GENERALIZED ALEXANDROFF-URYSOHN SQUARES AND A CHARACTERIZATION OF THE FIXED POINT PROPERTY

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Abstract. Given a Hausdorff continuum X, we introduce a topology on $X \times X$ that yields a Hausdorff continuum. We call the resulting space the Alexandroff-Urysohn Square of X and prove that X has the fixed point property if and only if the Alexandroff-Urysohn Square of X has the fixed point property.

Many topological properties (e.g., compact, connected, path connected) are preserved in certain basic topological constructions. The three mentioned, in particular, are preserved in products with the product topology. However, the fixed point property (fpp) is often not preserved in basic constructions. This is true for products, cones, quotients, and unions of two continua even when their intersection is an arc. Even the product of only two continua, each with the fpp, has proved to be unfriendly to the fpp. There are examples of

(1) a 1-dimensional continuum X with the fpp such that $X \times [0,1]$ does not have the fpp [9],
(2) a contractible continuum X with the fpp such that $X \times [0,1]$ does not have the fpp [7],
(3) a polyhedron X with the fpp such that neither $X \times [0,1]$ nor $X \times X$ has the fpp [8], and
(4) manifolds X and Y with the fpp such that $X \times Y$ does not have the fpp [6].

Still open is the question, "If X is a manifold with the fpp, does $X \times X$ have the fpp?" See [3] for a nice survey article concerning products and the fpp.

By a *continuum* we mean a compact, connected topological space. In §1, we define a construction that produces a compact, connected, Hausdorff topology on the set-theoretic product of a Hausdorff continuum with itself that not only preserves the fpp, but yields a characterization of the fpp for Hausdorff continua. In §2, we define a generalization of this construction to a product of two possibly different Hausdorff continua and determine necessary and sufficient conditions for the resulting space to have the fpp.

1. Generalized Alexandroff-Urysohn Squares

In 1929, P. S. Alexandroff and P. Urysohn [1] defined an interesting space that came to be known as the Alexandroff Square. A definition of the Alexandroff Square $([0,1]^2, \tau)$ is given by L. A. Steen and J. A. Seebach in [10, Ex. 101, p. 120-121]. The resulting topology on $[0,1]^2$ yields a non-metrizable, path connected, compact Hausdorff space.

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In [5], it is shown that the Alexandroff Square has the fpp. In the spirit of this example, we make the following definitions.

Let X be a non-degenerate Hausdorff continuum. We take the Cartesian product of X with itself, denoted by X^2, and define the *Alexandroff-Urysohn (AU) topology* on X^2 as follows. Let Δ denote the diagonal in X^2.

1) For each point $(x, y) \in X^2 - \Delta$, let U_y be a neighborhood of y in X that doesn't contain x, and define $V(x, y) = \{x\} \times U_y$. We will refer to $V(x, y)$ as a *basic vertical neighborhood of* (x, y).

2) For each point $(x, x) \in \Delta$, pick a finite set of points $\{x_i \in X \mid x_i \neq x,\ 1 \leq i \leq n\}$ (possibly empty) and let U_x be a neighborhood of x in X. Define $H(x, x) = (X \times U_x) - \bigcup_{i=1}^{n}(\{x_i\} \times U_x)$. We refer to $H(x, x)$ as a *basic horizontal strip neighborhood of* (x, x).

We refer to X^2 with the AU topology as the *AU square of* X and denote this space by $X \times_\Delta X$. Analogous proofs to those indicated in [10] gives us that $X \times_\Delta X$ is compact, Hausdorff, and non-metrizable.

We make a few additional observations about the AU square of X.

Observation 1. *The relative topology on* $\{x\} \times X$ *for* $x \in X$ *is the topology on* X.

Observation 2. *The relative topology on* Δ *is the topology on* X.

Proof. Observe that for a horizontal strip neighborhood $H(x, x) = (X \times U_x) - \bigcup_{i=1}^{n}(\{x_i\} \times U_x)$, we have $H(x, x) \cap \Delta = \left[(X \times U_x) - \bigcup_{i=1}^{n}(\{x_i\} \times U_x)\right] \cap \Delta = \{(y, y) \mid y \in U_x - \bigcup_{i=1}^{n}\{x_i\}\} \overset{T}{\approx} U_x - \bigcup_{i=1}^{n}\{x_i\}$. □

Observation 3. $X \times_\Delta X$ *is connected.*

Proof. Let (x, y) and (v, w) be points of $X \times_\Delta X$. Then $(\{x\} \times X) \cup \Delta \cup (\{v\} \times X)$ is a subcontinuum of $X \times_\Delta X$ containing (x, y) and (v, w). □

Observation 4. *Let* $\pi_i \colon X \times_\Delta X \to X$ *denote the coordinate projections for* $i = 1, 2$. *The second coordinate projection* π_2 *is continuous. The first coordinate projection* π_1 *is discontinuous at points of* Δ *and continuous at points not in* Δ.

Proof. To see that π_2 is continuous, simply observe that the preimage under π_2 of a neighborhood in X is a basic horizontal strip neighborhood in $X \times_\Delta X$.

Since each horizontal strip neighborhood of a point of Δ is projected by π_1 onto the complement of a finite set in X, it follows that π_1 is discontinuous at each point of Δ.

For each point $(x, y) \notin \Delta$, let U_x be a neighborhood of $x = \pi_1(x, y)$ in X. Let $V(x, y)$ be a basic vertical neighborhood of (x, y). Then $\pi_1(V(x, y)) = \{x\} \subseteq U_x$. So, π_1 is continuous at points not in Δ. □

Observation 5. *Let* $j \colon X \to \Delta$ *be defined by* $j(x) = (x, x)$. *Then* $j \circ \pi_2$ *is a retraction of* $X \times_\Delta X$ *onto* Δ.

Proof. By Observation 2, j is continuous. By Observation 4, it follows that $j \circ \pi_2$ is continuous. □

Lemma 1. *Let* $Y_0 = \Delta \overset{T}{\approx} X$. *For* $\{x_i \mid 1 \le i \le n\}$ *a finite set of points in* X, *let* $Y_n = \Delta \cup (\cup_{i=1}^n \{x_i\} \times X)$. *Define* $p_n \colon X \times_\Delta X \to Y_n$ *by*

$$p_n(x,y) = \begin{cases} (x,y) & \text{if } (x,y) \in Y_n, \\ (y,y) & \text{if } (x,y) \notin Y_n. \end{cases}$$

Then p_n *is a retraction onto* Y_n *and* $p_n((X \times_\Delta X) - Y_n) = \Delta$.

Proof. To show that p_n is continuous, we write $X \times_\Delta X$ as the union of closed sets $K = \Delta \cup (\cup_{x \neq x_i}(\{x\} \times X))$ and Y_n; and note that p_n is $j \circ \pi_2$ on K and is the identity map on Y_n. That $K \cup Y_n = X \times_\Delta X$ is clear. It follows from Observations 1 and 2 that Y_n is a subcontinuum of $X \times_\Delta X$. To see that K is closed, we note that its complement is $\cup_{i=1}^n(\{x_i\} \times X) - \Delta$, which is open by Observation 1. So, K and Y_n are closed.

It is easy to see, by applying the definitions to both sides, that $p_n = j \circ \pi_2$ on K. Also, by definition, we see that p_n is the identity map on Y_n. Furthermore, $j \circ \pi_2$ is the identity map on $K \cap Y_n = \Delta$. It follows that p_n is continuous. From the definition of p_n, we see that $p_n((X \times_\Delta X) - Y_n) = \Delta$. □

We note that Observation 5 is a special case of Lemma 1 when $n = 0$. J. Prajs has observed that the AU topology on X^2 is the smallest topology for which the maps p_1 are continuous onto $\Delta \cup (\{x_1\} \times X)$ with the wedge sum topology.

Theorem 1. *The continuum* X *has the fpp if and only if the AU square of* X *has the fpp.*

Proof. ⇒: Let $f \colon X \times_\Delta X \to X \times_\Delta X$ be a mapping. Since X has the fpp, it follows from Observation 2 that the map $j\pi_2 f|_\Delta \colon \Delta \to \Delta$ has a fixed point $z_1 = (u_1, u_1)$. So, $(u_1, u_1) = j\pi_2 f(u_1, u_1) = (\pi_2 f(u_1, u_1), \pi_2 f(u_1, u_1))$. Hence, $u_1 = \pi_2 f(u_1, u_1)$. Thus, $f(u_1, u_1) = (x_1, u_1)$ for some $x_1 \in X$.

Let Y_1 and $p_1 \colon X \times_\Delta X \to Y_1$ be defined as in Lemma 1 for the finite set $\{x_1\}$. Since Y_1 is a wedge sum of Δ and $\{x_1\} \times X$, Y_1 has the fpp. So, the map $p_1 f|_{Y_1} \colon Y_1 \to Y_1$ has a fixed point z_2. If $f(z_2) \in Y_1$, then since p_1 is a retraction, it follows that z_2 is a fixed point of f and we are done. So, we assume that $f(z_2) \notin Y_1$. By Lemma 1, $p_1 f(z_2) = z_2 \in \Delta$. Let $z_2 = (u_2, u_2)$. Since $f(z_2) \notin Y_1$, $(u_2, u_2) \neq (u_1, u_1)$. We get that $(u_2, u_2) = p_1 f(u_2, u_2) = (\pi_2 f(u_2, u_2), \pi_2 f(u_2, u_2))$. So, $u_2 = \pi_2 f(u_2, u_2)$. Thus, we let $f(u_2, u_2) = (x_2, u_2)$ for some $x_2 \in X$ with $x_2 \neq x_1$.

Let Y_2 and $p_2 \colon X \times_\Delta X \to Y_2$ be defined as in Lemma 1 for the finite set $\{x_1, x_2\}$. Since Y_2 is a wedge sum of Y_1 and $\{x_2\} \times X$, Y_2 has the fpp. So, the map $p_2 f|_{Y_2} \colon Y_2 \to Y_2$ has a fixed point z_3. Again, we may assume that $f(z_3) \notin Y_2$, for otherwise f has a fixed point. So, as in the previous paragraph, $z_3 \in \Delta$. We let $z_3 = (u_3, u_3)$ and it follows that $(u_3, u_3) \neq (u_i, u_i)$ for $i = 1, 2$. We get that $(u_3, u_3) = p_2 f(u_3, u_3) = (\pi_2 f(u_3, u_3), \pi_2 f(u_3, u_3))$. So, $u_3 = \pi_2 f(u_3, u_3)$. Thus, we let $f(u_3, u_3) = (x_3, u_3)$ for some $x_3 \in X$ with $x_3 \neq x_i$ for $i = 1, 2$.

Continuing, we get a sequence of points $\{(u_n, u_n)\}$ in Δ such that for each $n \neq m \geq 1$, $(u_n, u_n) \neq (u_m, u_m)$, $f(u_n, u_n) = (x_n, u_n)$, and $x_n \neq x_m$.

By Theorem 3.1.23 in [4], $\{u_n\}$ has a cluster point u in X. By Proposition 1.61 of [4], we let $\{u_{n_\alpha}\}$ be a net finer than $\{u_n\}$, with domain the directed set D, that converges to u in X. By Observation 2, $\{(u_{n_\alpha}, u_{n_\alpha})\}$ converges to $(u,u) \in \Delta$. If $H(u,u) = X \times U_u - \bigcup_{i=1}^{n}(\{v_i\} \times U_u)$ is a horizontal strip neighborhood of (u,u) in $X \times_\Delta X$, there is a $\beta \in D$ such that for $\alpha \geq \beta$, $u_{n_\alpha} \in U_u$ and $x_{n_\alpha} \neq v_i$ for $1 \leq i \leq n$. It follows that $(x_{n_\alpha}, u_{n_\alpha})$ is in $H(u,u)$ for $\alpha \geq \beta$. So, $\{(x_{n_\alpha}, u_{n_\alpha})\}$ also converges to (u,u). By continuity of f, $\{f(u_{n_\alpha}, u_{n_\alpha})\}$ converges to $f(u,u)$. Thus, by uniqueness of limits in Hausdorff spaces (see Proposition 1.67 in [4]), $f(u,u) = (u,u)$.

$\Leftarrow$: Suppose that the AU square of X has the fpp. It follows from Observation 5 that Δ has the fpp. By Observation 2, X has the fpp. □

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2. The AU Product of Two Possibly Different Continua

It is possible to define an AU topology on the product of two possibly different Hausdorff continua X and Y if there exists a surjective map g from Y to X. We simply use the graph of g in $X \times Y$ in the same manner we used Δ (the graph of $id\colon X \to X$) in $X \times_\Delta X$. Specifically, let g denote both the function $g\colon Y \to X$ and the graph of g in $X \times Y$. That is, $g = \{(g(y), y) \mid y \in Y\}$. One might view $X \times Y$ and g (set theoretically) as an inverse system with two factors and the graph of g as the inverse limit. However, the topology will be defined, not as the product topology, but as follows.

1) For each point $(x,y) \in (X \times Y) - g$, let U_y be a neighborhood of y in Y that misses $g^{-1}(x)$, and define $V(x,y) = \{x\} \times U_y$. We will refer to $V(x,y)$ as a *basic vertical neighborhood of* (x,y).

2) For each point $(g(y), y) \in g$, pick a finite set of points $\{x_i \mid x_i \neq g(y),\ 1 \leq i \leq n\}$ (possibly empty) and let U_y be a neighborhood of y in Y. Define $H(g(y), y) = (X \times U_y) - \bigcup_{i=1}^{n}(\{x_i\} \times U_y)$. We refer to $H(g(y), y)$ as a *basic horizontal strip neighborhood of* $(g(y), y)$.

We refer to the space $X \times Y$ with the topology so defined as the *AU(g) product of* X and Y, denoted $X \times_g Y$.

Once again $X \times_g Y$ is a Hausdorff continuum. It is easy to check that $X \times_g Y$ also satisfies the appropriately modified Observations 1 through 5, and Lemma 1. It should be noted that horizontal fibers $X \times \{y\}$ meet g in exactly one point, namely $(g(y), y)$; but vertical fibers $\{x\} \times Y$ will typically meet g in a non-degenerate set of points.

We give the modified statements of Observation 5 and Lemma 1 below.

Observation 6. *Let $j\colon Y \to g$ be defined by $j(y) = (g(y), y)$. Then $j \circ \pi_2$ is a retraction of $X \times_g Y$ onto g.*

Lemma 2. *Let $Y_0 = g \overset{T}{\approx} Y$. For $\{x_i \mid 1 \leq i \leq n\}$ a finite set of points in X, let $Y_n = g \cup (\cup_{i=1}^{n}\{x_i\} \times Y)$. Define $p_n\colon X \times_g Y \to Y_n$ by*

$$p_n(x,y) = \begin{cases} (x,y) & \text{if } (x,y) \in Y_n, \\ (g(y), y) & \text{if } (x,y) \notin Y_n. \end{cases}$$

Then p_n is a retraction onto Y_n and $p_n((X \times_g Y) - Y_n) = g$.

The proofs of Observation 6 and Lemma 2 are analogous to the proofs of Observation 5 and Lemma 1.

Since $g \overset{T}{\approx} Y$, the relative topology on Y_n is the topology on the sum of Y and each $\{x_i\} \times Y$ glued together at $g^{-1}(x_i)$. Equivalently, if we let $j_i \colon g^{-1}(x_i) \hookrightarrow Y$ be inclusion for each $1 \leq i \leq n$, then Y_n is homeomorphic to the adjunction space $Y \cup_{j_1} Y \cup_{j_2} Y \cup_{j_3} \ldots \cup_{j_n} Y$.

Theorem 2. *Let $X \times_g Y$ be the $AU(g)$ product of continua X and Y and let Y_n denote subspaces of $X \times_g Y$ as defined in Lemma 2. Then $X \times_g Y$ has the fpp if and only if each Y_n has the fpp.*

Proof. $\Rightarrow$: By Lemma 2, each Y_n is a retract of $X \times_g Y$ and, therefore has the fpp.

$\Leftarrow$: The proof of this implication is similar to the proof of Theorem 1 with g replacing Δ. One should also note that in the proof of Theorem 1, since Y has the fpp and each Y_n is a wedge sum of $n+1$ copies of Y, it follows that each Y_n has the fpp. In this direction of the proof of Theorem 2, each Y_n has the fpp by assumption. Otherwise, the proof is analogous. □

Corollary 1. *Let $I \times_g I$ be the $AU(g)$ square of $I = [0,1]$. Then $I \times_g I$ has the fpp if and only if g is monotone.*

Proof. $\Rightarrow$: Suppose $g \colon I \to I$ is a surjective non-monotone map. Let $s \in I$ with $g^{-1}(s)$ not connected. Let $t \in I - g^{-1}(s)$ be a point separating two components of $g^{-1}(s)$. Let (t_1, t_2) be the maximal open segment in the complement of $g^{-1}(s)$ that contains t. Consider $Y_1 = g \cup (\{s\} \times I)$. Now, $C = g|_{[t_1,t_2]} \cup (\{s\} \times I)$ is either a simple closed curve or a simple closed curve with one or two stickers attached. In any case, C does not have the fpp.

Let $h \colon I \to \{s\} \times I$ be the mapping defined by $h(x) = (s, x)$. It follows from Observation 1 that h is continuous. Let $p \colon Y_1 \to C$ be the retraction where $p = h \circ \pi_2$ on $g|_{I-(t_1,t_2)}$, and $p = \text{id}$ on C. Since $h \circ \pi_2 = \text{id}$ on $g|_{I-(t_1,t_2)} \cap C = g^{-1}(s)$, p is continuous. Since C does not have the fpp, neither does Y_1. It follows from Theorem 2 that $I \times_g I$ does not have the fpp.

$\Leftarrow$: Simply observe that if g is monotone, then each $I_n = g \cup (\cup_{i=1}^{n} \{x_i\} \times I)$ is a tree and therefore has the fpp. □

In light of Corollary 1, one might ask if, in general, $X \times_g Y$ having the fpp is equivalent to the map g being monotone. We give examples to show that neither implication holds.

Example 1. *There exist metric continua X and Y with the fpp and a monotone map $g \colon Y \to X$ such that not all Y_n have the fpp, and therefore, by Theorem 2, $X \times_g Y$ does not have the fpp.*

Proof. Let $D \subseteq \mathbb{R}^2$ be the unit disk in the plane given in polar coordinates by $D = \{(r, \theta) \mid 0 \leq r \leq 1\}$. Let $I = \{(r, 0) \mid 0 \leq r \leq 1\}$ be the unit interval in polar coordinates. Define $g \colon D \to I$ by $g(r, \theta) = (r, 0)$. Note that g is monotone.

Consider the space $I \times_g D$. Let $Y_1 = g \cup (\{(1,0)\} \times D)$ and let p_1 be the retraction as defined in Lemma 2. Then Y_1 is a topological 2-sphere, which does not have the fpp. Note that p_1 followed by the "antipodal" map on Y_1 is a fixed point free map on $I \times_g D$. □

Example 2. *There exist metric continua X and Y with the fpp and a non-monotone map $g\colon Y \to X$ such that each Y_n (and therefore $X \times_g Y$) has the fpp.*

Proof. Let L, T, and S be subsets of the plane $\mathbb{R}^2$ given by $L = \{(0,t) \mid -1 \le t \le 1\}$, $T = \{(x, \sin\frac{1}{x}) \mid 0 < x \le \frac{2}{3\pi}\}$, and $S = \{(x,y) \mid (x - \frac{1}{3\pi})^2 + (y+1)^2 = \frac{1}{9\pi^2}$ and $y \le -1\}$. Let $X = L \cup T \cup S$. Note that X is topologically the Warsaw circle (also called the $\sin\frac{1}{x}$ circle), which has the fpp (see [2, Th.13]). Let $Y = L \cup T$. Note that Y is the topologist's sine curve, which also has the fpp.

For convenience and clarity in describing Example 2 and its properties, we will think of X and Y as being disjoint, and we choose the following notations for points of X and Y.

Points $(0,t)$ of $L \subseteq Y$ will be denoted by a_t. Note that $T \subseteq Y$ is a topological ray. We denote its points by α_t for $t \ge 0$ with $\alpha_0 = (\frac{2}{3\pi}, \sin(\frac{3\pi}{2}))$.

Points $(0,t)$ of $L \subseteq X$ will be denoted by b_t. Note that $T \cup S \subseteq X$ is a topological ray. We denote its points by β_t for $t \ge 0$ with $\beta_0 = (0,-1) = b_{-1}$.

Define the map $g\colon Y \to X$ by $g(a_t) = b_t$ for $a_t \in L$ and $g(\alpha_t) = \beta_t$ for $\alpha_t \in T$. Note that $g^{-1}(\beta_0) = \{a_{-1}, \alpha_0\}$; otherwise, g is one-to-one. So, g is not monotone. The map g is equivalent to the quotient map on Y that identifies the points a_{-1} and α_0 and hence is continuous.

We will now show that each Y_n subspace of $X \times_g Y$ has the fpp, and hence, $X \times_g Y$ has the fpp. If $x \ne \beta_0$, then $g^{-1}(x)$ is degenerate. Since $g \overset{T}{\approx} Y$ is a topologist's sine curve, $g \cup (\{x\} \times Y)$ is the wedge sum of two copies of the topologist's sine curve, which has the fpp.

If we consider $Y_1 = g \cup (\{\beta_0\} \times Y)$, then Y_1 is the sum of two copies of Y glued together at $g^{-1}(\beta_0) = \{a_{-1}, \alpha_0\}$ (Recall the comments immediately after Lemma 2). In $X \times_g Y$, we have that $g \cap (\{\beta_0\} \times Y) = \{(b_{-1}, a_{-1}), (\beta_0, \alpha_0)\}$. Hence, topologically Y_1 is two topologist's sine curves glued together at the endpoints of their rays and glued together at an endpoint of their limit bars.

By observing that the image (under a mapping of Y_1 to itself) of the arc component of Y_1 that is a topological interval must either be a subset of itself or must be an interval lying in the non-compact arc component of Y_1, it is easy to see that Y_1 has the fpp.

So, any Y_n subcontinuum of $X \times_g Y$ is a finite wedge sum of topological copies of Y or of Y_1 and copies of Y. It follows that all Y_n and, by Theorem 2, $X \times_g Y$ have the fpp. □

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