.

# INTERNALLY $\mathcal{K}$-LIKE SPACES AND INTERNAL INVERSE LIMITS

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Abstract. We establish equivalences between compacta that admit mappings that limit to the identity, and compacta that are inverse limits of the images under these maps. Our results have relationships to Mardešić's and Segal's equivalence between polyhedra-like compacta and inverse limits of polyhedra, to the Anderson-Choquet Embedding Theorem, to approximative absolute neighborhood retracts, and to continua that are approximated from within as defined by C.A. Eberhart and J.B. Fugate.

## 1. Introduction and definitions

We study conditions that allow, up to homeomorphism, a compact metric space $X$ to be represented as the inverse limit of compact spaces "from within", that is, with factor spaces contained in $X$, and threads, as sequences in $X$, converging to their corresponding points in $X$. This idea was initiated in [2] and continued by others (see e.g. [17]). Our main result leads to the conclusion that such a representation is possible whenever $X$ has maps limiting to the identity map. If the factor spaces are much simpler than $X$ itself, this investigation may provide a useful tool to study such a space $X$. In a forthcoming paper, we will apply some of the results presented here to study the special case when the maps are retractions or $r$-maps.

A *compactum* is a compact metric space. A *continuum* is a connected compactum. A *mapping* (or *map*) is a continuous function. For composition of functions $f$ after $g$, we will use both the notations $f \circ g$ and $fg$.

We use *inverse sequences* and *inverse limits* throughout this paper. Definitions and general properties of these notions can be found in [10, p.7-14], [11, Sections 2.1-2.3], or [15, Part One, Sec.II]. Two inverse sequences $\{X_n, f_n^{n+1}\}$ and $\{Y_n, g_n^{n+1}\}$ are said to be *topologically equivalent* if for each $n \geq 1$, there exists a homeomorphism $h_n\colon X_n \to Y_n$ such that $g_n^{n+1} \circ h_{n+1} = h_n \circ f_n^{n+1}$.

If $A_n$ is a sequence of non-empty subsets of $X$ converging with respect to the Hausdorff distance, then $\operatorname{Lim} A_n$ denotes the Hausdorff limit of the $A_n$'s.

In the spirit of the Anderson-Choquet Embedding Theorem (see [15, Th.2.10]), the following definition was introduced in [17, p.104].

Let $\{X_n\}$ be a sequence of closed sets in a metric space $(X, d)$ and $\{f_n : X_{n+1} \to X_n\}$ be a sequence of maps. We say the inverse sequence $\{X_n, f_n\}$ *converges* in $X$ provided that

(1) Each thread $(x_1, x_2, \dots)$ of the inverse sequence is a convergent sequence in $X$,
(2) the assignment $f$ defined by $(x_1, x_2, \dots) \mapsto \lim x_n$ is a continuous map from $\varprojlim\{X_n, f_n\}$ to $X$, and
(3) the projections $\pi_n : \varprojlim\{X_n, f_n\} \to X_n$ converge uniformly to $f\colon \varprojlim\{X_n, f_n\} \to \operatorname{Lim} X_n$.

If, additionally, $f$ is an embedding, we say that $\{X_n, f_n\}$ *converges exactly* in $X$ to $f(\varprojlim\{X_n, f_n\})$.

2010 *Mathematics Subject Classification.* Primary 54F15, 54F65; Secondary 54A20, 54D80.

If $\{X_n, f_n\}$ converges exactly in $X$ and $f(\varprojlim\{X_n, f_n\}) = X$, that is, $f$ is a homeomorphism onto the entire space $X$, we call $\{X_n, f_n\}$ an *internal inverse limit structure* on $X$. Identifying $\varprojlim\{X_n, f_n\}$ with $X$ by $f$, in this case, we have the projection maps in (3) converging to the identity map on $X$.

We note that this terminology is slightly different from that used in [17]. What we call here an *exactly convergent inverse sequence*, in [17] was referred to as a *convergent inverse sequence*.

Let $X$ be a space and $A$ a non-empty subspace of $X$. For any map $f : A \to X$ we write $\tilde{d}(f) = \sup\{d(x, f(x)) \mid x \in A\}$.

Let $\mathcal{K}$ be a class of compacta. We say that $X$ is *internally $\mathcal{K}$-like* if for each $\epsilon > 0$, there is a $K \in \mathcal{K}$ with $K \subset X$, and a map $f \colon X \to K$ such that $\tilde{d}(f) < \epsilon$. We say that $X$ is *internally $\mathcal{K}$-representable* if $X$ has an internal inverse limit structure with factor spaces in $\mathcal{K}$.

We note that assigning, to any class $\mathcal{K}$ of compacta, the class of all internally $\mathcal{K}$-like compacta, is a closure operator defining a topology on the class of compacta. Such structures have been recently studied in [1], [6].

We point out that $X$ being internally $\mathcal{K}$-like is equivalent to Eberhart's and Fugate's [9] terminology that $X$ *can be approximated from within by compacta that belong to* $\mathcal{K}$.

Similar terminology was used in [5] and in [16]. Specifically, Oversteegen and Prajs [16] proved that for any family $\mathcal{K}$ of graphs, $X$ is confluently $\mathcal{K}$-like if and only if $X$ is confluently $\mathcal{K}$-representable. Let $\mathcal{LC}$ be the class of locally connected continua. Charatonik, Charatonik, and Prajs [5] proved that $X$ is atriodic and confluently $\mathcal{LC}$-like if and only if $X$ is confluently $\mathcal{LC}$-representable if and only if $X$ is either a Knaster type continuum or a solenoid.

A collection $\mathcal{S}$ of convergent sequences in a space $X$ is called *uniformly convergent* provided for every $\epsilon > 0$ there is an $N$ such that $d(s_m, \lim_{n\to\infty} s_n) < \epsilon$ for each $m > N$ and $\{s_n\} \in \mathcal{S}$. In a similar way, we define *uniformly Cauchy* collections of sequences. Clearly, in a complete metric space a collection of sequences is uniformly convergent if and only if it is uniformly Cauchy.

The proofs of the following three propositions are easy, and therefore, they are omitted.

**Proposition 1.** *For each $n \geq 1$, let $X_n$ be a compact non-empty subset of a complete metric space $X$, and $f_n : X_{n+1} \to X_n$ be a map. The inverse sequence $\{X_n, f_n\}$ converges in $X$ if and only if the threads of $\{X_n, f_n\}$ form a uniformly convergent collection of sequences in $X$.*

**Proposition 2.** *For each $n \geq 1$, let $X_n$ be a compact non-empty subset of a complete metric space $X$, and $f_n : X_{n+1} \to X_n$ be a map. The inverse sequence $\{X_n, f_n\}$ converges exactly in $X$ if and only if it converges in $X$ and the function $(x_1, x_2, \ldots) \mapsto \lim x_n$ from $\varprojlim\{X_n, f_n\}$ to $X$ is one-to-one.*

**Proposition 3.** *Let $\{X_n, f_n\}$ be an inverse sequence converging in a compact space $X$. If $\operatorname{Lim} X_n = Y \subset X$, then the map $f$ in the definition of the convergence of inverse sequences is a surjection onto $Y$.*

## 2. Internally $\mathcal{K}$-like and internally $\mathcal{K}$-representable compacta

In this section we present the main result of the paper, Theorem 2 below. This theorem allows the detection of exactly convergent inverse sequences in a compact metric space. Though we can think of many other possible applications of this result, our focus remains on internal inverse limit structures of spaces.

In [13], Mardešić and Segal proved that a compactum is polyhedra-like if and only if it is an inverse limit of polyhedra. In this section, we show that for any class of compacta $\mathcal{K}$, a compactum is internally $\mathcal{K}$-like if and only if it admits an internal inverse limit structure with factor spaces in $\mathcal{K}$. Furthermore, the internal structure of our setting allows for a direct relationship between the maps on $X$ that converge to the identity map and the bonding maps in the inverse limit. Specifically, if $X$ is internally $\mathcal{K}$-like with maps $\{f_n\colon X \to X\}$, the bonding maps in the resulting inverse limit structure are restrictions of some of the maps $f_n$. The bonding maps in Mardešić's and Segal's equivalence are constructed, using the triangulable structure of the polyhedra, so as to $\epsilon$-commute with the $\epsilon$-maps. So, the $\epsilon$-maps in their setting are not as closely related to the bonding maps in the inverse limit sequence.

The next theorem, originally presented in [9, Theorem 1.1], explains some of our interest in internally $\mathcal{K}$-like compacta. Furthermore, in this theorem the term "internally $\mathcal{K}$-like" can be equivalently replaced with "internally $\mathcal{K}$-representable" by Theorem 3 below. By Bellamy's classic example of a tree-like continuum without the fixed point property [3], the word "internally" cannot be omitted.

**Theorem 1** (Eberhart and Fugate)**.** *If $\mathcal{K}$ is a class of continua with the fixed point property, and $X$ is an internally $\mathcal{K}$-like continuum, then $X$ has the fixed point property.*

A set $\{f_\alpha\colon X_\alpha \to Y_\alpha \mid \alpha \in \Gamma\}$ of maps is *uniformly equicontinuous* if for each $\epsilon > 0$, there exists $\delta > 0$ such that $d(f_\alpha(x), f_\alpha(y)) < \epsilon$ for all $\alpha \in \Gamma$ and all $x, y \in X_\alpha$ such that $d(x, y) < \delta$.

The next theorem is the main result of the paper. Given a compact space $X$ and a collection of maps $\{f_\alpha\colon X_\alpha \to X \mid X_\alpha \in 2^X, \alpha \in \Gamma\}$, this theorem provides a general condition under which the maps $f_\alpha$ can be used to define inverse systems converging exactly in $X$.

**Theorem 2.** *Let $X$ be compact space and $\{Y_n\}$ a sequence of closed subsets of $X$ with $\operatorname{Lim} Y_n = Y \subset X$, and let $\mathcal{F}$ be a collection of maps. Suppose*

*for each $\epsilon > 0$ there is an $N(\epsilon) \in \mathbb{N}$ such that for each $n > N(\epsilon)$, there exists a uniformly equicontinuous sequence of maps $f_n^m : Y_m \to Y_n$ in $\mathcal{F}$, for $m > n$, with $\tilde{d}(f_n^m) < \epsilon$.*

*Then there are a subsequence $\{Y_{n_k}\}$ and maps $g_k : Y_{n_{k+1}} \to Y_{n_k}$ in $\mathcal{F}$ such that the inverse sequence $\{Y_{n_k}, g_k\}$ converges exactly in $X$ to $Y$. If, additionally, $Y_n \subset Y$ for each $n$, then $\{Y_{n_k}, g_k\}$ is an internal inverse limit structure on $Y$.*

*Proof.* Fix a number $\alpha > 0$ such that $\alpha < \operatorname{diam} X$. Assume the conditions of the theorem hold for $Y_n$. Note that any subsequence of $Y_n$ also satisfies these conditions for the same set of functions $\mathcal{F}$, and it suffices to prove the theorem for a subsequence of $Y_n$. Therefore, by replacing $Y_n$ with a properly selected subsequence of $Y_n$, without loss of generality, we assume $N(\alpha/2^{n+2}) = n$. Let $\mathcal{B}$ be a countable basis of the topology on $X$. List all pairs of the members of $\mathcal{B}$ having disjoint closures as $(A_1, B_1)$, $(A_2, B_2)$, $\ldots$, and define $\sigma_n = \inf\{d(a, b) \mid a \in A_n,\, b \in B_n\}$ for each $n \geq 1$. Note that $\sigma_n > 0$ for each $n \geq 1$ by assumption. By rearranging the order of this last sequence, we assume $\sigma_1 > \alpha$.

For each $m, n \in \mathbb{N}$ with $m > n$ let $f_n^m : Y_m \to Y_n$ be a fixed member of $\mathcal{F}$ such that $\tilde{d}(f_n^m) < \alpha/2^{n+2}$, and for each $n \geq 1$ the collection $\mathcal{F}_n = \{f_n^m \mid m > n\}$ is uniformly equicontinuous. Define $\mathcal{F}_0 = \bigcup\{\mathcal{F}_n \mid n \in \mathbb{N}\}$. In the construction of the desired inverse limit we only use members of $\mathcal{F}_0$. More specifically, if $n_j$ and $n_{j+1}$ are consecutive indexes for a selected subsequence, then $f_{n_j}^{n_{j+1}}$ is the corresponding bonding map. Thus, any subsequence $n_j$ of positive integers uniquely determines an inverse sequence having $Y_{n_j}$'s as factor spaces. Before we proceed with selecting such a subsequence, notice that for an arbitrary inverse sequence of this type, $\{Y_{n_j}, f_{n_j}^{n_{j+1}}\}$, and arbitrary thread $(x_1, x_2, \ldots)$ of it, for $m > n$ we have $d(x_m, x_n) < \sum_{i=n}^{m-1} \alpha/2^{j_i+2} < \sum_{i=n}^{\infty} \alpha/2^{j_i+2} = \alpha/2^{j_n+1}$. Thus all these threads are uniformly Cauchy, and they uniformly converge in $X$. Consequently, each such inverse sequence $\{Y_{n_j}, f_{n_j}^{n_{j+1}}\}$ converges in $X$ by Proposition 1, and each thread $(x_1, x_2, \ldots)$ of it satisfies

$$(*) \qquad d(x_i, \lim_{j\to\infty} x_j) < \alpha/2^{j_i+1} \text{ for each } i \in \mathbb{N}.$$

Moreover, by Proposition 3, for each such inverse limit, the map $f$ guaranteed in the definition of its convergence is surjective. To complete the proof, among these inverse sequences we need to select one that converges exactly. In view of Proposition 2, it suffices to ensure that the assignment $(x_1, x_2, \ldots) \mapsto \lim x_n$, for the threads $(x_1, x_2, \ldots)$ of a selected inverse sequence, is one-to-one.

Our definition is inductive. We introduce a sequence of subsequences of positive integers having the form
$\mathbf{s}_1 = (n_1, n_1+1, n_1+2, \ldots)$
$\mathbf{s}_2 = (n_1, n_2, n_2+1, n_2+2, \ldots)$
$\mathbf{s}_3 = (n_1, n_2, n_3, n_3+1, n_3+2, \ldots)$
$\cdots$

At each step we examine the inverse system $\mathbf{S}_k$ associated with the sequence $\mathbf{s}_k$. The sequences $\mathbf{s}_k$ produce the needed subsequence $\mathbf{s} = (n_1, n_2, n_3, \ldots)$, and its associated inverse system $\mathbf{S}$ defines the desired one that converges exactly.

Letting $n_1 = 1$, we have $\mathbf{s}_1 = (1, 2, \ldots)$ and $\mathbf{S}_1 = \{Y_n, f_n^{n+1}\}$. For each thread $(x_1, x_2, \ldots)$ in this inverse system, and each $i$ we have $d(x_i, \lim_{j\to\infty} x_j) < \alpha/4$ by $(*)$. This last inequality stands in any inverse system $\mathbf{S}_k$ and in $\mathbf{S}$, defined according to the pattern described above. Suppose $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ are threads in any of these inverse systems with $x_i \in A_1$ and $y_i \in B_1$ for some $i$. Then $d(\lim_{j\to\infty} x_j, \lim_{j\to\infty} y_j) \geq d(x_i, y_i) - d(x_i, \lim_{j\to\infty} x_j) - d(y_i, \lim_{j\to\infty} y_j) > \sigma_1 - \alpha/4 - \alpha/4 > \alpha/2 > 0$. Thus, if two threads in any of these inverse systems have their single coordinate in $A_1$ and $B_1$, respectively, then their limits are different.

Suppose sequences $\mathbf{s}_1, \ldots, \mathbf{s}_m$, and consequently the associated inverse systems $\mathbf{S}_1, \ldots, \mathbf{S}_m$, are already defined so that they satisfy the following. If two threads $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ of $\mathbf{S}_m$, or of any possible subsequent inverse system $\mathbf{S}_k$ for $k > m$ defined according to the above pattern, satisfy $x_i \in A_l$ and $y_i \in B_l$ for some $l \in \{1, \ldots, m\}$, then their limits are different. We select an integer $n_{m+1} > n_m$ satisfying the following two inequalities.

$$(1) \quad \alpha/2^{n_{m+1}} < \sigma_{m+1} \quad \text{and} \quad (2) \quad \alpha/2^{n_{m+1}} < \psi_{m+1}$$

We will comment momentarily on how $\psi_{m+1}$ is chosen. First, we note that Inequality (1) ensures that if two threads $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ of the to-be-constructed $\mathbf{S}_{m+1}$, or of any possible subsequent inverse system $\mathbf{S}_k$ for $k > m+1$ defined according to the above pattern, satisfy $x_i \in A_{m+1}$ and $y_i \in B_{m+1}$, for $i \geq m+1$, then their limits are different. Indeed, for such threads we have $d(x_i, \lim_{j\to\infty} x_j) < \alpha/2^{n_{m+1}+1}$ and $d(x_i, \lim_{j\to\infty} x_j) < \alpha/2^{n_{m+1}+1}$ by $(*)$. Consequently, $d(\lim_{j\to\infty} x_j, \lim_{j\to\infty} y_j) \geq d(x_i, y_i) - d(x_i, \lim_{j\to\infty} x_j) - d(y_i, \lim_{j\to\infty} y_j) > \sigma_{m+1} - \alpha/2^{n_{m+1}+1} - \alpha/2^{n_{m+1}+1} > \alpha/2^{n_{m+1}} - \alpha/2^{n_{m+1}} = 0$.

To choose $\psi_{m+1}$, consider the already defined maps $g_l = f_{n_l}^{n_{l+1}}$ for $l \in \{1, \ldots, m-1\}$, and their compositions $h_l = g_l \circ g_{l_1} \circ \cdots \circ g_{m-1}$. The maps $h_1, \ldots, h_{m-1}$, being a finite collection and having compact domains, are uniformly equicontinuous. Thus, there exists a $\xi > 0$ such that for each $x$ and $y$ in the domain of any $h_j$, if $d(x,y) < \xi$, then $d(h_j(x), h_j(y)) < \sigma_{m+1}/2$. Fix such a $\xi > 0$, additionally assuming $\xi < \sigma_{m+1}$. By assumption, the maps in $\mathcal{F}_m$ are uniformly equicontinuous, and thus there is a $\psi_{m+1} > 0$ such that if $x, y \in Y_{m+k}$ and $d(x,y) < \psi_{m+1}$, then $d(f_m^{m+k}(x), f_m^{m+k}(y)) < \xi$. We fix such a $\psi_{m+1}$.

Inequality (2) ensures that if two threads $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$ of the to-be-constructed $\mathbf{S}_{m+1}$, or of any possible subsequent inverse system $\mathbf{S}_k$ for $k > m+1$ defined according to the above pattern, satisfy $x_i \in A_{m+1}$ and $y_i \in B_{m+1}$, for $i \le m$, then their limits are different. Indeed, suppose $\lim_{j\to\infty} x_j = \lim_{j\to\infty} y_j$ for $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$ in $\mathbf{S}_{m+1}$ constructed according to inequality (2). Then $d(x_{m+1}, y_{m+1}) \le d(x_{m+1}, \lim_{j\to\infty} x_j) + d(y_{m+1}, \lim_{j\to\infty} y_j) < \alpha/2^{n_{m+1}+1} + \alpha/2^{n_{m+1}+1} = \alpha/2^{n_{m+1}} < \psi_{m+1}$. Consequently, $d(x_m, y_m) < \xi < \sigma_{m+1}$. Also, for $i < m$, $d(x_i, y_i) = d(h_i(x_m), h_i(y_m)) < \sigma_{m+1}$. Thus, by the definition of $\sigma_{m+1}$, it is impossible that $x_i \in A_{m+1}$ and $y_i \in B_{m+1}$ for some $i \le m$.

We choose $n_{m+1}$ to satisfy both inequalities (1) and (2). This ensures that if two threads $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$ of the associated $\mathbf{S}_{m+1}$, or of any possible subsequent inverse system $\mathbf{S}_k$ for $k > m+1$, satisfy $x_i \in A_{m+1}$ and $y_i \in B_{m+1}$, for all $i \ge 1$, then their limits are different.

By induction, we have defined a sequence $(n_1, n_2, \dots)$. The associated inverse system $\mathbf{S}$ has the property that for each $n$ and $i$ if two threads $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$ of $\mathbf{S}$ satisfy $x_i \in A_n$ and $y_i \in B_n$, then the limits of these threads are different. Hence the assignment $(x_1, x_2, \dots) \mapsto \lim x_n$ for this system is one-to-one. To complete the proof notice that if $Y_n \subset Y$ for each $n$, then the above constructed inverse sequence $\mathbf{S}$ is an internal inverse limit structure on $Y$. □

**Corollary 1.** *Let $X$ be a compactum. Each sequence of maps $\{f_n : X \to X\}$ converging uniformly to the identity map on $X$ has a subsequence $\{g_k = f_{n_k}\}$ such that $\{g_k(X), g_k|_{g_{k+1}(X)}\}$ is an internal inverse limit structure on $X$.*

*Proof.* Clearly, by hypothesis, $\operatorname{Lim} f_n(X) = X$. For $n \ge 1$, let $X_n = f_n(X)$. For $m > n$, let $f_n^m \colon X_m \to X_n$ be defined by $f_n^m = f_n|_{X_m}$. Let $\mathcal{F} = \{f_n^m \mid m > n\}$. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be large enough so that for $n \ge N$, $d(f_n, \mathrm{id}) < \epsilon$. Fix $n \ge N$. By uniform continuity of $f_n$, let $\delta > 0$ be chosen so that $d(f_n(x), f_n(y) < \epsilon$ whenever $x, y \in X$ and $d(x,y) < \delta$. It follows that the sequence $\{f_n^m\}_{m>n}$ of members of $\mathcal{F}$ is uniformly equicontinuous and that $\tilde{d}(f_n^m) < \epsilon$ for $m > n$. Thus, the corollary follows from Theorem 2. □

**Remark.** The above corollary can also be shown using Morton Brown's theorem, [4, Theorem 3, p. 481]. Indeed, we use the inverse sequence $X \xleftarrow{\mathrm{id}} X \xleftarrow{\mathrm{id}} \dots$, whose inverse limit is obviously homeomorphic to $X$, and replace the identity map with a subsequence $f_{n_k}$ of the sequence $f_n$. Then restrict each map $f_{n_k}$ to the set $f_{n_{k+1}}(X)$. The details of this argument are left to the reader.

The following theorem is an immediate consequence of Corollary 1.

**Theorem 3.** *Let $\mathcal{K}$ be any class of compacta and $X$ be any compactum. The compactum $X$ is internally $\mathcal{K}$-like if and only if $X$ is internally $\mathcal{K}$-representable.*

We restate the above result for several well studied classes $\mathcal{K}$, for which this application may be particularly meaningful.

**Corollary 2.** *For any continuum $X$, $X$ is internally arc-like (tree-like, graph-like, polyhedra-like, AR-like, ANR-like) if and only if $X$ is internally arc-representable (tree-representable, graph-representable, polyhedra-representable, AR-representable, ANR-representable).*

Internally AR-like and ANR-like compacta are natural, large classes of spaces with potential for applications. They are of special interest in relation to the past study, by various authors (see [7], [8], & [14]), of *approximative absolute (neighborhood) retacts*, generalizations of Borsuk's AR's and ANR's. A compactum $X$ is called an *approximative absolute retract* (written AAR) provided whenever $X$ is embedded in a compactum $Y$, and $X'$ is the embedded copy, we have that for each $\epsilon > 0$, there is a map $f : Y \to X'$ with $\tilde{d}(f|_{X'}) < \epsilon$. If the map $f$ can be defined on some neighborhood of $X'$ but not necessarily on entire $Y$, we say that $X$ in an *approximative absolute neighborhood retract* (written AANR). Observe that every internally AR-like continuum is internally ANR-like.

We rephrase a result and a question from [7]. Our terminology adds clarity and better underlines their significance.

**Proposition 4** ([7], Proposition 5.2)**.** *Every internally AR-like (internally ANR-like) compactum is an AAR (an AANR).*

The question about the converse is intriguing.

**Question 1** ([7], part of Question 5.3)**.** *Is every AAR (AANR) an internally AR-like (internally ANR-like) compactum?*

Below, Theorems 4 and 5, we answer this question in dimension 1. The general case is still open. Recall that $\mathcal{LC}$ stands for the class of locally connected continua. We make the following obvious observation.

**Proposition 5.** *Every connected AANR is internally $\mathcal{LC}$-like.*

*Proof.* Let $X$ be an AANR continuum. We embed $X$ into the Hilbert cube $H$ with the embedded copy $X'$. Given an $\epsilon > 0$, we take a locally connected, compact, connected neighborhood $N$ of $X'$ in $H$ such that there exist a map $f : N \to X'$ with $\tilde{d}(f|_{X'}) < \epsilon$. Note that $M = f(N)$ is a locally connected continuum in $X'$ and the restriction $g = f|_{X'}$ maps $X'$ into $M$ with $\tilde{d}(g) < \epsilon$. Hence $X'$ is internally $\mathcal{LC}$-like, and so is $X$. □

In the one-dimensional case we have the equivalences below. We provide a proof for the first set of equivalences. The proof for the second set is similar.

**Theorem 4.** *For each one-dimensional continuum $X$ the following conditions are equivalent.*

(1a) *$X$ is internally $\mathcal{LC}$-like.*
(2a) *$X$ is internally graph-like.*
(3a) *$X$ is internally graph-representable.*
(4a) *$X$ is internally ANR-like.*
(5a) *$X$ is an AANR.*

*Proof.* The equivalence (2a)⇔(3a) follows from Theorem 3. It is known that every locally connected curve $M$ admits, for every $\epsilon > 0$, a map $f$ into a graph in $M$ with $\tilde{d}(f) < \epsilon$. In short, $M$ is internally graph-like. This result was recently confirmed, in a stronger version, for retractions, by Mańka [12]. If $X$ is internally $\mathcal{LC}$-like, given $\epsilon > 0$, we take a map $g : X \to M \subset X$, where $M$ is a locally connected continuum and $\tilde{d}(g) < \epsilon/2$, and compose it with a map $f : M \to G \subset M$, where $G$ is a graph and $\tilde{d}(g) < \epsilon/2$. Thus (1a)⇒(2a) follows. The implication (2a)⇒(4a) is obvious. The implication (4a)⇒(5a) is a part of Proposition 5.2 of [7], also restated above as Proposition 4. Finally, (5a)⇒(1a) follows by Proposition 5. □

**Theorem 5.** *For each one-dimensional continuum $X$ the following conditions are equivalent.*

(1b) *$X$ is internally $\mathcal{LC}$-like and contains no simple closed curve.*
(2b) *$X$ is internally tree-like.*
(3b) *$X$ is internally tree-representable.*
(4b) *$X$ is internally AR-like.*
(5b) *$X$ is an AAR.*

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