PROJECTIONS ONTO CLOSED CONVEX SETS IN HILBERT SPACES

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Abstract. Let X be a real Hilbert Space. We give necessary and sufficient algebraic conditions for a mapping $F\colon X \to X$ with a closed image set to be the metric projection mapping onto a closed convex set. We provide examples that illustrate the necessity of each of the conditions. Our characterizations generalize several results related to projections onto closed convex sets.

1. Introduction

Let X be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A set C in X is a *Chebyshev set* if for each point x in X, there is a unique point of C that is nearest to x; that is, there is a point q in C such that $\|x - q\| < \|x - y\|$ for all $y \in C \setminus \{q\}$. So, there is a natural "nearest point" mapping $N\colon X \to C$ associated with each Chebyshev set C. Clearly, Chebyshev sets are closed. It is well-known that every closed convex set in a real Hilbert space is a Chebyshev set. If every Chebyshev set in an infinite dimensional Hilbert space must be convex is an old question that remains open. In finite dimensional normed linear spaces with smooth and strictly convex unit spheres, the closed convex sets coincide with the Chebyshev sets. There are many partial results, in Banach spaces of any dimension, that connect geometrical properties of the space to the convexity of Chebyshev sets. See [5, Ch.12], [22, §2], and [2]) for survey articles that give history, results, and questions related to the "Chebyshev problem", and see [1, 8, 9, 14, 15, 16, 19, 20, 21] for some well-known results in this area.

We consider closed convex sets and their associated *nearest point mappings*, which we also call *metric projection mappings*. The set C in X is *convex* if $tx + (1-t)y \in C$ for all $x, y \in C$ and $0 \le t \le 1$. The set C is a *convex cone* if C is closed under addition, and multiplication by non-negative scalars. Closed convex sets are fundamental geometric objects in Hilbert spaces. They have been studied extensively and are important in a variety of applications, including optimization, duality, linear programming, and robotics. There are many questions related to convex sets, even in $\mathbb{R}^n$, that remain unanswered. See

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[3, 5, 10, 11, 12, 18, 23] for basic properties, and surveys of results, applications, and open questions.

For nearest point mappings onto closed convex sets, we denote the mapping by either P or N. If we have a function defined on X with a closed image set C and satisfying certain algebraic properties, but we neither assume that the function is continuous nor that it is a nearest point mapping, we will denote such a function by F.

For $C \subset X$, we define the sets $C^{\perp} = \{x \mid \langle x, y\rangle = 0 \text{ for all } y \in C\}$, and $C^{\circ} = \{x \mid \langle x, y\rangle \leq 0 \text{ for all } y \in C\}$. We let $\overline{C}$ denote the topological closure, $\operatorname{int} C$ the topological interior, and $\operatorname{bd} C = \overline{C} - \operatorname{int} C$ the topological boundary of C. For points $x, y \in X$, we let $[x, y]$ denote the line segment from x to y. For points $x, y, z \in X$, we let $\angle xyz$ denote the angle formed by the vectors $x - y$ and $z - y$.

We begin by recalling some results about metric projection mappings onto closed convex sets, and observing a few immediate properties that they must have.

Theorem 1.1. [4, Lemma] *Let $P\colon X \to C$ be the metric projection onto a closed convex set C in X. Then for $x \in X$ and $y \in C$,*

$$\langle x - y, y\rangle \geq \langle x - y, q\rangle \text{ for all } q \in C \text{ if and only if } y = P(x).$$

Theorem 1.2 below exhibits two special cases of Theorem 1.1.

Theorem 1.2. [7, 1.12.4] *Let $P\colon X \to C$ be the metric projection mapping onto a closed convex set C.*

(a) *If C is a subspace of X, then for $x \in X$ and $y \in C$,*

$$x - y \in C^{\perp} \text{ if and only if } y = P(x).$$

(b) *If C is a cone, then for $x \in X$ and $y \in C$,*

$$\langle x - y, y\rangle = 0 \text{ and } \langle x - y, q\rangle \leq 0 \text{ for all } q \in C \text{ if and only if } y = P(x).$$

The next two theorems give algebraic characterizations for a mapping $F\colon X \to X$ to be the metric projection mapping onto its image. In contrast to Theorems 1.1 and 1.2, note that there is no assumption of convexity of the image set C. That is, the algebraic properties of F characterize both the convexity of C and F as the nearest point mapping onto C.

Theorem 1.3. [17, Theorem 13.5.1] *Let $F\colon X \to X$ be a mapping with closed image set $C = F(X)$. Then C is a subspace, and F is the metric projection of X onto C if and only if F satisfies the following properties.*

(a) *$F^2 = F$ (F is idempotent).*
(b) *F is linear.*
(c) *For all $x, y \in X$, $\langle x, F(y)\rangle = \langle F(x), y\rangle$ (F is symmetric).*

Theorem 1.4 below generalizes Theorem 1.3 to cones.

Theorem 1.4. [13, Theorem 2] *Let $F\colon X \to X$ be a mapping with closed image set $C = F(X)$. Then C is a closed convex cone and F is the metric projection of X onto C if and only if F satisfies the following properties.*

(1) $F^2 = F$.
(2) *For $x \in X$ and $\lambda \geq 0$, $F(\lambda x) = \lambda F(x)$.*
(3) *For $x, y \in X$, $F(x+y) = F(x) + F(y)$ if and only if $\langle x - F(x), F(y)\rangle = 0 = \langle y - F(y), F(x)\rangle$.*

Our objective hereafter is to determine algebraic properties of a mapping F (assumed to have a closed image set but not assumed to be continuous) on a real Hilbert space that will characterize F as the metric projection mapping onto a closed convex set, thusly generalizing Theorem 1.4 from convex cones to convex sets. In Section 2, we propose four algebraic properties, (F1), (F2), (F3), and (F4), of a mapping $F\colon X \to X$ with a closed image set, similar to those in Theorem 1.4, and we will show, in Sections 3 and 4, that various combinations of these properties give us the desired characterizations.

Generalizing Theorem 1.4 from Hilbert spaces to Banach spaces raises different kinds of questions. The first and third authors have obtained results in Banach spaces for some special cases of convex cones [6].

2. Preliminaries

Observation 2.1 below follows immediately from Theorem 1.1 above.

Observation 2.1. *Let C be a closed convex set in X with $0 \in C$, and let N be the nearest point mapping of X onto C. Then $\langle x - N(x), N(x)\rangle \geq 0$ for all $x \in X$.*

Observation 2.2. *Let C be a closed convex set in X with $0 \in C$, and let N be the nearest point mapping of X onto C. Then $\|x\| \geq \|N(x)\|$ for all $x \in X$. Moreover, if $x \notin C$, then $\|x\| > \|N(x)\|$.*

Proof. If $N(x) = 0$, then the observation follows immediately. So, we assume that $N(x) \neq 0$. By the Cauchy-Schwarz inequality and Observation 2.1, we get that $\|x\| \cdot \|N(x)\| \geq \langle x, N(x)\rangle \geq \|N(x)\|^2$. Hence, $\|x\| \geq \|N(x)\|$.

Suppose $x \notin C$. Then, $x - N(x) \neq 0$, and from Observation 2.1, we have two options. If $\langle x - N(x), N(x)\rangle > 0$, then the Cauchy-Schwarz inequality gives $\|x\| > \|N(x)\|$. If $\langle x - N(x), N(x)\rangle = 0$, then by the Pythagorean theorem

$$\|x\|^2 = \|x - N(x)\|^2 + \|N(x)\|^2 > \|N(x)\|^2.$$

□

Observation 2.3. *Let C be a closed convex set in X with $p \in C$, and let N be the nearest point mapping of X onto C. Since translations $T\colon X \to$*

X preserve convex sets, and the metric projection mapping onto $T(C)$ is the conjugate mapping $T \circ N \circ T^{-1}$, it follows from Observation 2.2 that $\|x - p\| \geq \|N(x) - p\|$ for all $x \in X$. Moreover, if $x \notin C$, then $\|x - p\| > \|N(x) - p\|$.

Let $F: X \to X$ be a mapping with closed image set $C = F(X)$. The properties of F, mentioned in Section 1, are as follows.

(F1) F is idempotent.

(F2) For $x \in X \setminus C$ and $y \in C$, if $\langle x - F(x), y - F(x) \rangle \geq 0$, then

$$F(tx + (1-t)y) = tF(x) + (1-t)y \text{ for all } 0 \leq t \leq 1.$$

(F3) For $x, y \in \overline{X \setminus C}$, if $F(tx+(1-t)y) = tF(x)+(1-t)F(y)$ for all $0 \leq t \leq 1$, then $\langle x - F(x), F(x) - F(y) \rangle = 0 = \langle y - F(y), F(x) - F(y) \rangle$.

(F4) For all $x, y \in X$, there exists $0 \leq s \leq 1$ such that for $z = sx + (1-s)y$,

$$\langle z - F(z), F(x) - F(y) \rangle = 0.$$

We offer a few comments related to these properties. A nearest point mapping onto a closed set must, of course, be idempotent. Properties (F2) and (F3) together relate inner product conditions between pairs of points x and y and their images under F to the linearity of F on the line segment $[x, y]$. In general, the nearest point mapping to a closed convex set will be highly non-linear. Property (F4) is a type of mean value theorem for points x and y that have different images. It ensures that for some point z in the segment $[x, y]$, either $z \in C$ or F projects z orthogonally to $F(x) - F(y)$. Properties (F2) and (F3) together, or (F2) and (F4) together, ensure that F is the metric projection onto C, as opposed to some other type of retraction of X onto C (see Examples 5.1, 5.2, and 5.3 in Section 5).

In Section 3, we will show that metric projections onto closed convex sets satisfy all four of these properties. In fact, they satisfy stronger versions of properties (F2) and (F3). In Section 4, we show that if $F: X \to X$ satisfies properties (F1), (F2), and one of either (F3) or (F4), then F is the nearest point mapping of X onto a closed convex set. As a consequence, we have two characterizations of a mapping that ensure it's image C is convex and that it is the metric projection mapping onto C (see Theorems 4.1, 4.2, and 4.3).

As mentioned, we assume that F has a closed image set, but not that F is continuous. Since Chebyshev sets are closed, if one could show that the nearest point mapping onto a Chebyshev set C satisfies properties (F2), and one of (F3) or (F4), then C would be convex, answering the Chebyshev question mentioned at the beginning of the paper. If the nearest point mapping onto a Chebyshev set C is continuous, it is known that C must be convex (see [5, 12.8(3)]). We obtain one additional characterization under the assumption that

F is idempotent, continuous, satisfies (F4), and satisfies a weaker version of (F2) (see Theorem 4.4).

3. Properties of metric projections onto closed convex sets

In this section, we show that if C is a closed convex set, and N is the nearest point mapping (metric projection) of X onto C, then N is continuous and satisfies all of properties (F1) through (F4). That N is idempotent is clear. It is well-known that N is continuous.

Theorem 3.1 below establishes a characterization of those points x and y for which the nearest point mapping N is linear on the segment $[x, y]$. Property (F3) is a special case of the right-to-left implication in this characterization.

Theorem 3.1. *Let C be a closed convex set in X, and let N be the nearest point mapping of X onto C. Then for $x, y \in X$,*

$$\langle x - N(x), N(x) - N(y)\rangle = 0 = \langle y - N(y), N(x) - N(y)\rangle$$

if and only if

$$N(tx + (1-t)y) = tN(x) + (1-t)N(y) \text{ for all } 0 \le t \le 1.$$

Proof. Let $x, y \in X$. Without loss of generality, by Observation 2.3, we may assume that $0 \in C$ and $N(x) = 0$.

$\Rightarrow$: Assume that $\langle x - N(x), N(x) - N(y)\rangle = 0 = \langle y - N(y), N(x) - N(y)\rangle$. So, with our assumption that $N(x) = 0$, this reduces to

(A) $\quad \langle x, N(y)\rangle = 0 = \langle y - N(y), N(y)\rangle$.

We wish to show that $N(tx + (1-t)y) = (1-t)N(y)$ for all $0 \le t \le 1$. If $t = 0$ or $t = 1$, the result follows immediately. So, assume that $0 < t < 1$.

Let $K_1 = \{\lambda x \mid \lambda \ge 0\}^\circ$, and $K_2 = \{\lambda(y - N(y)) \mid \lambda \ge 0\}^\circ$. Note that K_1 and K_2 are closed convex cones, so it follows that $K_1^\circ = \{\lambda x \mid \lambda \ge 0\}$, and $K_2^\circ = \{\lambda(y - N(y)) \mid \lambda \ge 0\}$, (see Lemma 2, (i) & (ii) in [13]). Let $K = K_1 \cap K_2$; K is also a closed convex cone. Let P_1, P_2, and P denote respectively, the metric projection maps of X onto K_1, K_2, and K.

By Theorem 1.1, if z is a point of C, then $\langle x - N(x), N(x)\rangle \ge \langle x - N(x), z\rangle$, and $\langle y - N(y), N(y)\rangle \ge \langle y - N(y), z\rangle$. So, from (A) and our assumption that $N(x) = 0$, we have that

(B) $\quad$ for $z \in C$, $\langle x, z\rangle \le 0$ and $\langle y - N(y), z\rangle \le 0$.

From (B), it follows that $C \subset K$. By Lemma 3 (viii) in [13], since $x \in K_1^\circ$, $P_1(x) = 0$. That is, 0 is the nearest point of K_1 to x. It follows that 0 is the nearest point of K to x. So, $P(x) = 0$.

From (A), $N(y) \in K$. Let q be the nearest point of $\{\lambda(y - N(y)) \mid \lambda \ge 0\}$ to y. Then $q = y - N(y)$, and by Lemma 3 (vii) in [13], $P_2(y) = y - q = N(y)$. Thus, we have that $N(y)$ is the nearest point of K_2 to y. So, $N(y)$ is also the nearest point of K to y; that is, $P(y) = N(y)$. By (A), $\langle x, P(y)\rangle = \langle x, N(y)\rangle = 0$; and since $P(x) = 0$, we also have that $\langle P(x), P(y)\rangle = 0 = \langle P(x), y\rangle$.

Recall that t has been chosen with $0 < t < 1$. Since K is a closed convex cone, P is the projection mapping onto K, and $P(x) = 0$, it follows, from Lemma 3 (iii) in [13], that

$$P(tx + (1-t)y) = tP(x) + (1-t)P(y) = (1-t)N(y).$$

We have that the nearest point of K to $tx + (1-t)y$ is $(1-t)N(y)$. Since $N(x) = 0$, and C is convex, it follows that $(1-t)N(y) \in C$. Recall that $C \subset K$. So, it follows that $(1-t)N(y)$ is the nearest point of C to $tx + (1-t)y$. That is, $N(tx + (1-t)y) = (1-t)N(y)$.

$\Leftarrow$: Suppose x and y are points where $N(tx+(1-t)y) = tN(x)+(1-t)N(y)$ for all $0 \le t \le 1$. Again, we may assume that $0 \in C$ and $N(x) = 0$. So, our assumption is $N(tx + (1-t)y) = (1-t)N(y)$ for all $0 \le t \le 1$. By Theorem 1.1, we have that $\langle x - N(x), N(x)\rangle \ge \langle x - N(x), N(y)\rangle$ and $\langle y - N(y), N(y)\rangle \ge \langle y - N(y), N(x)\rangle$. This gives us that

(C) $\quad \langle x, N(y)\rangle \le 0$ and $\langle y - N(y), N(y)\rangle \ge 0$.

We wish to show that $\langle x, N(y)\rangle = 0$ and $\langle y - N(y), N(y)\rangle = 0$. By Theorem 1.1, for all $0 \le s, t \le 1$, we have that

$$\langle tx + (1-t)y - (1-t)N(y), (1-t)N(y)\rangle \ge \langle tx + (1-t)y - (1-t)N(y), sN(y)\rangle.$$

So,

$$\langle tx + (1-t)(y - N(y)), (1 - (t+s))N(y)\rangle \ge 0.$$

This gives us that

(D) $\quad (1-(t+s))\big(t\langle x, N(y)\rangle + (1-t)\langle y - N(y), N(y)\rangle\big) \ge 0$ for all $0 \le s, t \le 1$.

Assume that $\langle x, N(y)\rangle \neq 0$. It follows from (C) that $\langle x, N(y)\rangle < 0$ and $\langle y - N(y), N(y)\rangle \ge 0$. So, there exists a number $0 < t < 1$ such that $t\langle x, N(y)\rangle + (1-t)\langle y - N(y), N(y)\rangle < 0$. Let s be chosen so that $0 < s < 1$ and $t + s < 1$. Then $1 - (t+s) > 0$, and the expression in (D) must be negative for t and s so chosen, which is a contradiction. Hence, $\langle x, N(y)\rangle = 0$.

So, (D) becomes $(1-(t+s))(1-t)\langle y - N(y), N(y)\rangle \ge 0$ for all $0 \le s, t \le 1$.

Pick any s and t where $\frac{1}{2} < s, t < 1$. Then $1 - (t+s) < 0$. Since $(1-t)\langle y - N(y), N(y)\rangle \ge 0$, it follows from (D) that $\langle y - N(y), N(y)\rangle = 0$. □

Property (F2) follows from Corollary 3.1 below.

Corollary 3.1. *Let C be a closed convex set in X, and let N be the nearest point mapping of X onto C. For $x \in X$ and $y \in C$, if $\langle x - N(x), y - N(x)\rangle \ge 0$, then*

$$N(tx + (1-t)y) = tN(x) + (1-t)y \text{ for all } 0 \le t \le 1.$$

Proof. Let $x \in X$ and $y \in C$ with $\langle x - N(x), y - N(x)\rangle \geq 0$. Note that $y = N(y)$. By Theorem 1.1, $\langle x - N(x), y - N(x)\rangle \leq 0$. So, $\langle x - N(x), y - N(x)\rangle = 0$. It follows from Theorem 3.1 that $N(tx + (1-t)y) = tN(x) + (1-t)y$. □

Theorem 3.2. *Let C be a closed convex set in X, and let N be the nearest point mapping of X onto C. Then property $(F4)$ holds.*

Proof. Let $x, y \in X$. Since $N(y) \in C$, it follows from Theorem 1.1 that $\langle x - N(x), N(y) - N(x)\rangle \leq 0$. If $\langle x - N(x), N(y) - N(x)\rangle = 0$, then we choose $s = 1$, and we are done. So, we assume that $\langle x - N(x), N(y) - N(x)\rangle < 0$. Similarly, we may assume that $\langle y - N(y), N(y) - N(x)\rangle > 0$. Since N is continuous, the function $\alpha\colon [0,1] \to \mathbb{R}$, given by $\alpha(t) = \langle tx + (1-t)y - N(tx + (1-t)y), N(y) - N(x)\rangle$ is continuous. Since $\alpha(0) > 0$ and $\alpha(1) < 0$, there exists s with $0 \leq s \leq 1$ such that $\alpha(s) = 0$. So, (F4) holds. □

4. Characterizations

The first two theorems of this section give sufficient conditions for a mapping F with a closed image set to be a nearest point mapping onto a closed convex set. Together with results from Section 3, this establishes two characterizations of such mappings. One additional characterization is obtained by assuming continuity of the mapping and relaxing property (F2).

Theorem 4.1. *Let $F\colon X \to X$ be a mapping with closed image set $C = F(X)$, and suppose that F satisfies properties $(F1)$, $(F2)$, and $(F4)$. Then C and F have the following properties.*

(i) C is the set of fixed points of F.

(ii) If $x \in X \setminus C$, then $F(x) \in \operatorname{bd} C$.

(iii) C is convex.

(iv) F is the metric projection of X onto C.

Proof. Property (i) follows from the idempotency of F.

For (ii), let $x \in X \setminus C$. By (i), $x \neq F(x)$. Now, $F(x) \in C$ and $\langle x - F(x), F(x) - F(x)\rangle = 0$. So, applying property (F2), we get that, for all $0 \leq t \leq 1$, $F(tx + (1-t)F(x)) = tF(x) + (1-t)F(x) = F(x)$. This equality implies that the only fixed point of F in the segment $[x, F(x)]$ is $F(x)$. It follows that $F(x) \in \operatorname{bd} C$.

For (iii), suppose that C is not convex. Let $y, z \in C$ with the open segment from y to z contained in $X \setminus C$. Let x be a point of the open segment from y to z. Suppose that $F(x)$ is collinear with y and z, and suppose, without loss of generality, that $F(x)$ belongs to the half-line starting at x and containing y. Then $\langle x - F(x), z - F(x)\rangle > 0$. So, by (F2), for all $0 \leq t \leq 1$, $F(tx + (1-t)z) = tF(x) + (1-t)F(z) = tF(x) + (1-t)z$. But this equality implies that all points in $[F(x), z]$ are in C, contradicting our choice of y and z. We point out that, under our assumption that C is not convex, we only needed property (F2) to get that $F(x)$ is not collinear with y and z. We will need both properties (F2) and (F4) to reach a contradiction, and establish the convexity of C.

Since $F(x)$ is not collinear with y and z, inside the triangle formed by the points y, z, and $F(x)$, one of the angles $\angle zF(x)x$ or $\angle yF(x)x$ must be acute. Assume, without loss of generality, that $\angle yF(x)x$ is acute. Then $\langle x - F(x), y - F(x)\rangle > 0$, giving us, by (F2), that F is linear on $[x, y]$. So, for all $0 < t \leq 1$, we get that

$$\langle tx + (1-t)y - F(tx + (1-t)y), F(y) - F(x)\rangle = t\langle x - F(x), y - F(x)\rangle > 0.$$

But then, by choosing $w = \frac{1}{2}y + \frac{1}{2}x$, we violate property (F4) for the points w and x, giving us a contradiction. Hence, C is convex.

To establish (iv), we also need both properties (F2) and (F4). Let $P\colon X \to C$ be the metric projection mapping onto C. Let x be in $X \setminus C$, and consider the points x, $F(x)$, and $P(x)$. Note that $F(P(x)) = P(x)$ and $P(F(x)) = F(x)$ since $P(x), F(x) \in C$. By Theorem 1.1, $\langle x - P(x), P(x) - F(x)\rangle \geq 0$. Clearly, we have that $\langle P(x) - F(x), P(x) - F(x)\rangle \geq 0$; so adding these two inequalities, we get that $\langle x - F(x), P(x) - F(x)\rangle \geq 0$. Hence, by property (F2), $F(tx + (1-t)P(x)) = tF(x) + (1-t)P(x)$ for all $0 \leq t \leq 1$.

Let $y = \frac{1}{2}x + \frac{1}{2}P(x)$. We have that $F(y) = \frac{1}{2}F(x) + \frac{1}{2}P(x)$. By property (F4), there exists a number s with $0 \leq s \leq 1$ such that $\langle sx + (1-s)y - F(sx + (1-s)y), F(y) - F(x)\rangle = 0$. Substituting, in this equation for y and $F(y)$, using the linearity of F between x and $P(x)$, and reducing, we get that $\frac{1}{4}(s+1)\langle x - F(x), P(x) - F(x)\rangle = 0$. Hence, $\langle x - F(x), P(x) - F(x)\rangle = 0$. It follows that $\|x - F(x)\|^2 + \|P(x) - F(x)\|^2 = \|x - P(x)\|^2$. From this, and recalling that P is the metric projection mapping, we have that $\|x - F(x)\| \leq \|x - P(x)\| \leq \|x - F(x)\|$. So, $F(x) = P(x)$.

□

Theorem 4.2. *Let $F\colon X \to X$ be a mapping with closed image set $C = F(X)$, and suppose that F satisfies properties $(F1)$, $(F2)$, and $(F3)$. Then C and F have the following properties.*

(i) C is the set of fixed points of F.

(ii) If $x \in X \setminus C$, then $F(x) \in \operatorname{bd} C$.

(iii) C is convex.

(iv) F is the metric projection of X onto C.

Proof. As in the proof of Theorem 4.1, properties (F1) and (F2) give us that F satisfies properties (i) and (ii).

For (iii), assume that C is not convex, and choose points y, z, and x as in the proof of Theorem 4.1. Again, we get that $F(x)$ is not collinear with y and

z, and as in the proof of Theorem 4.1, we assume that $\angle yF(x)x$ is acute. Then $\langle x - F(x), y - F(x)\rangle > 0$, giving us, by (F2), that F is linear on $[x, y]$. Since $x, y \in \overline{X \setminus C}$, it follows from property (F3) that $\langle x - F(x), y - F(x)\rangle = 0$, which is a contradiction. So, C is convex.

For (iv), let $P\colon X \to C$ be the metric projection mapping onto C. Let $x \in X \setminus C$, and consider the points x, $F(x)$, and $P(x)$. Similarly, as in the proof of Theorem 4.1, we get that $\langle x - F(x), P(x) - F(x)\rangle \geq 0$. Hence, by property (F2), $F(tx + (1-t)P(x)) = tF(x) + (1-t)P(x)$ for $0 \leq t \leq 1$. Since $P(x) \in \text{bd } C$, it follows that $\langle x - F(x), P(x) - F(x)\rangle = 0$, for otherwise, we violate property (F3). By Theorem 1.1, $\langle x - P(x), F(x) - P(x)\rangle \leq 0$. So, $0 \geq \langle x-P(x), F(x)-P(x)\rangle - \langle x-F(x), F(x)-P(x)\rangle = \langle F(x)-P(x), F(x)-P(x)\rangle$. It follows that $\|F(x) - P(x)\| = 0$, and therefore $F(x) = P(x)$.

□

Theorem 4.3 below follows from Corollary 3.1 and Theorems 3.1, 3.2, 4.1, and 4.2. It provides two generalizations of Theorem 1.4, where $C = F(X)$ is a closed convex cone, to C being a closed convex set.

Theorem 4.3. *Let $F\colon X \to X$ be a mapping with closed image set $C = F(X)$. Then C is a closed convex set and F is the metric projection of X onto C if and only if F satisfies properties* ($F1$)*,* ($F2$)*, and one of properties* ($F3$) *or* ($F4$)*.*

As the reader probably noticed, in Theorems 4.1, 4.2, and 4.3, we didn't assume continuity of F. However, all mappings in the examples in upcoming Section 5 are continuous. It would be of interest to know if, assuming the continuity of F, there is a weaker version of (F2) that could be used to obtain a result analogous to Theorem 4.1. We provide a weaker version of (F2) below that answers this question, providing a third characterization for a mapping to be the metric projection onto a closed convex set.

(F2′). If $x \in X \setminus C$, $y \in \text{bd } C$, $[x, y] \cap C = \{y\}$, and $\langle x - F(x), y - F(x)\rangle > 0$, then for all $0 \leq t \leq 1$, $F(tx+(1-t)y)$ belongs to the convex hull of $\{x, F(x), y\}$.

Theorem 4.4 below gives a generalization of Corollary 1 in [13] from closed convex cones to closed convex sets.

Theorem 4.4. *Let $F\colon X \to X$ be a continuous mapping with image set C. Then C is closed and convex, and F is the metric projection of X onto C if and only if F satisfies properties* ($F1$)*,* ($F2'$)*, and* ($F4$)*.*

Proof. The necessity of properties ($F1$), ($F2'$), and ($F4$) has been shown in Section 3. We establish sufficiency.

Observe first that the set C is closed, as the fixed point set of a continuous mapping.

Notice that for any $x \in X \setminus C$ we have that $F(x) \in \mathrm{bd}\, C$. If not, then there exists an $x \in X \setminus C$ such that $F(x)$ belongs to the interior of C. By the continuity of F, there exists open balls U and V such that $x \in U$, $F(x) \in V$, $U \subset X \setminus C$ and $F(U) \subset V \subset C$. Also, there exist a $y \in [x, F(x)] \cap \mathrm{bd}\, C$ such that $y \notin U \cup V$, and y is the closest such point to x. We apply property $(F2')$ for x and y, and find that for all $u \in [x, F(x)] \cap U$, we have $F(u) \in [x, F(x)] \cap V$. This contradicts $(F4)$.

Assume that C is not convex. Then there exists $y, z \in C$ such that $[y, z] \cap C = \{y, z\}$. By the continuity of F, there exists $x \in [y, z]$ such that $F(x)$ is neither y nor z. Then $x \notin C$ and we consider two cases.

If $F(x)$ is collinear with y and z, then we can assume that $F(x)$ belongs to the half-line starting at x and containing y. Applying property $(F2')$ for x and y, as before, we get a contradiction to $(F4)$.

If $F(x)$ is not collinear with y and z. Inside the triangle with vertices y, $F(x)$, and z, one of the angles $\angle zF(x)x$ and $\angle xF(x)y$ must be acute. Assume that $\angle xF(x)y$ is acute. Then $\langle x - F(x), y - F(x)\rangle = a > 0$. By the continuity of F, we can choose an open ball U with $x \in U$ such that $\overline{U} \subset X \setminus C$,

$$\langle u - F(u), y - F(x)\rangle \geq \frac{a}{2} > 0\,, \text{ for all } u \in \overline{U}\,, \tag{1}$$

$$\langle u - F(u), y - F(u)\rangle \geq \frac{a}{2} > 0\,, \text{ for all } u \in \overline{U}\,, \tag{2}$$

and

$$\langle u - F(u), F(u) - F(x)\rangle > 0\,, \text{ for all } u \in \overline{U} \cap [x, y]\,. \tag{3}$$

Let x_1 be the intersection of $[x, y]$ with the boundary of U. By property $(F2')$, for each $u \in [x, x_1]$, $F(u)$ belongs to the closed convex hull of $\{x, F(x), y\}$. Also, for each $u \in [x, x_1]$, since $x_1 \in [u, y]$, it follows from $(F2')$ and (2) that $F(x_1)$ belongs to the closed convex hull of $\{u, F(u), y\}$. Therefore, the angle made by the vectors $u - F(u)$ and $F(x_1) - F(u)$ is smaller than the angle made by $u - F(u)$ and $y - F(u)$, and hence

$$\langle u - F(u), F(x_1) - F(u)\rangle > 0\,. \tag{4}$$

Adding the inequalities (3) and (4) implies that for all $u \in [x, x_1]$ we have

$$\langle u - F(u), F(x_1) - F(x)\rangle > 0\,, \tag{5}$$

which contradicts property (F4) for the points x and x_1. Hence, C is convex.

Assume that F is not the nearest point mapping. Then, by Theorem 1.1, there exists $x \in X \setminus C$ and $y \in C$ such that $\langle x - F(x), y - F(x)\rangle = a > 0$. Without loss of generality, we assume that y belongs to the boundary of C, and $[x, y] \cap C = \{y\}$. By the continuity of F, we can choose an open ball W containing x with properties analogous to (1), (2) and (3) above. Choosing $x_1 = [x, y] \cap \mathrm{bd}\, W$ gives us, as before, that for all $u \in [x_1, x]$ inequality (5)

holds, which contradicts property (F4) for the points x and x_1.
Therefore, we conclude that F is the metric projection onto C. □

5. Examples

The following examples show that properties (F1), (F2), and one of (F3) or (F4) are necessary in the characterization of metric projections onto closed convex sets, in the sense that the omission of any one of the properties yields an example of a retraction onto a closed convex set that is not the metric projection mapping.

Example 5.1. *A mapping $F\colon X \to X$ can have a closed convex image, and satisfy properties* (F1) *and* (F3), *yet not be the nearest point mapping onto its image.*

Proof. Let C be the closed convex set in $\mathbb{R}^2$ whose boundary is the ellipse $E = \{(s,t) \mid s^2 + 4t^2 = 4\}$. Let F be radial projection of $\mathbb{R}^2$ onto E for points (s,t) where $s^2+4t^2 > 4$; and let F be the identity map on C. Specifically, for points (s,t) where $s^2 + 4t^2 > 4$, let

$$F(s,t) = \frac{2}{\sqrt{s^2+4t^2}}(s,t).$$

It is easy to see that F is idempotent, that $C = F(X)$ is closed and convex, and that F is not the metric projection mapping onto C.

To see that F satisfies property (F3), we let x and y be points of $\overline{X \setminus C}$. If either both x and y are in C or $F(x) = F(y)$, then clearly (F3) holds.

Suppose that $x \notin C$, and that x and y lie on the same line through the origin with $F(x) = -F(y)$. Then $\langle x - F(x), F(y) - F(x)\rangle = \langle x - F(x), -2F(x)\rangle \neq 0$. Choose $0 \le t \le 1$, where $tx + (1-t)y = F(x)$. Then $F(tx + (1-t)y) = F(x)$, and $tF(x) + (1-t)F(y) = tF(x) - (1-t)F(x) = (2t-1)F(x)$. So, if these points are the same point, we must have that $t = 1$, giving that $x = F(x)$; in which case $x \in C$, a contradiction. So, property (F3) holds in this case.

Lastly, suppose that $x \notin C$, and that $F(x)$ and $F(y)$ are on the ellipse E, but not on the same line through the origin. Since the open segment from $F(x)$ to $F(y)$ lies in the interior of C, and part of the open segment from x to y lies in the complement of C, it is not the case that $F(tx+(1-t)y) = tF(x)+(1-t)F(y)$ for all $0 \le t \le 1$.

Next we observe that neither property (F4) nor (F2) is satisfied.

To see that property (F4) is not satisfied, we pick the points $x = (2,\sqrt{3})$ and $y = (4,0)$, and will observe that for no point z between x and y is $z - F(z)$ orthogonal to $F(x) - F(y)$. Now, $F(2,\sqrt{3}) = (1, \frac{\sqrt{3}}{2})$ and $F(4,0) = (2,0)$. The slope of the line segment from $(1,\frac{\sqrt{3}}{2})$ to $(2,0)$ is $-\frac{\sqrt{3}}{2}$. Suppose there is a point $z = (s, \frac{2}{\sqrt{3}}s)$ and $z = t(2,\sqrt{3}) + (1-t)(4,0)$ for some number t. Equating the two, and simplifying, we get that $t = \frac{8}{7}$. So, no z between x and y can satisfy the orthogonal relation in property (F4).

To see that property (F2) is not satisfied, we pick the points $x = (2, \sqrt{3})$ and $y = F(y) = (2, 0)$. We observe that $\langle x - F(x), y - F(x) \rangle \geq 0$; specifically, $\langle (2, \sqrt{3}) - (1, \frac{\sqrt{3}}{2}), (2, 0) - (1, \frac{\sqrt{3}}{2}) \rangle = \frac{1}{4} \geq 0$. But clearly F does not project the segment $[x, y]$ linearly onto the segment $[F(x), F(y)]$. □

Example 5.2. *A mapping $F\colon X \to X$ can have a closed convex image, and satisfy properties (F1) and (F2), yet not be the nearest point mapping onto its image.*

Proof. Let C be the closed convex cone in $\mathbb{R}^2$ given by $C = \{(c, 0) \mid c \geq 0\}$. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined as follows.

$$F(s,t) = \begin{cases} (0,0) & \text{if } s \leq 0 \text{ and } -s \geq t \\ (s,0) & \text{if } s \geq 0 \text{ and } t \leq 0 \\ (s+t,0) & \text{if } t \geq \max\{0, -s\}. \end{cases}$$

Note that F is the identity mapping on C. It is clear that F is idempotent, that $C = F(X)$ is closed and convex, and that F is not the metric projection mapping onto C.

To see that F satisfies property (F2), we consider points in the three regions over which F has different rules.

Suppose that $y \in C$ and x is a point in the first region over which F is defined. Let $x = (s, t)$ with $s < 0$, $t \leq -s$, and let $y = (c, 0) \in C$. We note that $\langle (s, t) - F(s, t), (c, 0) - F(s, t) \rangle = \langle (s, t), (c, 0) \rangle = cs < 0$. So, property (F2) holds by default for each such pair of points.

For points in the remaining two domain regions of F, we observe that these two regions are convex and contain C. Also, by definition, F is linear on these regions. It follows that F is linear on each segment lying in these regions. So, F satisfies property (F2).

Letting $x = (0, 1)$ and $y = (1, 1)$, it is easy to see that neither property (F3) nor (F4) is satisfied by F. □

Example 5.3. *A mapping $F\colon X \to X$ can have a closed convex image, and satisfy properties (F1) and (F4), yet not be the nearest point mapping onto its image.*

Proof. Consider

$$C = \left\{ (s,t) \in \mathbb{R}^2 \mid \frac{s^2}{2} + t^2 \leq 1 \right\}.$$

The boundary of C is the ellipse with parametric equations

$$s = \sqrt{2} \cos v\,,\ t = \sin v\,,$$

which is the member of the orthogonal curvilinear coordinate system

$$s = \cosh u\, \cos v\,,\ t = \sinh u\, \sin v\,,$$

corresponding to $u = a$, where $\cosh a = \sqrt{2}$ and $\sinh a = 1$. We define the mapping $F : \mathbb{R}^2 \to C$ as follows.

$$F(s,t) = \begin{cases} (s,t) & \text{if } (s,t) \in C \\ (\frac{\sqrt{2}s}{|s|}, 0) & \text{if } t = 0 \text{ and } |s| > \sqrt{2} \\ (0, \frac{t}{|t|}) & \text{if } s = 0 \text{ and } |t| > 1 \\ (\sqrt{2}\cos v, \sin v) & \text{if } (s,t) \notin C,\ s \neq 0,\ t \neq 0, \\ & s = \cosh u \cos v, \text{ and } t = \sinh u \sin v. \end{cases}$$

It is clear that F is continuous and idempotent, but not the metric projection mapping onto C. To show that F satisfies property (F4), we consider $F(x)$ and $F(y)$ on the ellipse and in the first quadrant. The computations for the other cases are similar. Let $0 < v_1 < v_2 < \frac{\pi}{2}$, $F(x) = (\sqrt{2}\cos v_1, \sin v_1)$, $F(y) = (\sqrt{2}\cos v_2, \sin v_2)$. Then there exists v_3 such that $v_1 < v_3 < v_2$ and the tangent line at the point $(\sqrt{2}\cos v_3, \sin v_3)$ to the boundary of C is parallel to $F(x) - F(y)$. Then the normal line at the same point is perpendicular to $F(x) - F(y)$, which means

$$\langle (\cos v_3, \sqrt{2}\sin v_3)\,,\ F(x) - F(y) \rangle = 0\,.$$

Consider v_4 such that $v_3 < v_4 < v_2$, and

$$\frac{\sin v_3}{\cos v_3} < \frac{\sin v_4}{\cos v_4} < \frac{\sqrt{2}\sin v_3}{\cos v_3}\,.$$

Consider the function $f : [a, +\infty) \to \mathbb{R}$ defined as follows.

$$f(u) = \begin{cases} \dfrac{\sqrt{2}\sin v_4}{\cos v_4} & \text{if } u = a \\[2ex] \dfrac{(\sinh u - 1)\sin v_4}{(\cosh u - \sqrt{2})\cos v_4} & \text{if } u > a. \end{cases}$$

It is easy to see that f is continuous and decreasing. From the inequalities above, we get that

$$f(a) > \frac{\sqrt{2}\sin v_3}{\cos v_3} \quad \text{and} \quad \lim_{u\to+\infty} f(u) = \frac{\sin v_4}{\cos v_4} < \frac{\sqrt{2}\sin v_3}{\cos v_3}\,.$$

Therefore, there exists $u_1 > a$ such that

$$f(u_1) = \frac{\sqrt{2}\sin v_3}{\cos v_3}\,.$$

The point z satisfying the orthogonal property in property (F4) can be given as

$$z = (\cosh u_1 \cos v_4)\,,\ \sinh u_1 \sin v_4)\,.$$

It is clear that F doesn't satisfy property (F2), since F is not linear on the segment $[x, F(x)]$ when $x \notin C$. $\square$

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