Modular linear differential equations in general form

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VERTEX OPERATOR ALGEBRAS, NUMBER THEORY, AND RELATED TOPICS
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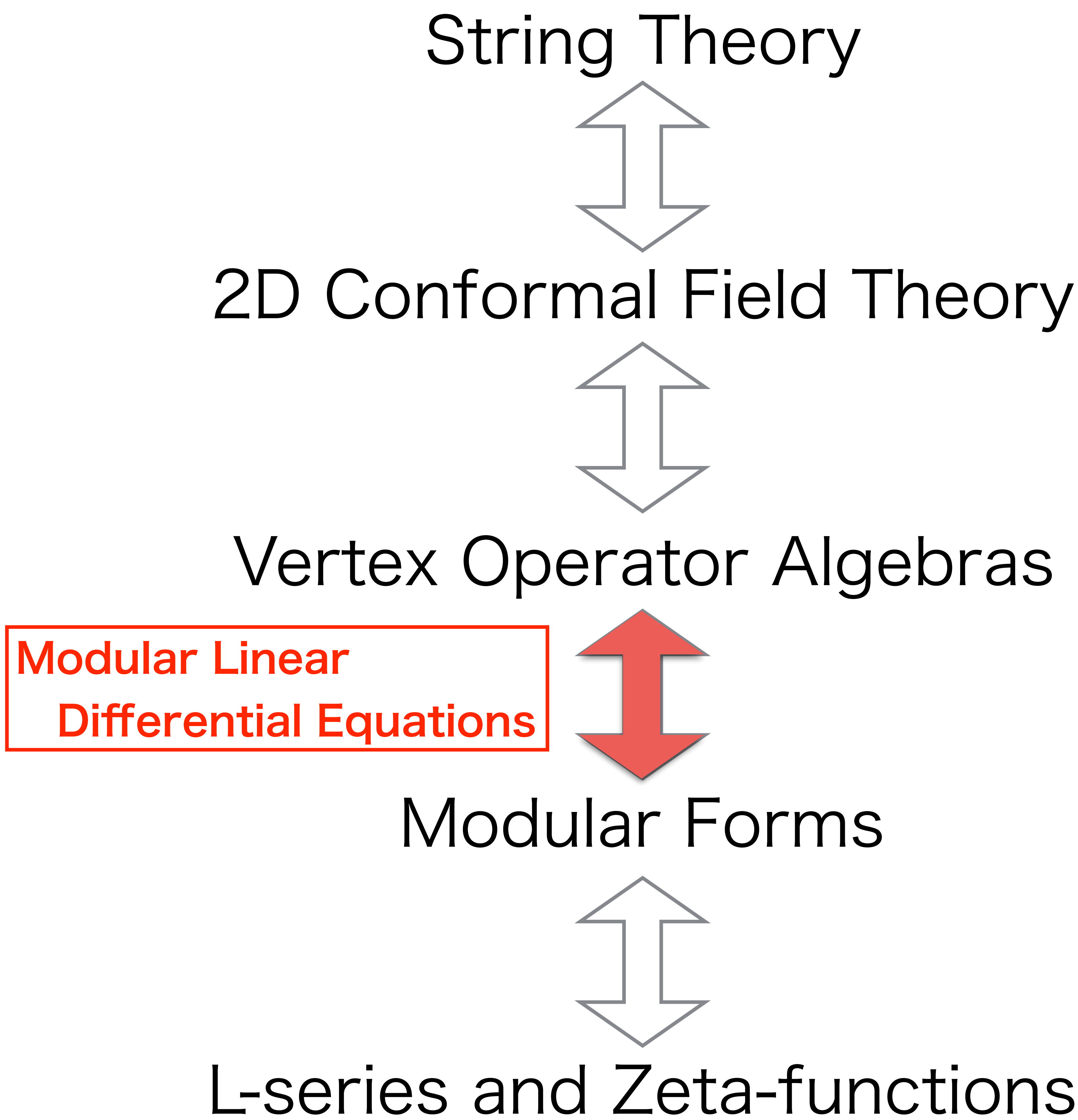
String Theory
2D Conformal Field Theory
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Contents

Introduction and motivations

Modular Linear Differential Equation(MLDE)

Definition(MLDE) Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$: $\mathrm{Vol}(\Gamma\backslash\mathbb{H}^*) < +\infty$.

For a fixed $k\,(\in \mathbb{Q})$, a (monic) ordinary linear differentail equation on the complex upper-half plane $\mathbb{H}$

$$f^{(n)}(\tau) + a_1(\tau) f^{(n-1)}(\tau) + \cdots + a_n(\tau) f(\tau) = 0$$

$$\left(\tau \in \mathbb{H},\ f' = \frac{df}{d(2\pi\sqrt{-1}\tau)} = q\frac{df}{dq},\ q = \exp(2\pi\sqrt{-1}\tau)\right)$$

is a modular linear differential equation of weight k on Γ if the space of solutions is invariant under the slash action $\big|_k\gamma \quad (\forall\gamma \in \Gamma)$.

i.e. $f(\tau)$: sol. $\Longrightarrow$ $f|_k\gamma := (c\tau+d)^{-k} f\left(\dfrac{a\tau+b}{c\tau+d}\right)$: sol. $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

We assume $a_i(\tau)$'s are holomorphic on $\mathbb{H}^* = \mathbb{H} \cup \{\text{cusps}\}$.

Modular Linear Differential Equation(MLDE)

Kaneko-Zagier equation $(\Gamma = \mathrm{SL}_2(\mathbb{Z}))$

$$f''(\tau) - \frac{k+1}{6} E_2(\tau) f'(\tau) + \frac{k(k+1)}{12} E_2'(\tau) f(\tau) = 0$$

where $E_k(\tau) = 1 - c_k \sum_{n=1}^{\infty} \Big(\sum_{d \mid n} d^{k-1} \Big) q^n \quad (c_2 = 24,\ c_4 = -240,\ c_6 = 504)$

$$\cdot\, k = 4 \quad \Longrightarrow \quad E_4(\tau),\ E_4(\tau) \int_{\tau}^{\sqrt{-1}\infty} \frac{\eta(\tau)^{20}}{E_4(\tau)^2} \frac{d\tau}{2\pi\sqrt{-1}}$$

modular　　mixed mock modular　　$\left(\eta(\tau) = q^{1/24} \prod_{n>0} (1 - q^n)\right)$

$$\cdot\, k = 5 \quad \Longrightarrow \quad E_4'(\tau) = \frac{E_2(\tau) E_4(\tau) - E_6(\tau)}{3},$$

quasimodular

Modular Linear Differential Equation(MLDE)

Kaneko-Zagier equation $(\Gamma = \mathrm{SL}_2(\mathbb{Z}))$

$$f''(\tau) - \frac{k+1}{6} E_2(\tau) f'(\tau) + \frac{k(k+1)}{12} E_2'(\tau) f(\tau) \;=\; 0$$

where $E_k(\tau) = 1 - c_k \sum_{n=1}^{\infty} \Big(\sum_{d \,|\, n} d^{k-1} \Big) q^n \quad (c_2 = 24,\ c_4 = -240,\ c_6 = 504)$

$\cdot\ \vartheta_{k+2} \circ \vartheta_k(f) = \dfrac{k(k+2)}{144} E_4 f\ :$ another expression

$\cdot$ the Serre-derivative $\vartheta_k(f) := f' - \dfrac{k}{12} E_2 f\ :\ M_k(\mathrm{SL}_2(\mathbb{Z})) \to M_{k+2}(\mathrm{SL}_2(\mathbb{Z}))$

$\cdot$ for $k \not\equiv 2 \pmod 3$, $E_4^{-1} \vartheta_{k+2} \circ \vartheta_2 \in \mathrm{End}\big(M_k(\mathrm{SL}_2(\mathbb{Z}))\big)$

$\cdot$ for $k = p - 1\,(p:$ prime), eigenfunctions of $E_4^{-1} \vartheta_{k+2} \circ \vartheta_k(f)$

give *supersingular j-polynomials.*

Modular Linear Differential Equation(MLDE)

Kaneko-Zagier equation $(\Gamma = \mathrm{SL}_2(\mathbb{Z}))$

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) \;=\; 0$$

$$g = f/\eta^{2k}$$

$$g''(\tau) - \frac{1}{6}E_2(\tau)g'(\tau) - \frac{k(k+2)}{144}E_4(\tau)g(\tau) \;=\; 0$$

MMS-Classification in 2D conformal field theory(VOA)

[Mathur-Mukhi-Sen (1988), Kaneko-Nagatomo-S. (2013)]

Modular Linear Differential Equation(MLDE)

For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the well-known "Serre"-form of MLDEs is

$$\vartheta_k^{(n)}(f) + \widetilde{b}_2 \vartheta_k^{(n-2)}(f) + \cdots + \widetilde{b}_{n-1} \vartheta_k(f) + \widetilde{b}_n f \ = \ 0 \,,$$

where $\vartheta_k^{(n)} = \vartheta_{k+2(n-1)} \circ \cdots \circ \vartheta_k$, $\widetilde{b}_i$: a modular form of wt. $2i$ on Γ.

· By $\vartheta_k\big(f|_k\gamma\big) = \vartheta_k(f)|_{k+2}\gamma$, the above eq. is a MLDE.

· This MLDE is used to construct the basis of VVMF by Prof. Mason.

$\Longrightarrow$ We want to know a general form of MLDEs in the standard forms.

$$f^{(n)}(\tau) + a_1(\tau) f^{(n-1)}(\tau) + \cdots + a_n(\tau) f(\tau) \ = \ 0$$

Aim in this talk

- Find the transformation property of coefficient of modular linear differential equations written in the standard form

$$f^{(n)}(\tau) + a_1(\tau) f^{(n-1)}(\tau) + \cdots + a_n(\tau) f(\tau) = 0$$

- Rewrite the modular linear differential equation in yet another form using certain "arithmetic" differential operators
  in both non-cocompact($\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$) and cocompact cases.

Main results

(non-cocompact case)

The general form of MLDEs(non-cocompact)

Suppose that eq. $(\sharp)_k^\Gamma$ is a MLDE of wt. $k\,(\in \mathbb{Q})$ on

$\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$: non-cocomact gp.

$$f^{(n)}(\tau) + a_1(\tau) f^{(n-1)}(\tau) + \cdots + a_n(\tau) f(\tau) = 0 \qquad (\sharp)_k^\Gamma$$

We assume such a ϕ exists:

$$\phi(\gamma\tau) = (c\tau+d)^2\phi(\tau) + \left(\frac{c}{2\pi\sqrt{-1}}\right)(c\tau+d) \quad (\forall\gamma\in\Gamma).$$

We introduce two differential operators:

Rankin-Cohen bracket: $[\,f, g\,]_{(n)}^{(k,\ell)} = \sum_{i=0}^{n} (-1)^i \binom{n+k-1}{n-i}\binom{n+\ell-1}{i} f^{(i)}\, g^{(n-i)}.$

The higher-Serre operator: $\Theta_k^{(n)}(f) = f^{(n)} - (k+n-1)[\,\phi,\ f\,]_{(n-1)}^{(2,k)}.$

The general form of MLDEs(non-cocompact)

Theorem A

Suppose $(\sharp)_k^\Gamma$ is a MLDE of wt. k on $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$: non-cocompact gp..

1) If we put $g_2(\tau) = a_1(\tau) + n(k+n-1)\phi(\tau)$,

$$g_{2m}(\tau) = \sum_{i=0}^{m-1} \binom{n-m+i}{i} \binom{k+n-m+i-1}{i} \binom{2m-2}{i}^{-1} a_{m-i}^{(i)}(\tau)$$

for $2 \leq m \leq n$,

then $g_{2m}(\tau)$ $(1 \leq m \leq n)$ is a modular form of wt. $2m$ on Γ,

and $(\sharp)_k^\Gamma$ can be written as $\Theta_k^{(n)}(f) + \sum_{i=1}^{n} \binom{n+i-1}{2i-1}^{-1} [\, g_{2i}, f\,]_{(n-i)}^{(2i,k)} = 0\,.$

2) Conversely, let g_{2m} be modular forms of wt. $2m$ on Γ.

Set $a_m(\tau) =(-1)^m \binom{n}{m} \binom{k+n-2}{m-1} (k+n-1)\phi^{(m-1)}(\tau)$

$$+ \sum_{j=1}^{m} (-1)^{m+j} \binom{n+j-1}{2j-1}^{-1} \binom{n+j-1}{m+j-1} \binom{k+n-j-1}{m-j} g_{2j}^{(m-j)}(\tau)$$

for $1 \leq m \leq n$.

Then $(\sharp)_k^\Gamma$ becomes a MLDE of wt. k on Γ.

The key to prove

(the transformation property of coefficients)

The transformation property of coefficients

Suppose that eq. $(\sharp)_k^\Gamma$ is a MLDE of wt. $k\,(\in \mathbb{Q})$ on

$\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$: non-cocomact gp.

$$f^{(n)}(\tau) + a_1(\tau) f^{(n-1)}(\tau) + \cdots + a_n(\tau) f(\tau) \;=\; 0 \qquad (\sharp)_k^\Gamma$$

By using followings :

$\cdot$ Substituting $f|_k\gamma\ (\gamma \in \Gamma)$ into f in $(\sharp)_k^\Gamma$.

$$\cdot \left(q\frac{d}{dq}\right)^n \Big(f|_k\gamma(\tau)\Big) = \sum_{i=0}^{n} \binom{n}{i} (-k-n+1)_i \left(\frac{c}{2\pi\sqrt{-1}}\right)^i (c\tau+d)^{-(k+2n-i)} f^{(n-i)}(\gamma\tau)$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ if $n>0$, $(\alpha)_n = 1$ if $n=0$.

$\cdot$ Comparing $f^{(n)}(\gamma\tau) + \sum_{i=1}^{n} a_i(\gamma\tau) f^{(n-i)}(\gamma\tau) \;=\; 0.$

The transformation property of coefficients

Proposition

Coefficients of the MLDE $(\sharp)_k^\Gamma$ satisfy the following transformation law:

$$a_m(\gamma\tau) = \sum_{i=0}^{m} \binom{n-m+i}{i} \left(-\frac{c}{2\pi\sqrt{-1}}\right)^i (k+n-m)_i (c\tau+d)^{2m-i} a_{m-i}(\tau)$$

for $1 \leq m \leq n$ and any $\gamma \in \Gamma$.

For $m=1$, $a_1(\gamma\tau) = (c\tau+d)^2 a_1(\tau) - n(k+n-1)\left(\frac{c}{2\pi\sqrt{-1}}\right)(c\tau+d)$.

More general, $a_m(\tau)$ has "quasi modularity" of depth m and weight $2m$.

For $k=0$, $a_n(\tau)$ of a MLDE $(\sharp)_0^\Gamma$ have the modularlity of wt. $2n$.

The structure of the space of quasimodular forms

· The space of modular forms of weight k on Γ:

$$M_k(\Gamma) = \{f : \text{hol. on } \mathbb{H}^*,\ f|_k\gamma(\tau) = f(\tau)\ (\gamma \in \Gamma)\}$$

· The space of quasimodular forms of weight k and depth r on Γ:

$$QM_k^{(r)}(\Gamma) = \left\{f : \text{hol. on } \mathbb{H}^*,\ f|_k\gamma(\tau) = \sum_{i=0}^{r} f_i(\tau)\Big(\frac{c}{c\tau+d}\Big)^i\ ({}^\exists f_i, \gamma \in \Gamma)\right\}$$

"quasimodularity"

· The following is hold:

$$M_k(\Gamma) = QM_k^{(0)}(\Gamma) \subset QM_k^{(1)}(\Gamma) \subset \cdots \subset QM_k^{(r)}(\Gamma) \qquad (r \le k/2)$$

$$DQM_k^{(i)}(\Gamma) \subset QM_{k+2}^{(i+1)}(\Gamma) \qquad \left(D = \frac{d}{d(2\pi\sqrt{-1}\tau)}\right)$$

The structure of the space of quasimodular forms

Theorem[Kaneko-Zagier(1995)]

For $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$: non-cocompact gp. of finite index,

$$QM_k^{([k/2])}(\Gamma) = \bigoplus_{i=0}^{[k/2]} D^i\big(M_{k-2i}(\Gamma)\big) \oplus \mathbb{C}D^{(\frac{k-2}{2})}(E_2)$$

where $\quad E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n.$

Any quasimodular form has a unique representation as a sum of derivatives of MFs and QMFs.

$\Longrightarrow \quad$ characterize $a_m(\tau)$ by a sum of derivatives of $a_i(\tau)$'s .

Derivatives of coefficients

Lemma

For any $s \geq 1$, the derivative of coefficients of $(\sharp)_k^\Gamma$ is given by

$$a_m^{(s)}(\gamma\tau) = \sum_{i=0}^{m} \binom{n-m+i}{i} (k+n-m)_i \times \sum_{j=0}^{s} \binom{s}{j} \Big(-\frac{c}{2\pi\sqrt{-1}} \Big)^{i+j} (i-2m-s+1)_j (c\tau+d)^{2(m+s)-i-j} a_{m-i}^{(s-j)}(\tau)$$

$$(1 \leq m \leq n)$$

$\because$) induction on s and a previous Proposition.

By using this lemma,

we show the modularity of a sum of derivatives of $a_i(\tau)$'s.

Modularity

Theorem B

Let $a_i(\tau)$'s be coefficients of the MLDE $(\sharp)_k^\Gamma$.

For $2 \leq \forall m \leq n$, we put

$$g_{2m}(\tau) = \sum_{i=0}^{m-1} \binom{n-m+i}{i} \binom{k+n-m+i-1}{i} \binom{2m-2}{i}^{-1} a_{m-i}^{(i)}(\tau)$$

We have $$g_{2m}(\gamma\tau) = (c\tau+d)^{2m} g_{2m}(\tau) \qquad (\forall \gamma \in \Gamma).$$

$\because$) The lemma of $a_{m-i}^{(s)}(\tau)$ of a previous slide, and

the Chu-Vandermonde identity ${}_2F_1(-n,b;c;1) = (c-b)_n/(c)_n$.

Modularity

By Theorem(Kaneko-Zagier), a hol. func. $\phi(\tau)$ exists s.t.

$$\phi(\gamma\tau) = (c\tau + d)^2\phi(\tau) + \Big(\frac{c}{2\pi\sqrt{-1}}\Big)(c\tau + d) \quad (\forall\gamma \in \Gamma).$$

Let $\Phi(\tau)$ be a modular form of weight ℓ on Γ.
For example, $\Phi'(\tau)/\{\ell\Phi(\tau)\}$ satisfies the above transformation-law.
If $\Phi'(\tau)/\{\ell\Phi(\tau)\}$ is holomorphic, we can take $\phi(\tau) = \Phi'(\tau)/\{\ell\Phi(\tau)\}$.

Recall $a_1(\gamma\tau) = (c\tau + d)^2 a_1(\tau) - n(k + n - 1)\Big(\frac{c}{2\pi\sqrt{-1}}\Big)(c\tau + d)$.

⇒ Setting $g_2(\tau) := a_1(\tau) + n(k + n - 1)\phi(\tau)$,

$$g_2(\gamma\tau) = (c\tau + d)^2 g_2(\tau) \quad (\gamma \in \Gamma).$$

Modularity

Similarly for $2 \leq m \leq n$, we set

$$g_{2m}(\tau) = \sum_{i=0}^{m-1} \binom{n-m+i}{i} \binom{k+n-m+i-1}{i} \binom{2m-2}{i}^{-1} a_{m-i}^{(i)}(\tau)$$

By Theorem B, we have $g_{2m}|_{2m}\gamma(\tau) = g_{2m}(\tau)$ $(1 \leq m \leq n,\ \gamma \in \Gamma)$.

→ Thus $g_{2m}(\tau)$ is a modular form of weight $2m$ on Γ.

Example

For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, we have

$\Phi(\tau) = \eta(\tau)^{24}, \phi(\tau) = E_2(\tau)/12, g_2(\tau) = 0$ and $g_{2m}(\tau) \in M_{2m}(\Gamma)$.

In general, the choice of $\phi(\tau)$ and $g_2(\tau)$ is not unique.

Modularity

Proposition

Let $\phi(\tau)$ be a quasimodular form of weight 2 and depth 1 on Γ:

$$\phi(\gamma\tau) = (c\tau+d)^2\phi(\tau) + \left(\frac{c}{2\pi\sqrt{-1}}\right)(c\tau+d) \quad (\forall\gamma\in\Gamma).$$

Let $g_{2m}(\tau)\,(1\le m\le n)$ be a modular form of weight $2m$ on Γ.

Then $a_m(\tau)$'s of a MLDE $(\sharp)_k^\Gamma$ can be written as

$$\begin{aligned} a_m(\tau) =& (-1)^m\binom{n}{m}\binom{k+n-2}{m-1}(k+n-1)\phi^{(m-1)}(\tau) \\ &+\sum_{j=1}^{m}(-1)^{m+j}\binom{n+j-1}{2j-1}^{-1}\binom{n+j-1}{m+j-1}\binom{k+n-j-1}{m-j}g_{2j}^{(m-j)}(\tau) \end{aligned}$$

$\because)$ by using the definition of $\phi(\tau)$ and $g_{2m}(\tau)$ of a previous slide.

Modularity

Now we recall the "Rankin-Cohen bracket"

$$[\,f, g\,]_{(n)}^{(k,\ell)} = \sum_{i=0}^{n} (-1)^i \binom{n+k-1}{n-i} \binom{n+\ell-1}{i} f^{(i)}\, g^{(n-i)}.$$

$\cdot$ Bi-linear differential operator on $M_k(\Gamma) \otimes M_\ell(\Gamma) \to M_{k+\ell+2n}(\Gamma)$.

$\cdot$ It is also on $QM_k^{(r)}(\Gamma) \otimes QM_\ell^{(s)}(\Gamma) \to QM_{k+\ell+2n}^{(r+s)}(\Gamma)$

by $[\,f,\, g]_{(n)}^{(k-r,\ell-s)}$ (Choie-Lee[2010]).

We also recall the "higher-Serre operator"

$$\Theta_k^{(n)}(f) = f^{(n)} - (k+n-1)[\,\phi,\ f]_{(n-1)}^{(2,k)}.$$

$\cdot$ For $n=1$, $\Theta_k^{(1)}(f) = \vartheta_k(f)$: the Serre-operator.

$\cdot$ Differential operator on $M_k(\Gamma) \to M_{k+2n}(\Gamma)$.

$\cdot$ It is also on $QM_k^{(r)}(\Gamma) \to QM_{k+2n}^{(r)}(\Gamma)$ by $\Theta_{k-r}^{(n)}(f)$.

(Kaneko-Koike[2006] for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, S. [2018] for $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$)

The general form of MLDEs

Algorithm $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$: non-cocomact gp.

Take $\phi(\tau)$ a quasimodular form of weight 2 and depth 1 on Γ:

$$\phi(\gamma\tau) = (c\tau+d)^2\phi(\tau) + \left(\frac{c}{2\pi\sqrt{-1}}\right)(c\tau+d) \quad (\forall\gamma\in\Gamma).$$

and $g_{2m}(\tau)\,(1 \le m \le n)$ modular forms of weight $2m$ on Γ.

$$\Rightarrow \quad \Theta_k^{(n)}(f) + \sum_{i=1}^{n} \binom{n+i-1}{2i-1}^{-1} [g_{2i}\,,\, f]_{(n-i)}^{(2i,k)} = 0$$

is a MLDE of weight k on Γ.

The general form of MLDEs

Example 1

For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $n = 2$, we have

$$\phi(\tau) = E_2(\tau)/12,\ g_2(\tau) = 0 \text{ and } g_4(\tau) = \alpha E_4(\tau)\ (\alpha \in \mathbb{C}).$$

Then the MLDE $(\sharp)_k^{\Gamma}$ has the form

$$f'' - \frac{k+1}{6} E_2 f' + \left(\frac{k(k+1)}{12} E_2' + \alpha E_4 \right) f = 0\,.$$

If this MLDE has q-series solution s.t. $f = 1 + O(q)$,

we have $\alpha = 0$, whose MLDE is Kaneko-Zagier equation.

The general form of MLDEs

Example 2

For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $n = 3$, we have

$\phi(\tau) = E_2(\tau)/12$, $g_2(\tau) = 0$, $g_4(\tau) = \alpha E_4$ and $g_6(\tau) = \beta E_6$ $(\alpha, \beta \in \mathbb{C})$.

Then the MLDE $(\sharp)_k^\Gamma$ has the form

$$f''' - \frac{k+2}{4} E_2 f'' + \left\{ \frac{(k+1)(k+2)}{4} E_2' + \alpha E_4 \right\} f' - \left\{ \frac{k(k+1)(k+2)}{24} E_2'' + \frac{k\alpha}{4} E_4' - \beta E_6 \right\} f = 0\,.$$

Putting $\alpha = k/4$ and $\beta = 0$, $(\sharp)_k^\Gamma$ has solutions $\theta_i(\tau)^{2k}$ $(i = 2, 3, 0)$,

$$\theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2}, \qquad \theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}, \qquad \theta_0(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2}\,.$$

The general form of MLDEs

Example 3 For $\Gamma = \Gamma_0(2)$ and $n = 2$,

$$M_*(\Gamma_0(2)) = \mathbb{C}[H_2,\ \Delta_2].$$

$H_2(\tau) = 2E_2(2\tau) - E_2(\tau)$, $\Delta_2(\tau) = \{\theta_2(\tau)/2\}^8$.

The choice of $\phi(\tau)$ is not unique. $\because)$ $QM_2^{(1)}(\Gamma_0(2)) = \mathbb{C}E_2 \bigoplus \mathbb{C}H_2$.

If $\Phi(\tau) = \Delta_2(\tau)$, then $\phi(\tau) = \{E_2(\tau) + 2H_2(\tau)\}/12$.

If $\Phi(\tau) = \Delta_2(\tau)(H_2(\tau)^2 - 64\Delta_2(\tau))$, then $\phi(\tau) = \{2E_2(\tau) + H_2(\tau)\}/24$.

Then the MLDE $(\sharp)_k^\Gamma$ has the form

$$f''(\tau) - \big(2(k+1)\phi(\tau) - g_2(\tau)\big) f'(\tau) + \Big(k(k+1)\phi'(\tau) - \frac{k}{2} g_2'(\tau) + g_4(\tau)\Big) f = 0$$

$$g_2(\tau) = C_1 H_2(\tau),\ g_4(\tau) = C_2 H_2(\tau)^2 + C_3 \Delta_2(\tau)\ (C_i \in \mathbb{C}).$$

Putting $\phi(\tau) = \{2E_2(\tau) + H_2(\tau)\}/24$ for $C_i = 0$ $(i = 1, 2, 3)$,

we have Kaneko-Zagier equation for the Fricke gp. of level 2.

Main results

(cocompact case)

The ring structure of (cocompact)quasimodular forms

Theorem[N. O. Azaiez(2005)]

Γ : cocomapct group (which has no cusps.)

$$QM_k^{([k/2])}(\Gamma) = \bigoplus_{i=0}^{[k/2]} D^i\bigl(M_{k-2i}(\Gamma)\bigr),$$

In particular, $QM_2^{(1)}(\Gamma) = M_2(\Gamma)$.

・Any quasimodular forms has a representation as a sum of derivatives of MFs.

・There are no hol. quasimodular forms of wt.2 and depth1.

The $a_1(\tau)$ of the MLDE $(\sharp)_k^{\Gamma}$ is a meromorphic QMF of wt.2 and depth1.

The ring structure of (cocompact)quasimodular forms

Theorem[N. O. Azaiez(2005)]

Γ : cocomapct group (which has no cusps.)

$$QM_k^{([k/2])}(\Gamma) = \bigoplus_{i=0}^{[k/2]} D^i\big(M_{k-2i}(\Gamma)\big),$$

In particular, $QM_2^{(1)}(\Gamma) = M_2(\Gamma)$.

- Any quasimodular forms has a representation as a sum of derivatives of MFs.
- There are no hol. quasimodular forms of wt.2 and depth1.

$\Longrightarrow$ We have to multiply a MF $K(\tau)$ of wt. ξ to $a_m(\tau)$'s s.t. $K(\tau)a_1(\tau)$ is holomorphic.

The MLDEs for cocompact case

We now set a (non-monic)MLDE $(\flat)_k^\Gamma$:

$$b_0(\tau)f^{(n)}(\tau) + b_1(\tau)f^{(n-1)}(\tau) + \cdots + b_n(\tau)f(\tau) \;=\; 0 \qquad (\flat)_k^\Gamma.$$

$$\big(b_0(\tau) = K(\tau),\ b_i(\tau) = K(\tau)a_i(\tau)\big)$$

By similar discussions to the MLDE $(\sharp)_k^\Gamma$, we have :

Proposition

Coefficients of the MLDE $(\flat)_k^\Gamma$ satisfy the following transformation law:

$$b_m(\gamma\tau) = \sum_{i=0}^{m} \binom{n-m+i}{i} \Big(-\frac{c}{2\pi\sqrt{-1}}\Big)^i (k+n-m)_i (c\tau+d)^{\xi+2m-i} b_{m-i}(\tau)$$

for $1 \le m \le n$ and any $\gamma \in \Gamma$.

The general form of MLDEs(non-cocompact)

Theorem A'

Suppose $(\flat)_k^\Gamma$ is a MLDE of wt. k on $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$: cocompact gp..

1) If we put for $2 \le m \le n$,

$$p_{2m}(\tau) = \sum_{i=0}^{m} \binom{n-m+i}{i} \binom{k+n-m+i-1}{i} \binom{\xi+2m-2}{i}^{-1} b_{m-i}^{(i)}(\tau),$$

then $p_{2m}(\tau)$ $(1 \le m \le n)$ is a modular form of wt. $2m+\xi$ on Γ,

and $(\flat)_k^\Gamma$ can be written as $\displaystyle\sum_{i=0}^{n} \binom{n+\xi-i-1}{n-i} [\, p_{2i} \,, f \,]_{(n-i)}^{(2i+\xi,k)} = 0\,.$

2) Conversely, let p_{2m} be a modular form of wt. $2m+\xi$ on Γ.

Set $$b_m(\tau) = \sum_{j=0}^{m} (-1)^{m+j} \binom{n-j}{m-j} \binom{k+n-j-1}{m-j} \binom{\xi+m-1+j}{m-j}^{-1} p_{2j}^{(m-j)}(\tau)$$

for $1 \le m \le n$.

Then $(\flat)_k^\Gamma$ becomes a MLDE of wt. k on Γ.

The general form of MLDEs

Algorithm

Γ : cocompact gp.

Take $p_{2m}(\tau)\,(1 \le m \le n)$ modular forms of weight $2m+\xi\,(\xi > 0)$ on Γ.

$$\Rightarrow \quad \sum_{i=0}^{n} \binom{n+\xi-i-1}{n-i} [\, p_{2i} \, , f \,]^{(2i+\xi,k)}_{(n-i)} = 0 \,.$$

is a MLDE of weight k on Γ.

Summary(non-cocompact case)

$$f^{(n)}(\tau) + a_1(\tau) f^{(n-1)}(\tau) + \cdots + a_n(\tau) f(\tau) \;=\; 0 \qquad (\sharp)_k^{\Gamma}$$

- $a_i(\tau)$'s of a MLDE $(\sharp)_k^{\Gamma}$ have the quasimodularlity of wt. $2i$ and depth i.
- For $k=0$, $a_n(\tau)$ of a MLDE $(\sharp)_0^{\Gamma}$ have the modularlity of wt. $2i$.

- For $\phi(\tau)$ and $g_{2m}(\tau)\,(1 \le m \le n)$ s.t.

$$\phi(\gamma\tau) = (c\tau + d)^2 + \left(\frac{c}{2\pi\sqrt{-1}}\right)(c\tau + d),$$

$$g_2(\tau) := a_1(\tau) + n(k+n-1)\phi(\tau),$$

$$g_{2m}(\tau) := \sum_{i=0}^{m} \binom{n-m+i}{i}\binom{k+n-m+i-1}{i}\binom{2m-2}{i}^{-1} a_{m-i}^{(i)}(\tau) \qquad (2 \le m \le n).$$

we have $\displaystyle \Theta_k^{(n)}(f) + \sum_{i=1}^{n} \binom{n+i-1}{2i-1}^{-1} [\, g_{2i}, f\,]_{(n-i)}^{(2i,k)} = 0\,.$

Summary(cocompact case)

$$b_0(\tau)f^{(n)}(\tau)+b_1(\tau)f^{(n-1)}(\tau)+\cdots+b_n(\tau)f(\tau)\;=\;0 \qquad (\flat)^{\Gamma}_{k}.$$

- $b_i(\tau)$'s of a MLDE $(\flat)^{\Gamma}_{k}$ have the quasimodualrlity of wt. $2i+\xi$ and depth i.
- For $k=0$, $b_n(\tau)$ of a MLDE $(\flat)^{\Gamma}_{0}$ have the modularity of wt. $2i+\xi$.
- We take $p_{2m}(\tau)$'s $(1\le m\le n)$ s.t.

$$p_{2m}(\tau)=\sum_{i=0}^{m}\binom{n-m+i}{i}\binom{k+n-m+i-1}{i}\binom{\xi+2m-2}{i}^{-1}b^{(i)}_{m-i}(\tau).$$

we have $$\sum_{i=0}^{n}\binom{n+\xi-i-1}{n-i}[\,p_{2i}\,,\,f\,]^{(2i+\xi,k)}_{(n-i)}=0\,.$$

Thank you for your attention.

Happy birthday Professor Mason !!

Appendix(The MLDEs for cocompact case)

Lemma

For any $s \geq 1$, the derivative of coefficients of the MLDE $(\flat)_k^\Gamma$ is given by

$$b_m^{(s)}(\gamma\tau) = \sum_{i=0}^{m} \binom{n-m+i}{i} (k+n-m)_i \times \sum_{j=0}^{s} \binom{s}{j} \Big(-\frac{c}{2\pi\sqrt{-1}}\Big)^{i+j} (i-\xi-2m-s+1)_j (c\tau+d)^{\xi+2(m+s)-i-j} b_{m-i}^{(s-j)}(\tau)\,.$$

$(1 \leq m \leq n)$

$\because$) induction on s.

By using this lemma,

we show the modularity of a sum of derivatives of $b_i(\tau)$'s.

Appendix(The MLDEs for cocompact case)

Theorem B' Let $b_i(\tau)$'s be coefficients of the MLDE $(\flat)_k^{\Gamma}$.

For $1 \leq \forall m \leq n$, we put

$$p_{2m}(\tau) = \sum_{i=0}^{m} \binom{n-m+i}{i} \binom{k+n-m+i-1}{i} \binom{\xi+2m-2}{i}^{-1} b_{m-i}^{(i)}(\tau).$$

We have $$p_{2m}(\gamma\tau) = (c\tau+d)^{2m+\xi} p_{2m}(\tau) \quad (\forall \gamma \in \Gamma).$$

Proposition

Let $p_{2m}(\tau)\,(1 \leq m \leq n)$ be a modular form of weight $2m+\xi$ on Γ.

Then $b_m(\tau)$'s of the MLDE $(\flat)_k^{\Gamma}$ can be written as

$$b_m(\tau) = \sum_{j=0}^{m} (-1)^{m+j} \binom{n-j}{m-j} \binom{k+n-j-1}{m-j} \binom{\xi+m-1+j}{m-j}^{-1} p_{2j}^{(m-j)}(\tau).$$