Lattices and Vertex Operator Algebras

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Vertex Operator Algebras, Number Theory, and Related Topics
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How all began

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From gerald Mon Apr 24 18:49:10 1995
To:  gem@cats.ucsc.edu
Subject:  Santa Cruz / Babymonster twisted sector
Dear Prof. Mason,
Thank you for your detailed letter from Thursday.
It is probably the best choice for me to come to Santa Cruz in the year
95-96.  I see some projects which I can do in connection with my thesis.
The main project is the classification of all self-dual SVOA's with rank
smaller than 24.  The method to do this is developed in my thesis, but
not all theorems are proven.  So I think it is really a good idea to
stay at a place where I can discuss the problems with other people.
[...]
I must probably first finish my thesis to make an application [to the
DFG] and then they need around 4 months to decide the application, so in
principle I can come September or October.
[...]
I have not all of this proven, but probably the methods you have
developed are enough.  I hope that I have at least really constructed my
Babymonster SVOA of rank 23 1/2.
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(*edited for spelling and grammar*)

Theorem (H. & Mason 2014)

Let X be the Fano variety of the smooth cubic fourfold

$$S = \{(x_0 : \ldots : x_5) \mid x_0^3 + \cdots + x_5^3 + \lambda \cdot \widetilde{\sigma}_3(x_0, \ldots, x_5)\} \subset \mathbf{CP}^5.$$

It is a complex 4-dimensional Hyperkähler manifold of deformation type $K3^{[2]}$ with the group M_{10} as automorphism group. Here, $K3^{[2]}$ denotes the second Hilbert scheme of a $K3$ surface.

Its equivariant complex elliptic genus $\chi_{-y}(g, q, \mathcal{L}X))$ for $g \in M_{10}$ equals the second coefficient of the equivariant second quantized complex elliptic genus

$$\chi_{-y}(g; q, \mathcal{L}\exp(pX)) = \prod_{n>0,\, m\geq 0,\, \ell} \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} c_{g^k}(4nm - \ell^2)\,(p^n q^m y^\ell)^k \right),$$

where $\chi_{-y}(g, q, \mathcal{L}K3)) = \sum_{n,\, \ell \in \mathbb{Z}} c_g(4n - \ell^2) q^n y^\ell$ as formally predicted by Mathieu Moonshine.

Theorem (Borcherds (1986), Frenkel/Lepowsky/Meurman (1988))

There exists a vertex operator algebra (VOA) $V^\natural$ of central charge 24 called the **Moonshine module** *which has the* **Monster**, *the largest sporadic group, as its automorphism group.*

Definition

A VOA V of central charge c is a tuple $(V, \mathbf{1}, \omega, Y)$ consisting of a graded complex vector space $V = \bigoplus_{n=0}^{\infty} V_n$ with two elements $\mathbf{1} \in V_0$ and $\omega \in V_2$ and a linear map $Y : V \longrightarrow \mathrm{End}(V)[[z, z^{-1}]]$ satisfying several axioms motivated from CFT.

The **character** of a VOA V is the series

$$\chi_V := q^{-c/24} \left(\sum_{n=0}^{\infty} \dim V_n \cdot q^n \right).$$

One has

$$\chi_{V^\natural} = j - 744 = q^{-1}\,(1 + 196884\,q^2 + 21493760\,q^3 + \cdots)$$

where j is Klein's j-invariant, a modular function of weight zero for the modular group $\mathrm{SL}(2,\mathbf{Z})$ defined on the upper half-plane for $q = e^{2\pi i\tau}$.

Conjecture (FLM)

The Moonshine module $V^\natural$ is the unique VOA with character $j - 744$.

It is unclear if any essential progress has been made in the last 30 years.

One idea: Look at "similar" VOAs.
This leads to the notion of the **genus** of VOAs.

Theorem (Huang)

For a "nice" rational VOA V, the representation category $\mathcal{T}(V)$ has the structure of a **modular tensor category**.

Simple objects of $\mathcal{T}(V)$ are the irreducible V-modules.

$\mathcal{T}(V^\natural) = \mathbf{1}$ since the only irreducible $V^\natural$-module (up to isomorphism) is $V^\natural$ itself (C. Dong).

Theorem

If $\mathcal{T}(V) = \mathbf{1}$, the central charge c of V is an integer multiple of 8.

This follows from Zhu's result about the genus 1 correllation functions of V which implies here that χ_V is projectively invariant under the action of $\mathrm{SL}(2, \mathbf{Z})$ on the upper half-plane.

Definition

The **genus** *of a nice rational VOA V is the pair $(\mathcal{T}(V), c)$. We denote by $\mathrm{gen}(\mathcal{T}(V), c)$ the set of isomorphism classes of VOAs W with the same genus as V.*

- $\mathrm{gen}(\mathbf{1}, 8) = \{V_{E_8}\}$
- $\mathrm{gen}(\mathbf{1}, 16) = \{V_{E_8 \oplus E_8},\ V_{D_{16}^+}\}$
- $\mathrm{gen}(\mathbf{1}, 24)$, the genus of $V^\natural$: Conjecture of Schellekens (1992)

For sufficiently nice VOAs one has:

- V_1 has the structure of a reductive Lie algebra $\mathbf{g} = \bigoplus_i \mathbf{g}_i$.
- Let $\langle V_1 \rangle$ be the subVOA of V generated by V_1. $\langle V_1 \rangle$ is a fundamental integrable highest weight representation of the affine Kac-Moody algebra $\widetilde{\mathbf{g}} = \bigoplus_i \widetilde{\mathbf{g}}_i$, i.e. $\langle V_1 \rangle = \bigotimes_i V_{\widetilde{\mathbf{g}}_i, k_i}$.

Theorem (Schellekens (1992))

Let V be a VOA in the genus of $V^\natural$. Then the Lie algebra $V_1 = \bigoplus_i \mathbf{g}_i$ must be one of 71 possibilities:

- $V_1 = 0$ *realized by* $V = V^\natural$,
- $V_1 = \mathbf{t}_{24}$ *realized by* $V = V_\Lambda$,
- V_1 *is semisimple and* $\langle V_1 \rangle$ *has central charge* 24 *(69 cases).*

Furthermore, in each case, $\langle V_1 \rangle$ *and the decomposition of* V *as* $\langle V_1 \rangle$*-module is uniquely determined.*

Goals:

- Construct VOAs in all cases.
- Show uniqueness in all cases.
- Find a better description.

Theorem (1985–2018)

There **exists** *at least* 71 *self-dual VOAs of central charge* 24 *realizing all* 71 *possibilities of* $\langle V_1 \rangle$. *In all cases besides possibly for* $\langle V_1 \rangle = 0$, *these VOAs are* **unique.**

The **proof** is a joint effort of many people (Griess, Borcherds, Frenkel, Lepowsky, Meurmann, Goddard, Dolan, Montague, Carnahan, Dong, Lam, Mason, Miyamoto, Möller, Scheithauer, Shimakura, van Ekeren, Yamauchi, Xu, ...).

Mostly Case-by-case study.

In the following, I will discuss **three novel lattice descriptions** of Schellekens' list.

Definition

An even integral lattice is a free $\mathbf{Z}$*-module* L *of finite rank together with a symmetric non-degenerate bilinear form* $b: L \times L \longrightarrow \mathbf{Z}$ *such that* $b(v,v)$ *is even for all lattice vectors* v*.*

In the following the lattices are assumed to be positive definite. One sets $L^* = \{v \in L \otimes \mathbf{Q} \mid b(v,w) \in \mathbf{Z} \text{ for all } \ w \in L\}$. Then the pair $L^*/L = (L^*/L, \bar{b})$ where $\bar{b}: L^*/L \longrightarrow \mathbf{Q}/2\mathbf{Z}$ is a finite quadratic space.

Definition

The **genus** *of a lattice* L *is the pair* $(L^*/L, \mathrm{rank}\, L)$*.*
We denote by $\mathrm{gen}(L^*/L, \mathrm{rank}\, L)$ *the set of isomorphism classes of lattices* K *with the same genus as* L*.*

Remark: The so defined genus is the same as the collection of p-adic lattice $L \otimes \mathbf{Z}_p$ for $p = 2, 3, 5, \ldots, \infty$.

To an even lattice L, one can associate a VOA V_L called a lattice VOA. It has the character

$$\chi_V = \Big(\sum_{v\in L} q^{b(x,x)/2}\Big)\Big/\Big(q^{1/24}\prod_{n=0}^{\infty}(1-q^n)\Big)^{\mathrm{rank}\, L}.$$

To a finite quadratic space $A=(A,q)$, one can associate a modular tensor category $\mathcal{Q}(A)$.

This results in the commutative diagram

$$\begin{array}{ccc} \text{even integral lattice } L & \longrightarrow & \text{genus } (L^*/L, \mathrm{rank}\, L) \\ \downarrow & & \downarrow \\ \text{VOA } V_L & \longrightarrow & \text{genus } (\mathcal{T}(V_L), c) = (\mathcal{Q}(L^*/L), \mathrm{rank}\, L) \end{array}$$

where the vertical maps are injective.

$\mathcal{T}(V_L) = \mathbf{1} \;\Leftrightarrow\; L^*/L = \mathbf{0} \;\Leftrightarrow\; L^* = L \;\Leftrightarrow:\; L$ is unimodular

- $\mathrm{gen}(\mathbf{0}, 8) = \{E_8\}$
- $\mathrm{gen}(\mathbf{0}, 16) = \{E_8 \oplus E_8,\ D_{16}^+\}$ (Witt 1941)
- $\mathrm{gen}(\mathbf{0}, 24)$ contains 24 lattices (Niemeier 1973)

To describe the 24 Niemeier lattices one uses:

- $L_2 = \{v \in L \mid b(v,v) = 2\}$ is a **root system** with components of type A_i, D_i, E_i.
- Let $R = \langle L_2 \rangle$ be the sublattice of L generated by L_2 called the **root lattice** of L.
- Given a root lattice R, the unimodular lattices L having R as root lattice and satisfying $\mathrm{rank}\, R = \mathrm{rank}\, L$ are determined by the the isotropic subspaces $C = L/R \subset R^*/R$ with $|C|^2 = |R^*/R|$. We call C the **glue code**.

Theorem (Niemeier (1973))

Let L be a lattice in the genus formed by the even unimodular lattices of rank 24. Then the root lattice R must be one of 24 possibilities:

- *$R = 0$ realized by $L = \Lambda$, the Leech lattice,*
- *R has rank 24 (23 cases).*

Furthermore, in each of the $\mathrm{rank}\, R = 24$ cases, the glue code $C = L/R$ is (up to equivalence under $O(R)$) uniquely determined.

- The **Leech lattice** Λ (Leech 1967) is the unique positive-definite, even, unimodular lattice of rank 24 without roots.
- It provides the densest sphere packing in dimension 24 (Cohn, Kumar, Miller, Radchenko, Viazovska (2016)).
- Its automorphism group is the **Conway group** $\mathrm{Co}_0 = O(\Lambda)$, its group quotient $\mathrm{Co}_1 = O(\Lambda)/\{\pm 1\}$ is one of the 26 sporadic simple groups (Conway 1968).

The Niemeier lattices

	R	C	$\lvert C\rvert$	$\lvert G_1\rvert$	$\lvert G_2\rvert$	
$A1$	$D_{24,1}$	(s)	2	1	1	1
$A2$	$D_{16,1}E_{8,1}$	$(s,0)$	2	1	1	1
$A3$	$E_{8,1}^3$	–	1	1	6	1
$A4$	$A_{24,1}$	(5)	5	2	1	1
$A5$	$D_{12,1}^2$	$([s,v])$	4	1	2	1
$A6$	$A_{17,1}E_{7,1}$	$(3,1)$	6	2	1	1
$A7$	$D_{10,1}E_{7,1}^2$	$(s,1,0),\ (c,0,1)$	4	1	2	1
$A8$	$A_{15,1}D_{9,1}$	$(2,s)$	8	2	1	1
$A9$	$D_{8,1}^3$	$([s,v,v])$	8	1	6	1
$A10$	$A_{12,1}^2$	$(1,5)$	13	2	2	1
$A11$	$A_{11,1}D_{7,1}E_{6,1}$	$(1,s,1)$	12	2	1	1
$A12$	$E_{6,1}^4$	$(1,[0,1,2])$	9	2	24	1
$A13$	$A_{9,1}^2D_{6,1}$	$(2,4,0),\ (5,0,s),\ (0,5,c)$	20	2	2	1
$A14$	$D_{6,1}^4$	$(0,s,v,c),\ (1,1,1,1)$	16	1	24	1
$A15$	$A_{8,1}^3$	$([1,1,4])$	27	2	6	1
$A16$	$A_{7,1}^2D_{5,1}^2$	$(1,1,s,v),\ (1,7,v,s)$	32	2	4	1
$A17$	$A_{6,1}^4$	$(1,[2,1,6])$	49	2	12	1
$A18$	$A_{5,1}^4D_{4,1}$	$(2,[0,2,4],0),\ (3,3,0,0,s),\ (3,0,3,0,v),\ (3,0,0,3,c)$	72	2	24	1
$A19$	$D_{4,1}^6$	$(s,s,s,s,s,s),\ (0,[0,v,c,c,v])$	64	3	720	1
$A20$	$A_{4,1}^6$	$(1,[0,1,4,4,1])$	125	2	120	1
$A21$	$A_{3,1}^8$	$(3,[2,0,0,1,0,1,1])$	256	2	1344	1
$A22$	$A_{2,1}^{12}$	$(2,[1,1,2,1,1,1,2,2,2,1,2])$	729	2	$\lvert M_{12}\rvert$	1
$A23$	$A_{1,1}^{24}$	$(1,[0,0,0,0,0,1,0,1,0,0,1,1,0,0,1,1,0,1,0,1,1,1,1])$	4096	1	$\lvert M_{24}\rvert$	1
$A24$	$U(1)^{24}=\mathbf{R}^{24}/\Lambda$	–	1	1	Co_0	1

There are at least four different methods to show the theorem:

1. Computation of (enough parts of) the neighborhood graph (Niemeier 1973).
2. Classify root lattices and possible glue codes (Venkov 1978).
3. Mass formula (Conway/Sloane $<=$ 1982).
4. Deep holes of Leech lattice and Lorentian picture (Conway/Parker/Sloane 1982 and Borcherds' thesis)

Theorem (H. 2016)

The possible 69 *non-abelian affine Kac-Moody structures of VOAs* V *in the genus of* $V^\natural$ *are in a* **natural bijective correspondence** *with the equivalence classes of cyclic subgroups* Z *of positive type of the glue codes of the twenty-three Niemeier lattices.*
These are precisely the cases for Z *where* $R(Z)$ *is a root lattice for a Lie algebra* $\mathbf{g}$ *with* $\dim \mathbf{g} > 24$*.*

The simple current code D *for the affine Kac-Moody algebra of* V *is induced from* C *and* Z *under the above the bijective correspondence for all possible* 69 *cases of non-abelian affine Kac-Moody structures.*

To N and $Z = \langle c \rangle \subset C$, $c = (c_1, \ldots, c_r)$, we assign the **orbit lattice** $N(Z)$ as follows. For each component S of $R = R_1 \oplus \cdots \oplus R_r$ use $S(0) = S$ and

S	A_n	D_{2k}	D_{2k}	D_{2k+1}	D_{2k+1}	E_6	E_7
d	i	s	v	s	v	1	1
$S(d)$	$\sqrt{\frac{n+1}{i}}A_{i-1}$	B_k	C_{2k-2}	B_{k-1}	C_{2k-1}	G_2	F_4
type	$1^{-1}(\frac{n+1}{i})^i$	2^k	$1^{2k-4}2^2$	$1^{-1}2^{k-1}4^1$	$1^{2k-3}2^2$	3^2	1.2^3

Let ℓ be the order of c in C, m_i be the order of c_i in S_i^*/S_i. Set $R(Z) = \sqrt{k_1}S(c_1) \oplus \cdots \oplus \sqrt{k_r}S(c_r)$, where the scaling factors k_i are given by $k_i = \frac{\ell}{m_i}\alpha$ with $\alpha = 1$, if $\mathrm{norm}(Z) = 4$ and $\alpha = 2$ if $\mathrm{norm}(Z) = 6$ and $\mathrm{norm}(Z)$ denotes the largest minimal norm of a coset $R + v$ for all cosets with $(R + v)/R \subset Z$.

The **orbit lattice** $N(Z)$ itself is $R(Z)$ extended by the orbit glue code $C(Z) \subset R(Z)^*/R(Z)$ defined by the component-wise projection of the elements of C onto the glue groups of $S(d)$.

Lattice	generator	order	norms	$R(Z)$	$\dim \mathbf{g}$	No.in[S]	type
$D_{16}E_8$	$(0,0)$	1	0	$D_{16,1}E_{8,1}$	744	69	1^{24}
	$(s,0)$	2	$0,4$	$B_{8,1}E_{8,2}$	384	62	$1^8 2^8$
A_3^8	(0^8)	1	0	$A_{3,1}^8$	120	30	1^{24}
	$(0^4,2^4)$	2	$0,4$	$A_{3,2}^4 A_{1,1}^4$	72	16	$1^8 2^8$
	(2^8)	2	$0,8$	$A_{1,*}^8$	24	–	$[2^{16}/1^8]$
	$(0^3,2^1,(\pm 1)^4)$	4	$0,4$	$A_{3,4}^3 A_{1,2}$	48	7	$1^4 2^2 4^4$
	$(0^1,2^3,(\pm 1)^4)$	4	$0,4,6$	$A_{3,8}A_{1,4}^3$	24	–	$[2^6 4^4/1^4]$
	$((\pm 1)^8)$	4	$0,6,8$	–	0	–	$[4^8/1^8]$
A_1^{24}	(0^{24})	1	0	$A_{1,1}^{24}$	72	15	1^{24}
	$(0^{16},1^8)$	2	$0,4$	$A_{1,2}^{16}$	48	5	$1^8 2^8$
	$(0^{12},1^{12})$	2	$0,6$	$A_{1,4}^{12}$	36	2	2^{12}
	$(0^8,1^{16})$	2	$0,8$	$A_{1,*}^8$	24	–	$[2^{16}/1^8]$
	(1^{24})	2	$0,12$	–	0	–	$[2^{24}/1^{24}]$

Theorem (H.)

Nice VOAs V with $\mathcal{T}(V) = \mathbf{1}$ of central charge c are up to isomorphisms in one-to-one correspondence to quadruples $(\mathcal{G},\, L,\, W,\, [i])$ consisting of the following data:

(a) *A genus $\mathcal{G} = ((A, q),\, k)$ of positive definite lattices of rank k.*

(b) *An isometry class of lattices L in the genus $\mathcal{G}$.*

(c) *An isomorphism class of vertex operator algebras W of central charge $c - k$ with $\mathcal{T}(W) = \mathcal{Q}(A, -q)$ and $W_1 = 0$.*

(d) *An equivalence class $[i]$ of anti-isometries $i : (A_L, q_L) \longrightarrow (A_W, q_W)$ under the double coset action of $\overline{O}(L) \times i^* \overline{\mathrm{Aut}}(W)$ on $O(A, q)$.*

Proof: Set $V = \bigoplus_{a \in A_L} V_{L+a} \otimes W(i(a))$.

Name	type	rank	(A,q)	h	$O(A,q)$	$\#V$
A	1^{24}	24	1	24	1	24
B	$1^8 2^8$	16	2_{II}^{+10}	17	$O_{10}^{+}(2).2$	17
C	$1^6 3^6$	12	3^{-8}	6	$O_8^{-}(3).2^2$	6
D	2^{12}	12	$2_{II}^{-10} 4_{II}^{-2}$	2	$2.2^{20}.[6].O_{10}^{-}(2).2$	9
E	$1^4 2^2 4^4$	10	$2_2^{+2} 4_{II}^{+6}$	5	$2^{14}.2^6.2.2.O_7(2)$	5
F	$1^4 5^4$	8	5^{+6}	2	$2.O_6^{+}(5).2.2$	2
G	$1^2 2^2 3^2 6^2$	8	$2_{II}^{+6} 3^{-6}$	2	$2.O_6^{+}(3).O_6^{+}(2).[2^2]$	2
H	$1^3 7^3$	6	7^{-5}	1	$2.O_5(7).2$	1
I	$1^2 2.4.8^2$	6	$2_5^{+1} 4_1^{+1} 8_{II}^{+4}$	1	$[2^{21}].O_5(2)'.2$	1
J	$2^3 6^3$	6	$2_{II}^{+4} 4_{II}^{-2} 3^{+5}$	1	$[2^{14} 3^3].O_5(3).2$	2
K	$2^2 10^2$	4	$2_{II}^{-2} 4_{II}^{-2} 5^{+4}$	1	$[2^8 3^2].(O_3(5) \times O_3(5)).[2^2]$	1
L	$2^{24}/1^{24}$	0	1	1	1	1

Given $N(Z)$, interpret Z as an element g of $O(\Lambda) \cong \mathrm{Co}_0$ by using type as frame shape. Set

$$W = W(g) := V_{(\Lambda^g)^\perp}^{\langle \hat{g} \rangle}.$$

Conjecture (H.)

The fixed-point VOAs W are VOAs with a modular tensor category $\mathcal{T}(W)$ isomorphic to the modular tensor category $\mathcal{Q}(A, -q)$ where (A, q) is the discriminant space of the lattice $N(Z)$.

The subgroup $\overline{\mathrm{Aut}}(W) < O(A, -q) \cong O(A, q)$ induced by the action of $\mathrm{Aut}(W)$ on the set of irreducible modules has the following property:

Let K be an orbit lattice $N(Z)$ with discriminant space (A, q). Then there is only one orbit under the double coset action of $O(K) \times \overline{\mathrm{Aut}}(W)$ on $\overline{O}(A, q)$ for all genera besides the two genera D and J.

Remarks:

- Conjecture is true for g of frame shape 1^{24} (trivial), $1^8 2^8$ (Griess, $V^+_{E_8(2)}$), $1^6 3^6$ (Chen, Lam and Shimakura, V^τ_K) and 2^{12} (Scheithauer and H., $V^+_{D_{12}(2)}$); aditional cases: Lam, Shimakura 2017/2018.
- At least the modular tensor category structure of $\mathcal{T}(W(g))$ is correct for the additional cases $1^4 5^4$, $1^2 2^2 3^2 6^2$ and $1^3 7^3$ (S. Möller 2016).
- Uniqueness of $W(g)$ can be proven under certain general VOA assumptions by using uniqueness of V_Λ.
- The uniqueness statement of Schellekens **requires** that $\overline{\mathrm{Aut}}(W(g)) < O(A, -q)$ have enough elements.

The two pictures joined

Let $M = \Lambda \oplus II_{1,1} \cong II_{25,1}$. Then $O(M) \cong \Lambda.O(\Lambda).W_\Lambda$ (Conway).

Theorem (H. 2016)

The 69 orbit lattices $N(Z)$ are in one-to-one correspondence to $O(M)$-orbits of pairs (g, v) where g is an element in $O(M)$ arising from an element in $O(\Lambda)$ with a frame shape as in cases A to K, v is an isotropic vector of M where the Niemeier lattice $v^\perp/\mathbf{Z}v$ is not the Leech lattice and g fixes v. Here, we let $O(M)$ act on the first component of the pair (g, v) by conjugation and use the natural $O(M)$-action on the second.

Reference: G. Höhn: *On the genus of the Moonshine Module. Preprint 2016.*

Ingredients of the Orbifold picture

- Dong/Nagatomo (1998): $G := \mathrm{Aut}(V_N) \cong G_0 * \widehat{O}(N)$ where G_0 is the connected Lie group associated to $R < N < R^*$.
- van Ekeren/Möller/Scheithauer (2015): Orbifolding of self-dual VOAs by cyclic groups $\langle g \rangle$ works.
- Lie theory: Description of conjugacy classes of elements g in G for a given class $\bar{g}$ in $G/G_0 = \pi_0(G)$.
- Dong/Lepowsky (1996): Structure of g-twisted modules of V_N.
- van Ekeren/Möller/Scheithauer (2017): Dimension formula for $(V_N^{\mathrm{orb}(g)})_1$.
- Höhn/Mason (2015): The 290 fixed-point sublattices of the Leech lattice.
- Höhn/Möller (2018): $100 + 6217$ lines of MAGMA code.

Definition

An element g of a compact Lie group $G = \mathrm{Aut}(V_N)$ is called **good** if it has finite order n, its projection $\bar{g} \in \pi_0(G)$ has order n, is of the form $g = e^{2\pi i v}\bar{g}$ where $v \in (\frac{1}{n}(N^{\bar{g}})^*)/(N^{\bar{g}})^*$ has order n and the fixed-point VOA $V_N^{\langle g \rangle}$ has $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$ fusion.

Theorem (H. & Möller (last week))

There are exactly 226 conjugacy classes of good elements in the Lie groups arising from the 24 Niemeier lattice VOAs. The corresponding orbifolds consist of all VOAs from Schellekens list besides the Moonshine module.

The proof consists of two parts:

Kansas: 24 hours of computations with MAGMA.

```
Message from syslogd@euler at Jun 5 10:07:54 ...
kernel:[1538342.464350] Uhhuh.  NMI received for unknown reason 2c on
CPU 9.
kernel:[1538342.464352] Do you have a strange power saving mode enabled?
kernel:[1538342.464352] Dazed and confused, but trying to continue
```

Final Output: list of 226 quadruples $(N, \bar{g}, v, V_N^{\mathrm{orb}(g)})$, where N is a Niemeier lattice, $\bar{g} \in O(N)$, $v \in N \otimes \mathbf{Q}$ and $V_N^{\mathrm{orb}(g)}$ is the **predicted** orbifold.

New Jersey: 1 hour of computations with MAGMA.

Output: In all 226 cases the orbifold exists and the predicted orbifold **is** the actual orbifold.

Good automorphisms of Niemeier lattice VOAs

Case	A	B	C	D	E	F	G	H	I	J	K
order $\langle g\rangle$	1	2	3	2	4	5	6	7	8	6	10
$O(\Lambda)$ class of $\Lambda^{\langle g\rangle}$ in [HM]	1	2	4	5	9	20	18	52	55	63	149
lift type	1	1	1	2	1	1	1	1	1	2	2
# class number of $\Lambda^{\langle g\rangle}$	24	24	10	3	8	5	8	3	4	3	2
$[O(\Lambda'_{\langle g\rangle}/\Lambda_{\langle g\rangle}):\overline{O}(\Lambda_{\langle g\rangle})]$	1	1	1	>1	1	1	1	1	1	>1	>1
# of classes $\langle g\rangle$	24	24	10	15	8	5	8	3	4	5	4
# good $O(N^{\langle g\rangle})$-orbits	24	76	27	15	31	9	26	3	6	5	4
# good stabilizer orbits	24	76	27	15	31	9	26	3	6	6	4
# of different orbifold VOAs	24	17	6	9	5	2	2	1	1	2	1

Genus D — Orbifolding by 2^{12} type elements

	N	$A4$	$A10$	$A15$	$A17$	$A20$	$A22$	$A5$	$A14$	$A19$	$A21$	$A23$	$A24$	$A15$	$A20$	$A22$
	N_2	A_{24}	A_{12}^2	A_8^3	A_6^4	A_4^6	A_2^{12}	D_{12}^2	D_6^4	D_4^6	A_3^8	A_1^{24}	Λ	A_8^3	A_4^6	A_2^{12}
	N^g															
$D1a$	$B_{12,2}$	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.
$D1b$	$B_{6,2}^2$	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.
$D1c$	$B_{4,2}^3$	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.
$D1d$	$B_{3,2}^4$	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.
$D1e$	$B_{2,2}^6$	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.
$D1f$	$A_{1,4}^{12}$	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.
$D2a$	$A_{8,2}F_{4,2}$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
$D2b$	$C_{4,2}A_{4,2}^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
$D2c$	$D_{4,4}A_{2,2}^4$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

Genus E — Orbifolding by $1^4 2^2 4^4$ type elements

	N	$A13$	$A18$	$A19$	$A20$	$A21$	$A22$	$A23$	$A24$
	N_2	$A_9^2 D_6$	$A_5^4 D_4$	D_4^6	A_4^6	A_3^8	A_2^{12}	A_1^{24}	Λ
	N^g								
$E1$	$C_{7,2} A_{3,1}$	1	2	.	1	1	.	.	.
$E2$	$E_{6,4} B_{2,1} A_{2,1}$	1	.	1	2	1	1	.	.
$E3?$	$A_{7,4} A_{1,1}^3$	.	1	.	1	1	2	1	.
$E4?$	$D_{5,4} C_{3,2} A_{1,1}^2$	.	1	1	1	3	1	2	.
$E5$	$A_{3,4}^3 A_{1,2}$	.	.	.	.	1	1	2	1