f in [seq(i, i = 1 ..  n-4-a-c-d-e)] do
for g in [seq(i, i = 1 ..  n-3-a-c-d-e-f)] do
for h in [seq(i, i = 1 ..  n-2-a-c-d-e-f-g)] do
for x in [seq(i, i = 1 ..  n-1-a-c-d-e-f-g-h)] do
for y in [seq(i, i = 1 ..  n-a-c-d-e-f-g-h-x)] do
for i in [seq(i, i = 0 ..  m+z-8)] do
for j in [seq(i, i = 1 ..  m+z-8-i)] do
for k in [seq(i, i = 1 ..  m+z-7-i-j)] do
for l in [seq(i, i = 1 ..  m+z-6-i-j-k)] do
for o in [seq(i, i = 1 ..  m+z-5-i-j-k-l)] do
for p in [seq(i, i = 1 ..  m+z-4-i-j-k-l-o)] do
for q in [seq(i, i = 1 ..  m+z-3-i-j-k-l-o-p)] do
for r in [seq(i, i = 1 ..  m+z-2-i-j-k-l-o-p-q)] do
for s in [seq(i, i = 1 ..  m+z-1-i-j-k-l-o-p-q-r)] do
for t in [seq(i, i = 1 ..  m+z-i-j-k-l-o-p-q-r-s)] do
if 0 < n-a-c-d-e-f-g-h-x-y and 0 < m+z-i-j-k-l-o-p-q-r-s-t then
temp := temp+Narayana(a, i+j)*Narayana(c, k)*Narayana(d, l)*
Narayana(e, o)*Narayana(f, p)*Narayana(g, q)*Narayana(h, r)*
Narayana(x, s)*Narayana(y, t)*
Narayana(n-a-c-d-e-f-g-h-x-y, m+z-i-j-k-l-o-p-q-r-s-t)
end if end do end do end do end do end do end do end do end do
end do end do end do end do end do end do end do end do end do
end do end do; temp; end proc:
```

```
> CompNineCombinedThricePlus := proc (n, m, z)
  local a,b,c,d,e,f,g,h,x,y,i,j,k,l,o,p,q,r,s,t,temp;temp := 0;
  for a in [seq(i, i = 1 ..  n-7)] do
  for d in [seq(i, i = 1 ..  n-6-a)] do
  for e in [seq(i, i = 1 ..  n-5-a-d)] do
  for f in [seq(i, i = 1 ..  n-4-a-d-e)] do
  for g in [seq(i, i = 1 ..  n-3-a-d-e-f)] do
  for h in [seq(i, i = 1 ..  n-2-a-d-e-f-g)] do
  for x in [seq(i, i = 1 ..  n-1-a-d-e-f-g-h)] do
  for y in [seq(i, i = 1 ..  n-a-d-e-f-g-h-x)] do
  for i in [seq(i, i = 0 ..  m+z-8)] do
  for j in [seq(i, i = 1 ..  m+z-8-i)] do
  for k in [seq(i, i = 1 ..  m+z-7-i-j)] do
  for l in [seq(i, i = 1 ..  m+z-6-i-j-k)] do
  for o in [seq(i, i = 1 ..  m+z-5-i-j-k-l)] do
  for p in [seq(i, i = 1 ..  m+z-4-i-j-k-l-o)] do
  for q in [seq(i, i = 1 ..  m+z-3-i-j-k-l-o-p)] do
  for r in [seq(i, i = 1 ..  m+z-2-i-j-k-l-o-p-q)] do
  for s in [seq(i, i = 1 ..  m+z-1-i-j-k-l-o-p-q-r)] do
  for t in [seq(i, i = 1 ..  m+z-i-j-k-l-o-p-q-r-s)] do
  if 0 < n-a-d-e-f-g-h-x-y and 0 < m+z-i-j-k-l-o-p-q-r-s-t then
  temp := temp+Narayana(a, i+j+k)*Narayana(d, l)*Narayana(e, o)*
  Narayana(f, p)*Narayana(g, q)*Narayana(h, r)*Narayana(x, s)*
  Narayana(y, t)*Narayana(n-a-d-e-f-g-h-x-y, m+z-i-j-k-l-o-p-q-r-s-t)
  end if end do end do end do end do end do end do end do end do
  end do end do end do end do end do end do end do end do end do
  end do; temp; end proc:
```

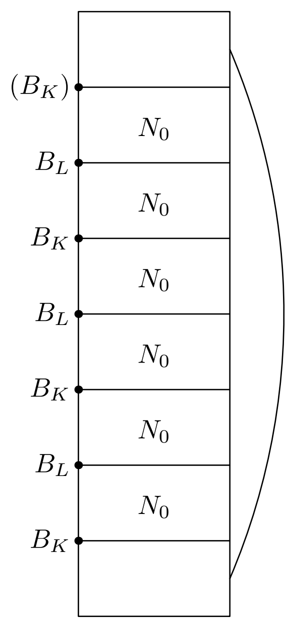
```
> CompNineCombinedTwicePlus := proc (n, m, z)
   local a, b, c, d, e, f, g, h, x, y, i, j, k, l, o, p, q, r, s, t,
temp;
   temp := 0;
   for a in [seq(i, i = 1 ..  n-7)] do
   for c in [seq(i, i = 1 ..  n-6-a)] do
   for e in [seq(i, i = 1 ..  n-5-a-c)] do
   for f in [seq(i, i = 1 ..  n-4-a-c-e)] do
   for g in [seq(i, i = 1 ..  n-3-a-c-e-f)] do
   for h in [seq(i, i = 1 ..  n-2-a-c-e-f-g)] do
   for x in [seq(i, i = 1 ..  n-1-a-c-e-f-g-h)] do
   for y in [seq(i, i = 1 ..  n-a-c-e-f-g-h-x)] do
   for i in [seq(i, i = 0 ..  m+z-8)] do
   for j in [seq(i, i = 1 ..  m+z-8-i)] do
   for k in [seq(i, i = 1 ..  m+z-7-i-j)] do
   for l in [seq(i, i = 1 ..  m+z-6-i-j-k)] do
   for o in [seq(i, i = 1 ..  m+z-5-i-j-k-l)] do
   for p in [seq(i, i = 1 ..  m+z-4-i-j-k-l-o)] do
   for q in [seq(i, i = 1 ..  m+z-3-i-j-k-l-o-p)] do
   for r in [seq(i, i = 1 ..  m+z-2-i-j-k-l-o-p-q)] do
   for s in [seq(i, i = 1 ..  m+z-1-i-j-k-l-o-p-q-r)] do
   for t in [seq(i, i = 1 ..  m+z-i-j-k-l-o-p-q-r-s)] do
   if 0 < n-a-c-e-f-g-h-x-y and 0 < m+z-i-j-k-l-o-p-q-r-s-t then
   temp := temp+Narayana(a, i+j)*Narayana(c, k+l)*Narayana(e, o)*
   Narayana(f, p)*Narayana(g, q)*Narayana(h, r)*Narayana(x, s)*
   Narayana(y, t)* Narayana(n-a-c-e-f-g-h-x-y, m+z-i-j-k-l-o-p-q-r-s-t)
   end if end do end do end do end do end do end do end do end do
   end do end do end do end do end do end do end do end do end do
   end do; temp; end proc:
```

```
> DeltaFourSum := proc (n, m)
  local temp;
  temp := 3*CompSixCombinedPlus(n, m, 3)+
  CompSixCombinedPlus(n, m, 4)+
  10*CompSevenCombinedPlus(n, m, 4)+
  CompSevenCombinedPlus(n, m, 5)+
  11*CompEightCombinedPlus(n, m, 5)+
  4*CompNineCombinedPlus(n, m, 6)+
  CompSevenCombinedThricePlus(n, m, 4)+
  3*CompSevenCombinedTwicePlus(n, m, 4)+
  2*CompEightCombinedThricePlus(n, m, 5)+
  7*CompEightCombinedTwicePlus(n, m, 5)+
  CompNineCombinedThricePlus(n, m, 6)+
  4*CompNineCombinedTwicePlus(n, m, 6)
  end proc:
> NPrimitiveDeltaFour(n, m)-DeltaFourSum(n, m);
  0
```

Appendix B

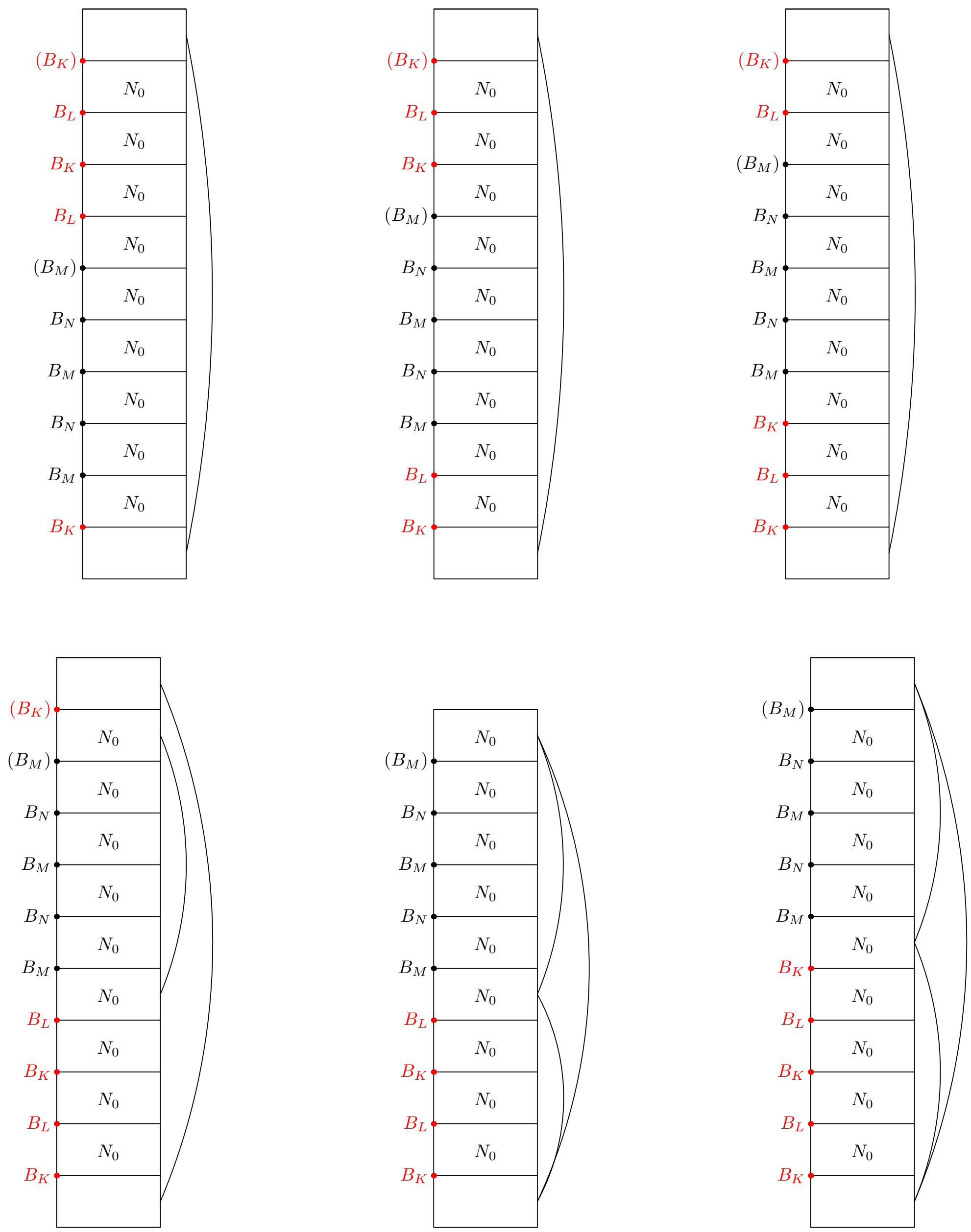
$\delta = 4$ Juggling Diagrams

B.1 Two AC balls



B.2 Three AC balls

B.3 Four AC balls



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