Schrödinger’s equation for a quantum harmonic oscillator is
\[
\frac{1}{2} \left(-\frac{d^2}{dx^2} + x^2\right) u = \epsilon u.
\]
This says \( u \) is an eigenfunction with eigenvalue \( \epsilon \). In physics, \( \epsilon \) represents an energy value.

Let
\[
a = \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx}\right)
\]
the “annihilation” operator,
and
\[
a^\dagger = \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx}\right)
\]
the “creation” operator.

Note that, as an operator,
\[
a^\dagger a + aa^\dagger = -\frac{d^2}{dx^2} + x^2,
\]
which we can use to rewrite Schrödinger’s equation.

**Fact.** The function
\[
u_0(x) = \pi^{-1/4} e^{-x^2/2}
\]
represents a quantum particle in its “ground state.” (Its square is the probability density function of a particle in its lowest energy state.)

Note that
\[
a u_0(x) = 0,
\]
which justifies calling the operator \( a \) the “annihilation” operator! On the other hand,
\[
u_1(x) := a^\dagger u_0(x)
\]
\[
= \pi^{-1/4} 2^{-1/2}(2x)e^{-x^2/2}
\]
represents a quantum particle in its next highest energy state. So the operator \( a^\dagger \) “creates” one unit of energy!
Each application of $a^\dagger$ creates a unit of energy, and we get the “Hermite functions"
\[ u_n(x) := (n!)^{-1/2}(a^\dagger)^nu_0(x) \]
\[ = \pi^{-1/4}2^{-n/2}(n!)^{-1/2}H_n(x)e^{-x^2/2}, \]
where $H_n(x)$ is a polynomial of order $n$ called “the $n$th Hermite polynomial.” In particular, $H_0(x) = 1$ and $H_1(x) = 2x$.

**Fact:** $H_n$ satisfies the ODE

\[ \left( \frac{d^2}{dx^2} - 2x \frac{d}{dx} + 2n \right) H_n = 0. \]

For review, let’s use the power series method: We look for $H_n$ of the form

\[ H_n(x) = \sum_{k=0}^{\infty} A_k x^k. \]

We plug in and compare coefficients to find:

\[ A_2 = -nA_0 \]
\[ A_{k+2} = \frac{2(k-n)}{(k+2)(k+1)} A_k \quad \text{for} \quad k \geq 1. \]

Note how there is a natural split into odd and even terms, similar to what happened for Legendre’s equation in §5.2. We can use this recurrence relation to find formulas for the $H_n$, and compare with wikipedia. [Note: Our $H_n$ are sometimes called “physicists’ Hermite polynomials.” Physicists use different conventions than probabilists.]

Now consider the Sturm-Liouville problem

\[ \frac{d}{dx} \left( e^{-x^2} \frac{dH}{dx} \right) + \lambda e^{-x^2} H = 0. \]

with the boundary condition that $H$ not grow faster than a polynomial as $x \to \pm \infty$.

**Fact.** When $\lambda = n$ is a non-negative integer, the Hermite polynomial $H_n(x)$ is a solution. Then Sturm-Liouville theory, with $p = e^{-x^2}$, $q = 0$, and $r = e^{-x^2}$, gives orthogonality:

\[ \int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x) \, dx = \begin{cases} 0 & \text{when } n \neq m \\ \frac{\pi}{\sqrt{2}} & \text{when } n = m. \end{cases} \]

This can be rewritten in terms of the $u_n$:

\[ \int_{-\infty}^{\infty} u_n(x)u_m(x) \, dx = \begin{cases} 0 & \text{when } n \neq m \\ 1 & \text{when } n = m. \end{cases} \]

We chose the constants above ($\pi^{-1/4}$, for example) so that we get $= 1$. 

We can use this orthogonality to find the Fourier-Hermite representation of a given function $f$:

$$f(x) = \sum_{n=0}^{\infty} A_n u_n(x).$$

This is useful because the Fourier transform acts very simply on Hermite functions:

$$\mathcal{F}[u_n](w) = (-i)^n u_n(w).$$

That is, the Hermite function $u_n$ is an eigenfunction of the Fourier transform, with eigenvalue $(-i)^n$. 