A Uniform Construction of the 71 Holomorphic VOAs of $c = 24$ from the Leech Lattice

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(joint work with Nils Scheithauer)

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Holomorphic VOAs of Small Central Charge

Proposition (Consequence of [Zhu96])

Let V be a strongly rational, holomorphic VOA. Then the central charge c of V is in $8\mathbb{Z}_{\geq 0}$.

- $c = 8$: V_{E_8}, $c = 16$: $V_{E_8^2}, V_{D_{16}^+}$ (only lattice theories) [DM04]

Theorem ([Sch93, DM04, EMS15])

Let V be a strongly rational, holomorphic VOA of central charge $c = 24$. *Then the Lie algebra* V_1 *is isomorphic to one of the 71 Lie algebras on Schellekens' list (*$V^\natural$*, 24 Niemeier lattice theories, etc. with* $\mathrm{ch}_V(\tau) = j(\tau) - 744 + \dim(V_1)$*).*

- $c = 32$: already more than 1 160 000 000 lattice theories

Classification

- Orbifold constructions give all 71 cases on Schellekens' list. [FLM88, DGM90, Don93, DGM96, Lam11, LS12, LS15, Miy13, SS16, EMS15, Mö16, LS16b, LS16a, LL16]

Theorem (Classification I)

There is a strongly rational, holomorphic VOA V of central charge $c = 24$ with Lie algebra V_1 if and only if V_1 is isomorphic to one of the 71 Lie algebras on Schellekens' list.

Conjecture (Classification II)

There are up to isomorphism exactly 71 strongly rational, holomorphic VOAs V of central charge $c = 24$.

- Uniqueness proved for all cases except $V^\natural$. [DM04, LS16c, KLL16, LS15, LL16, EMS17, LS17, LS18]

Schellekens' List

Rk. \ D.	0	24	36	48	60	72	84	96	108	120	132	144	156	168	192	216	240	264	288	300	312	336	360	384	408	456	552	624	744	1128
0	0																													
4			$C_{4,10}$																											
6			$A_{2,6}D_{4,12}$	$A_{6,7}$, $A_{1,2}D_{5,8}$	$A_{2,2}F_{4,6}$																									
8				$A_{1,2}A_{5,6}B_{2,3}$, $A_{4,5}^2$		$A_1^2D_{6,5}$, $A_1C_{5,3}G_{2,2}$																								
10				$A_{1,2}A_{3,4}^3$		$A_1^3A_{7,4}$, $A_1^2C_{3,2}D_{5,4}$		$A_2B_2E_{6,4}$		$A_3C_{7,2}$																				
12			$A_{1,4}^{12}$	$A_{2,3}^6$	$A_{2,2}^4D_{4,4}$, $B_{2,2}^6$	$A_1^3A_{5,3}D_{4,3}$	$A_{4,2}^2C_{4,2}$, $B_{3,2}^4$	$A_2^2A_{8,3}$	$B_{4,2}^3$	$A_3D_{7,3}G_2$, $E_{6,3}G_2^3$	$A_{8,2}F_{4,2}$		$B_{6,2}^2$	$A_5E_{7,3}$						$B_{12,2}$										
16				$A_{1,2}^{16}$		$A_1^4A_{3,2}^4$		$A_2^2A_{5,2}^2B_2$, $B_2^4D_{4,2}^2$		$A_3^2D_{5,2}^2$, $A_3A_{7,2}C_3^2$		$A_4A_{9,2}B_3$, $B_3^2C_4D_{6,2}$, C_4^4		$A_5C_5E_{6,2}$	$B_4^2D_{8,2}$, $B_4C_6^2$	$A_7D_{9,2}$	$B_5E_{7,2}F_4$, $C_8F_4^2$		B_6C_{10}					$B_8E_{8,2}$						
24		$\mathbb{C}^{24}$				A_1^{24}		A_2^{12}		A_3^8		A_4^6		$A_5^4D_4$, D_4^6	A_6^4	$A_7^2D_5^2$	A_8^3	$A_9^2D_6$, D_6^4			$A_{11}D_7E_6$, E_6^4	A_{12}^2	D_8^3		$A_{15}D_9$	$A_{17}E_7$, $D_{10}E_7^2$	D_{12}^2	A_{24}	E_8^3, $D_{16}E_8$	D_{24}

Cyclic Orbifold Construction

- Let V be a strongly rational, holomorphic VOA and let $G = \langle g \rangle$ with $g \in \operatorname{Aut}(V)$ of order n and type $n\{0\}$, i.e. $\rho(V(g)) \in (1/n)\mathbb{Z}$.
- The fusion algebra of V^G is the group algebra of the finite quadratic space $\mathbb{Z}_n \times \mathbb{Z}_n$ with $q((i,j)) = ij/n + \mathbb{Z}$.
- Assume that V^G satisfies the positivity condition. Then the direct sum of irreducible V^G-modules

$$V^{\operatorname{orb}(g)} := \bigoplus_{i \in \mathbb{Z}_n} V(g^i)^G$$

is again a strongly rational, holomorphic VOA.

Dimension Formula I

Conjecture (Dimension Formula I)

In the orbifold situation with $c = 24$:

$$\sum_{d|n} \frac{\phi((d,n/d))}{(d,n/d)} \left(24 + \frac{n}{d} \dim(V_1^{g}) - \dim(V_1^{\mathrm{orb}(g^d)}) \right) = 24 + R$$

with

$$R = \frac{24}{\phi(n)} \sum_{k=1}^{n-1} \sum_{\substack{i,j \in \mathbb{Z}_n \\ ij = k \pmod{n}}} d_{i,j,k} \dim(W_{k/n}^{(i,j)})$$

and $d_{i,j,k} \in \mathbb{Z}_{>0}$.

- Proved if n prime, $g(\Gamma_0(n)\backslash\mathbb{H}^*) = 0$ [EMS17] or $n = 14, \ldots$

Dimension Formula II

Conjecture (Dimension Formula II)

In the orbifold situation with $c = 24$:

$$\dim(V_1^{\mathrm{orb}(g)}) = 24 + \sum_{d|n} c_d \dim(V_1^{g^d}) - \tilde{R}$$

with the c_d determined by $\sum_{d|n}(t,d)c_d = n/t$ for all $t \mid n$ and

$$\tilde{R} = \frac{24}{\phi(n)} \sum_{k=1}^{n-1} \sum_{\substack{i,j\in\mathbb{Z}_n \\ ij=k \pmod{n}}} \tilde{d}_{i,j,k} \dim(W_{k/n}^{(i,j)})$$

with $\tilde{d}_{i,j,k} \in \mathbb{Z}_{\geq 0}$. Moreover, $\tilde{R} \geq 24$ if n prime and $g(\Gamma_0(n)) \neq 0$.

- Proved if n prime, $g(\Gamma_0(n)\backslash\mathbb{H}^*) = 0$ or $n = 14, \ldots$

Extremal Orbifolds

Corollary (Upper Bound)

In the orbifold situation with $c = 24$:

$$\dim(V_1^{\mathrm{orb}(g)}) \leq 24 + \sum_{d|n} c_d \dim(V_1^{g^d}).$$

Definition

We call g *extremal* if equality holds, i.e. if $\tilde{R} = 0$.

- This is the case for example if $\rho(V(g^i)) \geq 1$ for all $i \in \mathbb{Z}_n \setminus \{0\}$ (equivalence for n prime and $g(\Gamma_0(n)) = 0$).
- No extremal orbifolds for n prime and $g(\Gamma_0(n)) \neq 0$.

Deep-Hole Construction

- Construction of the 23 Niemeier lattices $N(\Phi)$ with $\Phi \neq \emptyset$ from the deep holes of the Leech lattice Λ [CS99].
- Inner automorphism of V_Λ of the form $g = \mathrm{e}^{-(2\pi\mathrm{i})h_0}$ for $h \in \mathfrak{h} = \Lambda \otimes_\mathbb{Z} \mathbb{C} \cong (V_\Lambda)_1$ a deep hole (of order $n = h^\vee$, the dual Coxeter number of Φ, i.e. $nh \in \Lambda$).
- Then

$$\rho(V_\Lambda(g)) = \min_{\alpha \in \Lambda + h} \langle \alpha, \alpha \rangle / 2 = 1$$

  and g is extremal, i.e. $\dim((V_\Lambda^{\mathrm{orb}(g)})_1) = 24 + 24n$.
- Indeed, $V_\Lambda^{\mathrm{orb}(g)} \cong V_{N(\Phi)}$ where Φ is the root system from the deep-hole construction.
- (Note that $V_\Lambda^g = V_{\Lambda^h}$ with $\Lambda^h := \{\alpha \in \Lambda \mid \langle \alpha, h \rangle \in \mathbb{Z}\}$.)

Generalised Deep-Hole Construction

- All finite-order automorphisms in $\mathrm{Aut}(V_\Lambda)$ are conjugate to $g = \hat{\nu}\mathrm{e}^{-(2\pi\mathrm{i})h_0}$ for $\nu \in \mathrm{Aut}(\Lambda) \cong \mathrm{Co}_0$ and $h \in \pi_\nu(\mathfrak{h})$ [DN99].
- The dimension formula yields for ν with cycle shape $\prod_{t|m} t^{b_t}$

$$\dim(V_1^{\mathrm{orb}(g)}) = 24 + n\sum_{t|m} b_t/t - \tilde{R}.$$

- Take ν from list [Hö17] of 11 (12) conjugacy classes $\langle\nu\rangle$ in $O(\Lambda)$ arising from certain cyclic subgroups of the glue codes of the 23 Niemeier lattices with roots.
- Search for $h \in \pi_\nu(\mathfrak{h})$ such that $\mathrm{rk}((V_\Lambda^g)_1) = \mathrm{rk}((V_\Lambda^{\mathrm{orb}(g)})_1)$ and g is extremal, i.e. has large conformal weights

$$\rho(V_\Lambda(g^i)) = \rho_{\nu^i} + \min_{\alpha\in\pi_{\nu^i}(\Lambda)+ih} \langle\alpha,\alpha\rangle/2.$$

- Expect that $V_\Lambda^{\mathrm{orb}(g)} \cong U$ for all U on Schellekens' list (observe that $n = (h_j^\vee/k_j)\,\mathrm{ord}(\hat{\nu})$ for simple components of U_1).
- (Note $V_{\Lambda^{\nu,h}} \subseteq V_\Lambda^g$ with $\Lambda^{\nu,h} := \{\alpha \in \Lambda \mid \nu\alpha = \alpha, \langle\alpha, h\rangle \in \mathbb{Z}\}$.)

Automorphisms

	cycl. shp.	orders n	#	orb. rk.	orb. dim.
A	1^{24}	$1, 2, \ldots, 25, 30, 46$	24	24	$24 + 24n$
B	$1^8 2^8$	$2, 4, \ldots, 18, 22, 30$	17	16	$24 + 12n$
C	$1^6 3^6$	$3, 6, 9, 12, 18$	6	12	$24 + 8n$
D	2^{12}	$2, 6, 10, \ldots, 22, 46$	9	12	$24 + 6n$
E	$1^4 2^2 4^4$	$4, 8, 12, 16$	5	10	$24 + 6n$
F	$1^4 5^4$	$5, 10$	2	8	$24 + (24/5)n$
G	$1^2 2^2 3^2 6^2$	$6, 12$	2	8	$24 + 4n$
H	$1^3 7^3$	7	1	6	$24 + (24/7)n$
I	$1^2 2^1 4^1 8^2$	8	1	6	$24 + 3n$
J	$2^3 6^3$	$6, 18$	2	6	$24 + 2n$
K	$2^2 10^2$	10	1	4	$24 + (6/5)n$
L	$1^{-24} 2^{24}$	2	1	0	$24 - 12n$

Results and Outlook

- Have candidate automorphism g for each of the 71 cases.
- Status: Proof of orbifold construction from the Leech lattice for 63 of the 71 cases.
- Application: Uniform proof of the uniqueness conjecture via inverse orbifolds.
- Related project with Gerald Höhn: Another uniform construction of Schellekens' list with "same-order lifts" of outer automorphisms of the 23 Niemeier lattices with roots (again all extremal).

Thank you for your attention!

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