On the Tensor Structure of Modules for Compact Orbifold Vertex Operator Algebras

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Vertex operator algebras and automorphisms

Vertex operator algebra $(V, Y, \mathbf{1}, \omega)$:

- V is a graded vector space $V = \bigoplus_{n \geq N} V_{(n)}$, $V_{(n)}$ finite dimensional.
- $Y(\cdot, x) : V \to (\mathrm{End}\, V)[[x, x^{-1}]]$ given by $v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v(n)\, x^{-n-1}$.
- $Y(\mathbf{1}, x) = 1_V$, $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2}$ gives action of the Virasoro algebra.
- Jacobi identity, etc.

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If G is a topological group of automorphisms of V, G acts continuously on V if it acts continuously on the finite-dimensional $V_{(n)}$, with respect to the usual Euclidean topology.

The setting

- V is a simple vertex operator algebra
- G is a compact Lie group of automorphisms acting continuously on V.
- $V^G = \{v \in V \,|\, g \cdot v = v \text{ for all } g \in G\}$ is the fixed-point vertex operator subalgebra, also called the orbifold subalgebra.

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Theorem (Dong-Li-Mason, 1996)

V is semisimple as a $G \times V^G$-module. Specifically,

$$V = \bigoplus_{\chi \in \widehat{G}} M_\chi \otimes V_\chi$$

where χ runs over all irreducible characters of G, M_χ is the corresponding irreducible G-module, and the V_χ are (non-zero) distinct irreducible V^G-modules.

An example

Take $Q = \mathbb{Z}\alpha$ the $\mathfrak{sl}_2$ root lattice ($\langle \alpha, \alpha \rangle = 2$) and $V = V_Q$ the lattice vertex operator algebra.

- The weight-1 subspace $V_{(1)}$ is a copy of $\mathfrak{sl}_2$. The zero modes $a(0) = \mathrm{Res}_x\, Y(a, x)$ for $a \in V_{(1)}$ give an action of $\mathfrak{sl}_2$ on V.
- Exponentiate: the operators $\exp a(0)$ for $a \in V_{(1)}$ generate a (faithful) action of the complex Lie group $PSL(2, \mathbb{C})$ on V. This restricts to a faithful action of the compact subgroup $SO(3)$.

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- Exponentiate: the operators $\exp a(0)$ for $a \in V_{(1)}$ generate a (faithful) action of the complex Lie group $PSL(2, \mathbb{C})$ on V. This restricts to a faithful action of the compact subgroup $SO(3)$.
- (Dong-Griess, 1998) The orbifold subalgebra $V_Q^{SO(3)}$ is generated by the conformal vector ω, and is the the central charge $c = 1$ Virasoro vertex operator algebra $L(1, 0)$.
- As a $SO(3) \times L(1, 0)$-module,

$$V_Q = \bigoplus_{n=0}^{\infty} M_{2n} \otimes L(1, n^2),$$

where M_{2n} is the $(2n+1)$-dimensional $SO(3)$-module and $L(1, n^2)$ is the irreducible $L(1, 0)$-module with lowest conformal weight n^2.

The categories

Given V and G, let $\mathrm{Rep}\, G$ be the category of finite-dimensional G-modules and let $\mathcal{C}_V$ be the abelian category of V^G-modules generated by the V_χ for $\chi \in \widehat{G}$.

- Dong, Li, and Mason's theorem implies that the correspondence $M_\chi \mapsto V_\chi$ determines an equivalence of abelian categories $\mathrm{Rep}\, G \to \mathcal{C}_V$.

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- Dong, Li, and Mason's theorem implies that the correspondence $M_\chi \mapsto V_\chi$ determines an equivalence of abelian categories $\mathrm{Rep}\, G \to \mathcal{C}_V$.
- But $\mathrm{Rep}\, G$ is much more than an abelian category: it is a rigid symmetric tensor category: tensor product $M \otimes N$, tensor unit $\mathbb{C}$, associativity isomorphisms, symmetry $\mathcal{R}_{M,N} : M \otimes N \to N \otimes M$, duals M^*, $i_M : \mathbb{C} \to M \otimes M^*$, $e_M : M^* \otimes M \to \mathbb{C}$.

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- Suppose we know that V^G has a braided tensor category $\mathcal{C}$ of modules, as constructed by Huang-Lepowsky-(Zhang), that contains $\mathcal{C}_V$. Does the equivalence $\mathrm{Rep}\, G \to \mathcal{C}_V$ of abelian categories also preserve tensor structure?

The first theorem

Theorem (M. 2018)

Assume V^G *has a braided tensor category of modules* $\mathcal{C}$ *that contains* $\mathcal{C}_V$*. Then there is a braided tensor functor* $\Phi : \mathrm{Rep}\, G \to \mathcal{C}$ *such that* $V_\chi \cong \Phi(M_\chi^*)$*. In particular,* Φ *gives an equivalence of symmetric tensor categories between its image* $\mathcal{C}_V$ *and* $\mathrm{Rep}\, G$*.*

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- This result was obtained by Kirillov in a categorical setting, *but* he assumed $\mathcal{C}$ was rigid. However, rigidity is difficult to prove and often not known for braided tensor categories of vertex operator algebra modules.
- Rigidity of modules in $\mathcal{C}_V$, and thus essentially the equivalence, was obtained in the compact abelian case by Carnahan-Miyamoto and Creutzig-Kanade-Linshaw-Ridout.

The second theorem

When does V^G actually have a braided tensor category of modules that includes the V_χ?

One case: If V is C_2-cofinite and G is finite solvable, then V^G is also C_2-cofinite (Miyamoto, 2015), and hence the full category of (grading-restricted, generalized/logarithmic) V^G-modules has braided tensor category structure (Huang, 2009). But this won't apply to general compact groups.

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Theorem (M. 2018)

If the fusion rules for intertwining operators among modules in $\mathcal{C}_V$ agree with dimensions of spaces of G-module intertwiners, $\mathrm{Hom}_{\mathbb{C}}(M_\chi \otimes M_\psi, M_\rho)$, then $\mathcal{C}_V$ itself has braided tensor category structure as given by Huang-Lepowsky.

Idea of proof: In this setting, associativity for intertwining operators in $\mathcal{C}_V$ follows from associativity of the vertex operator for V.

The functor, after Kirillov

How to get a V^G-module from a G-module naturally:

- Given M in $\mathrm{Rep}\, G$, $M \otimes V$ is a $G \times V$-module (not necessarily an object of $\mathcal{C}_V$ unless G is finite).
- The G-invariants $(M \otimes V)^G$ form a V^G-module.
- Define $\Phi(M) = (M \otimes V)^G$ and for $f : M_1 \to M_2$, define $\Phi(f) = (f \otimes 1_V)|_{(M_1 \otimes V)^G}$.

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Calculate $\Phi(M_\chi^*)$:

$$\Phi(M_\chi^*) = \bigoplus_{\psi \in \widehat{G}} (M_\chi^* \otimes (M_\psi \otimes V_\psi))^G = \bigoplus_{\psi \in \widehat{G}} (M_\chi^* \otimes M_\psi)^G \otimes V_\psi = V_\chi.$$

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Specific isomorphism $\varphi_\chi : V_\chi \to \Phi(M_\chi^*)$:

$$\varphi_\chi(v_\chi) = \sum_i m'_{\chi,i} \otimes (m_{\chi,i} \otimes v_\chi),$$

using a basis of M_χ and the dual basis of M_χ^* (this is just the coevaluation in $\mathrm{Rep}\, G$).

The natural transformation

For Φ to be a tensor functor, we need a natural isomorphism

$$J_{M_1,M_2} : \Phi(M_1) \boxtimes \Phi(M_2) \to \Phi(M_1 \otimes M_2).$$

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To construct this, we use the following intertwining operator $\mathcal{Y}_{M_1,M_2}$ of type $\binom{\Phi(M_1 \otimes M_2)}{\Phi(M_1)\,\Phi(M_2)}$:

$$(M_1 \otimes V)^G \otimes (M_2 \otimes V)^G \hookrightarrow ((M_1 \otimes M_2) \otimes (V \otimes V))^G$$
$$\xrightarrow{1_{M_1 \otimes M_2} \otimes Y(\cdot,x)\cdot} ((M_1 \otimes M_2) \otimes V)^G[[x, x^{-1}]].$$

By the universal property of tensor products of vertex operator algebra modules, $\mathcal{Y}_{M_1,M_2}$ induces a unique homomorphism J_{M_1,M_2} of the desired type.

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But is J_{M_1,M_2} an isomorphism?

Surjectivity

Showing J is surjective amounts to showing that if

$$V_\chi \boxtimes V_\psi \xrightarrow{\varphi_\chi \boxtimes \varphi_\psi} \Phi(M_\chi^*) \boxtimes \Phi(M_\psi^*) \xrightarrow{J_{M_\chi^*, M_\psi^*}} \Phi(M_\chi^* \otimes M_\psi^*) \xrightarrow{\Phi(f)} \Phi(M_\rho^*)$$

equals 0, then $f = 0$.

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Equivalently, we need to show that if the intertwining operator

$$\mathcal{Y}_f = \Phi(f) \circ \mathcal{Y}_{M_\chi^*, M_\psi^*} \circ (\varphi_\chi \otimes \varphi_\psi) = 0,$$

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By the definitions,

$$0 = \mathcal{Y}_f(v_\chi, x)v_\psi = \sum_{i,j} f(m'_{\chi,i} \otimes m'_{\psi,j}) \otimes Y(m_{\chi,i} \otimes v_\chi, x)(m_{\psi,j} \otimes v_\psi)$$

Surjectivity, continued

Write $f(m'_{\chi,i} \otimes m'_{\psi,j}) = \sum_k \langle m_{\rho,k}, f(m'_{\chi,i} \otimes m'_{\psi,j})\rangle m'_{\rho,k}$ (this is the rigidity of $\mathrm{Rep}\, G$). Then

$$\sum_{i,j,k} \langle m_{\rho,k}, f(m'_{\chi,i} \otimes m'_{\psi,j})\rangle m'_{\rho,k} \otimes Y(m_{\chi,i} \otimes v_\chi, x)(m_{\psi,j} \otimes v_\psi) = 0$$

$$\rightarrow \sum_j Y\left(\sum_i \langle m_{\rho,k}, f(m'_{\chi,i} \otimes m'_{\psi,j})\rangle (m_{\chi,i} \otimes v_\chi), x\right)(m_{\psi,j} \otimes v_\psi) = 0 \;\; \forall k$$

$$\rightarrow \sum_i \langle m_{\rho,k}, f(m'_{\chi,i} \otimes m'_{\psi,j})\rangle (m_{\chi,i} \otimes v_\chi) = 0 \;\; \forall j, k$$

$$\rightarrow \langle m_{\rho,k}, f(m'_{\chi,i} \otimes m'_{\psi,j})\rangle = 0 \;\; \forall i, j, k \rightarrow f = 0.$$

The second implication follows from a lemma of Dong and Mason: if V is a simple vertex operator algebra and $\sum_i Y(u_i, x)v_i = 0$ where the v_i are linearly independent, then the u_i must be 0.

Application to V_Q and $SO(3)$

Recall the action of $SO(3)$ on the $\mathfrak{sl}_2$-root lattice vertex operator algebra V_Q, with Virasoro orbifold subalgebra $L(1,0)$, where $V_Q = \bigoplus_{n=0}^{\infty} M_{2n} \otimes L(1, n^2)$.

- By (Milas, 2002), the fusion rules for intertwining operators among the $L(1, n^2)$ agree with the dimensions of the spaces of intertwiners for the $\mathfrak{sl}_2/SO(3)$-modules M_{2n}. So by the second theorem, $\mathcal{C}_{V_Q}$ is a braided tensor category.
- Then by the first theorem, $\mathcal{C}_{V_Q}$ is a symmetric (and rigid!) tensor category equivalent to $\mathrm{Rep}\, SO(3)$.

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What about the other finite-dimensional $\mathfrak{sl}_2$-modules? They are contained in $V_{\frac{1}{2}\alpha+Q}$, the non-trivial irreducible V_Q-module. The abelian intertwining algebra $V_P = V_Q \oplus V_{\frac{1}{2}\alpha+Q}$ admits a faithful action of $SU(2)$.
Then generalizing the first and second theorems to abelian intertwining algebras shows that $\mathcal{C}_{V_P}$ is equivalent to $\mathrm{Rep}\, SU(2)$, except that the symmetry isomorphisms in $\mathrm{Rep}\, SU(2)$ have to be modified.