The Algebra and Arithmetic of Vector-Valued Modular Forms on $\Gamma_0(2)$

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Thanks Geoff for being a fantastic advisor!

Notation

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$q = e^{2\pi i\tau}$
H = complex upper half plane.
G = finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$
$\rho : G \to \mathrm{GL}_n(\mathbf{C})$ is a n-dim. complex representation
$k \in \mathbf{Z}$
If $F : H \to \mathbf{C}^n$ is a function then

$$F|_k \begin{bmatrix} a & b \\ c & d \end{bmatrix}(z) := (cz+d)^{-k} F(\frac{az+b}{cz+d})$$

Definition of a vector-valued modular form (vvmf) on a subgroup

A vector-valued modular form (vvmf) of dimension n and weight k with respect to a complex n-dimensional representation ρ of a subgroup G of $\mathrm{SL}_2(\mathbf{Z})$ is:

A holomorphic function $F : H \to \mathbf{C}^n$ such that F has a holomorphic q-series expansion at all the cusps of G and

$$F|_k\gamma = \rho(\gamma)F \quad \text{for all } \gamma \in G.$$

$M_k(\rho) := \{\text{vector-valued modular forms of weight } k \text{ for } \rho\}$.
$M(\rho) := \bigoplus_{k \in \mathbf{Z}} M_k(\rho)$
$M_k(G) := \{\text{modular forms of weight k on } G\}$.
$M(G) := \bigoplus_{k \in \mathbf{Z}} M_k(\rho)$

If $F \in M_k(\rho)$ and if $m \in M_t(G)$ then $Ft \in M_{k+t}(\rho)$. In this way, we view $M(\rho)$ as a **Z**-graded $M(G)$-module.

The transformation equation

Let $F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}$.

The transformation equation is:

$$\begin{bmatrix} F_1|_k\gamma \\ F_2|_k\gamma \\ \vdots \\ F_n|_k\gamma \end{bmatrix} = \rho(\gamma) \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}.$$

Motivation

Goal: Understand the module structure of vector-valued modular forms and use it to understand the arithmetic of vector valued and scalar valued modular forms.

Noncongruence subgroups

A subgroup of $\mathrm{SL}_2(\mathbf{Z})$ is *congruence* if membership in the subgroup is determined by congruence conditions on the entries of the matrix.

Atkin and Swinnerton-Dyer noticed examples of modular forms on noncongruence subgroups whose sequence of Fourier coefficients have unbounded denominators. One way to attack and to generalize the unbounded denominator conjecture (i.e. "Modular forms on non-congruence subgroups have unbounded denominators") is via vector-valued modular forms.

The Unbounded Denominator Conjecture for Vector Valued Modular Forms

Definition

Let α denote an algebraic number. The denominator of α is the smallest positive integer N such that $N\alpha$ is an algebraic integer. The number $N\alpha$ is the numerator of α.

Conjecture

Let ρ be a representation of a finite index subgroup G of $SL_2(\mathbf{Z})$. If $ker\ \rho$ is noncongruence then for any vector valued modular form X with respect to ρ, the sequence of the denominators of the Fourier coefficients of at least one of the component functions of X is unbounded. (Provided the Fourier coefficients are algebraic numbers.)

Main Result

Theorem

(Gottesman) Let ρ denote an irreducible representation of $\Gamma_0(2)$ of dimension two such that $\rho(T)$ has finite order, ρ is induced from a character of $\Gamma(2)$, and such that a certain mild technical hypothesis on ρ holds. Then for any vector-valued modular form V with respect to ρ, the sequence of the denominators of the Fourier series coefficients of each component function of V is unbounded provided these Fourier coefficients are algebraic numbers.

Remark: This approach should allow us to eliminate the assumption that ρ is induced from a character of $\Gamma(2)$.

If ρ is an irreducible representation of $\Gamma_0(2)$ of dimension two, if $\rho(T)$ has finite order and if a mild hypothesis on ρ holds then for each $k \in \mathbf{Z}$, there is a basis of $M_k(\rho)$ consisting of vector-valued modular forms which can be normalized so that the Fourier coefficients of their normalization are elements of a certain quadratic field $\mathbf{Q}(\sqrt{M})$, which is determined by ρ.

In this context, normalizing means scaling each of the two component functions by a complex number so that the leading Fourier coefficient of each scaled component function is equal to one.

Equivalent formulation: There is a representation ρ', which is conjugate to ρ, such that if ρ is an irreducible representation of $\Gamma_0(2)$ of dimension two, if $\rho(T)$ has finite order and if a mild hypothesis on ρ holds then for each $k \in \mathbf{Z}$, there is a basis of $M_k(\rho')$ consisting of vector-valued modular forms whose Fourier coefficients belong to a certain quadratic field $\mathbf{Q}(\sqrt{M})$, which is determined by ρ.

Theorem

(Gottesman) Assume that ρ is an irreducible representation of $\Gamma_0(2)$ of dimension two such that $\rho(T)$ has finite order, ρ is induced from a character of $\Gamma(2)$, and a mild hypothesis on ρ holds. Let p denote a sufficiently large prime number such that M is not a quadratic residue mod p. Let X denote a vector-valued modular form for ρ' whose component functions have algebraic Fourier coefficients which lie in the quadratic field $\mathbf{Q}(\sqrt{M})$. Then p divides the denominator of at least one Fourier coefficient of the first component function and of the second component function of X.

Remark: For each $k \in \mathbf{Z}$, we can always find a basis of vector-valued modular forms for $M_k(\rho')$ whose Fourier coefficients lie in the quadratic field $\mathbf{Q}(\sqrt{M})$.

Remark: The density of prime numbers p for which M is not a quadratic residue mod p is one half.

It follows that at least one half of the prime numbers divide the denominator of at least one Fourier coefficient of the first and of the second component functions of X. In particular, the sequence of the denominators of the Fourier coefficients of the first and of the second component functions of X are both unbounded.

Remark: I can prove a similar result without the assumption that ρ is induced from a character of $\Gamma(2)$. However, I have not shown yet that the set of primes in this situation are an infinite set. The difficulty lies in computing the quadratic field $Q(\sqrt{M})$.

Method of Proof:

Step 1. Prove that the module $M(\rho)$ of vector valued modular forms for a representation ρ of $\Gamma_0(2)$ is a free graded module over the ring $M(\Gamma_0(2))$, the ring of modular forms on $\Gamma_0(2)$, (Note: Not true for many other subgroups of $\mathrm{SL}_2(\mathbf{Z})$.)

Step 2. Use Step 1 and the modular derivative to compute a basis for this module when ρ is irreducible and two-dimensional. Let k_0 denote the integer for which $M_{k_0}(\rho) \neq 0$ and $M_k(\rho) = 0$ if $k < k_0$. Let F denote a nonzero element in $M_{k_0}(\rho)$. Then F and $D_{k_0}F = q\frac{d}{q}F - \frac{k_0}{12}E_2F$ form a basis for $M(\rho)$ as a $M(\Gamma_0(2))$-module. In particular, $M_{k_0}(\rho) = \mathbf{C}F$ and so F is unique up to scaling by a complex number.

Method of Proof:

Step 3. Make this basis even more explicit by solving a differential equation satisfied by a vector-valued modular form of minimal weight. The coefficients of this differential equation are modular forms on $\Gamma_0(2)$. We solve this equation by using a Hauptmodul of $\Gamma_0(2)$, the Dedekind η-function and the hypergeometric function ${}_2F_1$. In particular, we then obtain formulas for the Fourier series coefficients involve rising factorials or Pochhammer symbols

$(r)_n := r(r+1)\cdots(r+n-1) =$ product of the first n consecutive numbers starting with r.

Step 4. Apply the arithmetic of quadratic fields to show that certain sets of prime numbers never divide $(r)_n$ for any n where r is a certain type of element in a quadratic field. Then prove that a vector-valued modular form of least weight whose Fourier coefficients are algebraic numbers has unbounded denominators.

Step 5. Use the module structure of vector-valued modular forms and step 4 to show that all vector-valued modular forms whose Fourier coefficients are algebraic numbers have unbounded denominators.

Arithmetic of Quadratic Fields

Here is the lemma we use in Step 4:

Lemma

Let M denote a square-free integer. Let p denote an odd prime number for which M is not a quadratic residue mod p. Let $X \in \mathbf{Q}(\sqrt{M})$ such that $X \notin \mathbf{Q}$. Let Z denote the smallest positive integer such that ZX is an algebraic integer and let $Y := ZX$. (We think of Z as the denominator and Y as the numerator of X.) Let y and z denote the integers for which $Y = \frac{x+y\sqrt{M}}{2}$. If $p \nmid y$ then p does not divide the numerator of any element in the set $\{(X)_t : t \geq 1\}$. (i.e. p does not divide $(X)_t$ in the ring of algebraic integers.)

The proof of this lemma is elementary. This lemma allows us to show that all sufficiently large primes p for which M is not a quadratic residue mod p divide the denominator of least one Fourier coefficient of the vector-valued modular form F of minimal weight provided ρ satisfies certain conditions, including the condition that ρ is induced from a character of $\Gamma(2)$.

If we drop the condition that ρ is induced from a character of $\Gamma(2)$, we can prove a similar result but this time the primes p must also satisfy a certain congruence condition in addition to having the property that M is not a quadratic residue mod p. Next step: Compute M explicitly.

Thank you!!

If there is time, I would be happy to discuss more details.