# BRUSH SPACES AND THE FIXED POINT PROPERTY

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Abstract. We introduce the notions of a brush space and a weak brush space. Each of these spaces has a compact connected core with attached connected fibers and may be either compact or non-compact. Many spaces, both in the Hausdorff non-metrizable setting and in the metric setting, have realizations as (weak) brush spaces. We show that these spaces have the fixed point property if and only if subspaces with core and finitely many fibers have the fixed point property. This result generalizes the fixed point result for generalized Alexandroff-Urysohn squares in [4]. We also look at some familiar examples, with and without the fixed point property, from [1, 3, 7] and note the brush space structures related to these examples.

We introduce the notion of a *weak brush space* and a *brush space*, and prove that these spaces have the fixed point property (fpp) if and only if certain subspaces have the fpp. Weak brush spaces are connected Hausdorff spaces, which may not be compact. Our definition suggests a procedure for constructing both compact and non-compact spaces with the fpp. By a *continuum* we mean a compact, connected Hausdorff space. If a continuum $X$ is metrizable, we refer to $X$ as a *metric continuum*.

**Definition 1.** *Let $X = X_0 \cup \bigcup_{\alpha\in\Gamma} X_\alpha$ be a Hausdorff space, where $X_0$ is compact and connected in $X$, and all $X_\alpha$ are connected. Suppose also that*

1. *for each $\alpha \in \Gamma$, $X_\alpha \cap X_0 \neq \emptyset$*
2. *for each $\alpha \in \Gamma$, $X_\alpha - X_0$ is open in $X$,*
3. *for each $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$, $X_\alpha \cap X_\beta \subseteq X_0$, and*
4. *there exists a retraction $r\colon X \to X_0$.*

*We call $X$ a* **weak brush space**. *We refer to $X_0$ as the* **core** *of $X$ and to the $X_\alpha'$s as* **fibers** *of $X$. If additionally for each $\alpha \in \Gamma$, the fiber $X_\alpha$ is closed in $X$ and $X_0 \cap X_\alpha$ is a degenerate set, which we will denote by $\{x_\alpha\}$, then we call $X$ a* **brush space**.

Each continuum $X$ has a trivial realization as a brush space by taking $X$ as the core $X_0$ with no fibers. Also, each continuum $X$ has a trivial realization as a weak brush space by taking any degenerate set (or absolute retract) in $X$ as the core and $X$ itself as a single fiber. If we say that a certain class of spaces are brush spaces, we mean that each such space has a non-trivial realization as a brush space. Each generalized Alexandroff/Urysohn Square, as introduced by Hagopian and Marsh in Section 1 of [4], is a brush space and each AU(g) product of two Hausdorff continua as introduced in Section 2 of [4] is a weak brush space. In [3], Connell studied relations between the fpp and compactness. He gave three examples of non-compact spaces with the fpp. Each of

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his examples is a weak brush space. A straightforward argument shows that all dendrites are brush spaces.

We will give some specific examples of brush spaces after a few propositions and after Theorem 1, which establishes a relationship between the brush space structure and the fixed point property. The first two propositions follow immediately from the definition.

**Proposition 1.** *Each weak brush space is connected.*

**Proposition 2.** *Let* $X = X_0 \cup \bigcup_{\alpha\in\Gamma} X_\alpha$ *be a weak brush space and* $Y$ *a continuum. Then* $Z = X \times Y$ *is a weak brush space with core* $X_0 \times Y$*, fibers* $X_\alpha \times Y$ *for* $\alpha \in \Gamma$*, and retraction* $r \times \mathrm{id}\colon X \times Y \to X_0 \times Y$*. Moreover, if* $X$ *has closed fibers, then* $Z$ *does also.*

**Proposition 3.** *Suppose* $X$ *is a brush space and* $\alpha \neq \beta \in \Gamma$*. If* $X_\alpha \cap X_\beta \neq \emptyset$*, then* $X_\alpha \cap X_\beta = \{x_\alpha\}$*. Therefore,* $x_\alpha = x_\beta$*.*

*Proof.* By definition of a brush space, $X_\alpha \cap X_\beta \subseteq X_0$. So, $X_\alpha \cap X_\beta = X_\alpha \cap (X_\alpha \cap X_\beta) \subseteq X_\alpha \cap X_0 = \{x_\alpha\}$. Since $X_\alpha \cap X_\beta \neq \emptyset$, $X_\alpha \cap X_\beta = \{x_\alpha\}$. □

**Proposition 4.** *Suppose* $A$ *is a set in the weak brush space* $X$ *such that* $(A\cap X_\alpha) - X_0$ *is finite for each* $\alpha \in \Gamma$*. Then every limit point of* $A$ *is in* $X_0$*. Equivalently,* $\mathrm{cl}A - A \subseteq X_0$*.*

*Proof.* Suppose $x \in X - X_0$. Then $x \in X_\beta - X_0$ for some $\beta \in \Gamma$ and by (2) of Definition 1, $X_\beta - X_0$ is open in $X$. By assumption, $X_\beta - X_0$ contains finitely many points of $A$, so $x$ is not a limit point of $A$. □

**Definition 2.** *Let* $X$ *be a weak brush space and* $\{\alpha_i \mid 1 \leq i \leq n\}$ *be a finite subset of* $\Gamma$*. Define* $Y_n = X_0 \cup \bigcup_{i=1}^{n} X_{\alpha_i}$*. We refer to each such* $Y_n$ *as an* $n$**-fibered brush** *in* $X$*. Note that there is exactly one* 0*-fibered brush in* $X$*, namely* $Y_0 = X_0$*.*

**Proposition 5.** *Let* $Y_n$ *be an* $n$*-fibered brush in the weak brush space* $X$*. Define* $r_n\colon X \to Y_n$ *by*

$$r_n(x) = \begin{cases} x & \text{if } x \in Y_n, \\ r(x) & \text{if } x \notin Y_n. \end{cases}$$

*Then* $r_n$ *is a retraction.*

*Proof.* Notice that $X$ is the union of $K = X_0 \cup \bigcup_{\alpha\neq\alpha_i} X_\alpha$ and $Y_n$.

For $\beta \neq \alpha_i$, $Y_n$ does not have limit points in $X_\beta - X_0$ by property (2). It follows that $Y_n$ is closed. The complement of $K$ is $\bigcup_{i=1}^{n}(X_{\alpha_i} - X_0)$, which is open. So, $K$ is closed. We have that $K$ and $Y_n$ are closed and $K \cup Y_n = X$. Note that $r_n = r$ on $K$ and $r_n = id$ on $Y_n$. Also, $r = id$ on $K \cap Y_n = X_0$. So, each $r_n$ is continuous. □

**Proposition 6.** *Let* $X$ *be a brush space. For each* $\alpha \in \Gamma$*,* $X_\alpha$ *is a retract of* $X$*.*

*Proof.* Note that for each $x_\alpha \in X_0$, the 1-fibered brush $Y_1 = X_0 \cup X_\alpha$ can be retracted to $X_\alpha$ since $X_\alpha$ is closed. Let $q\colon Y_1 \to X_\alpha$ be the retraction where $q(X_0) = \{x_\alpha\}$. Let $r_1\colon X \to Y_1$ be the retraction as defined in Proposition 5. Then $qr_1\colon X \to X_\alpha$ is a retraction. □

**Theorem 1.** *Let* $X$ *be a weak brush space. Then* $X$ *has the fpp if and only if for each* $n \geq 0$ *and each* $n$*-fibered brush* $Y_n$ *in* $X$*,* $Y_n$ *has the fpp.*

*Proof.* $\Rightarrow$: Assume that $X$ has the fpp. By Proposition 5, each $Y_n$ is a retract of $X$; so each $Y_n$ has the fpp.

$\Leftarrow$: Let $f\colon X \to X$ be a mapping. Since $X_0$ has the fpp by assumption, $rf|_{X_0}\colon X_0 \to X_0$ has a fixed point, say $z_1 \in X_0$. So, $rf(z_1) = z_1$. If $f(z_1) \in X_0$, then $z_1$ is a fixed point of $f$ and we are done. So, we assume that $f(z_1) \notin X_0$. Let $\alpha_1 \in \Gamma$ where $f(z_1) \in X_{\alpha_1}$. Let $Y_1$ be the 1-fibered brush $X_0 \cup X_{\alpha_1}$ and let $r_1$ be the retraction onto $Y_1$ as defined in Proposition 5. It follows from our hypothesis that $Y_1$ has the fpp. So, the map $r_1 f|_{Y_1}\colon Y_1 \to Y_1$ has a fixed point, say $z_2$. If $f(z_2) \in Y_1$, then $z_2$ is a fixed point of $f$ and we are done. So, assume $f(z_2) \notin Y_1$; say $f(z_2) \in X_{\alpha_2}$. It follows that $\alpha_1 \neq \alpha_2$ and that $z_1 \neq z_2$. Furthermore, we note that $z_2$ must be in $X_0$ since $z_2 = r_1 f(z_2) = rf(z_2) \in X_0$.

Let $Y_2$ be the 2-fibered brush $X_0 \cup X_{\alpha_1} \cup X_{\alpha_2}$ and let $r_2$ be the retraction onto $Y_2$ as defined in Proposition 5. Since $Y_2$, by hypothesis, has the fpp, $r_2 f|_{Y_2}\colon Y_2 \to Y_2$ has a fixed point, say $z_3$. As above, we get that $z_3 \in X_0 - \{z_1, z_2\}$, $f(z_3)$ is in some $X_{\alpha_3}$ with $\alpha_3 \neq \alpha_2$ and $\alpha_3 \neq \alpha_1$, and $z_3 = r_2 f(z_3) = rf(z_3)$.

Continuing this process, we get a sequence of points $\{z_n\}$ in $X_0$ such that for $n \geq 1$, $f(z_n) \in X_{\alpha_n}$ and $z_n = r_{n-1} f(z_n) = rf(z_n)$; and for $n \neq m$, $z_n \neq z_m$ and $\alpha_n \neq \alpha_m$. Let $A = \{z_1, z_2, \ldots\}$. Since $A \subseteq X_0$, $\mathrm{cl}A$ is compact by compactness of $X_0$. The map $r \circ f|_A\colon A \to A$ is the identity map on $A$ and thus, $r \circ f|_{\mathrm{cl}A}\colon \mathrm{cl}A \to \mathrm{cl}A$ is the identity map as well. Consequently, $f|_{\mathrm{cl}A}\colon \mathrm{cl}A \to f(\mathrm{cl}A)$ is a homeomorphism. So, the set $f(A)$ is infinite and has a limit point $z$ in the compact set $f(\mathrm{cl}A)$. By the definition of $A$, the set $(f(A) \cap X_\alpha) - X_0$ has at most one element for each $\alpha \in \Gamma$. Therefore, $z \in X_0$ by Proposition 4. We have that $z = r(z) \in rf(\mathrm{cl}A) = \mathrm{cl}A$, and thus, $rf(z) = z = r(z)$. Since $r|_{f(\mathrm{cl}A)}$ is one-to-one, $f(z) = z$. □

**Corollary 1.** *Let $X$ be a brush space. Then $X$ has the fpp if and only if $X_0$ and each fiber $X_\alpha$ has the fpp.*

*Proof.* $\Rightarrow$: Since $X_0$ is a retract of $X$ by property (4), it follows that $X_0$ has the fpp. It follows from Proposition 6 that each fiber $X_\alpha$ has the fpp.

$\Leftarrow$: Since each $X_\alpha$ is closed and $X_0$ is compact (and therefore closed, since $X$ is Hausdorff), it follows that each $n$-fibered brush $Y_n$ in $X$ is a finite wedge sum of spaces with the fpp. Hence, each $Y_n$ has the fpp, and by Theorem 1, $X$ has the fpp. □

We now look at several specific examples of brush spaces. The examples illuminate how the brush space structure gives immediate fixed point results for spaces that presently do not fit into any particular fixed point theory. The second example illustrates how slight changes to a space without the fpp may produce one with the fpp.

**Example 1.** *The harmonic comb $H$ and related spaces.*

**Discussion.** In the plane $\mathbb{R}^2$, let $I_0 = \{(x, 0) \mid 0 \leq x \leq 1\}$ and $L = \{(0, y) \mid 0 \leq y \leq 1\}$. For $n \geq 1$, let $I_n = \{(\frac{1}{n}, y) \mid 0 \leq y \leq 1\}$ and $J_n = \{(\frac{1}{n}, y) \mid 0 \leq y \leq n\}$. Let $H = I_0 \cup L \cup (\cup_{n\geq 1} I_n)$, $H' = I_0 \cup (\cup_{n\geq 1} I_n)$, and $H'' = I_0 \cup (\cup_{n\geq 1} J_n)$.

It is easy to see that $H$ is a compact metric brush space with core $I_0 \cup L$ and fibers $I_n$. Each of $H'$ and $H''$ is a non-compact metric brush space with core $I_0$. The examples $H'$ and $H''$ can also be found in [3, Example 3]. Connell provides a proof that $H'$ and $H''$ have the fpp. That all three of $H$, $H'$, and $H''$ have the fpp follows from our general theory, specifically Corollary 1.

**Remark.** One can similarly construct more exotic weak brush spaces by replacing the arcs in Example 1 with absolute retracts (ARs) and insisting that the intersection of each AR fiber with the core AR also be an AR.

**Example 2.** *The cone over a spiral to a circle, the cone over the spiral to a simple triod, and related spaces.*

**Discussion.** It has been shown by Knill [7, Theorem 3.4] that the cone over a spiral to a circle admits a fixed-point-free map and by Illanes [6] that the cone over a spiral to a simple triod admits a fixed-point-free map. We give examples of two spaces similar to these cones that are weak brush spaces and must have the fpp. We confine our discussion to the cone over the spiral to a circle and use the description of this space given by Bing in [1, Theorem 21].

In $\mathbb{R}^3$, using cylindrical coordinates $(r, \theta, z)$, let $C = \{(1, \theta, 1) \mid 0 \leq \theta \leq 2\pi\}$, $S = \{(1 + \frac{1}{1+\theta}, \theta, 1) \mid \theta \geq 0\}$, and $v = (0, \theta, 0)$. Let $X$ be the union of all convex intervals in $\mathbb{R}^3$ from $v$ to points of $C \cup S$. In [1], Bing refers to $X$ as the *cone-with-a-skirt* and describes Knill's fixed-point-free map on $X$. We also denote $X$ by $cone(C \cup S)$ and if $T \subseteq C \cup S$, we let $cone(T)$ denote the union of all convex intervals in $\mathbb{R}^3$ from $v$ to points of $T$.

For notational convenience, if $p$ is a point in $C \cup S$, we denote the convex interval from $v$ to $p$ by $I_p$. If $0 \leq \theta_1 < \theta_2$, we denote the set $\{(1 + \frac{1}{1+\theta}, \theta, 1) \mid \theta_1 \leq \theta \leq \theta_2\}$ by $S(\theta_1, \theta_2)$. Note that $S(\theta_1, \theta_2)$ is an arc in the spiral $S$.

For $n \geq 0$, let $S_n = S(2n(n+1)\pi, 2(n+1)^2\pi)$. Let $C' = C \cup (\cup_{n \geq 0} S_n)$ and let $X' = cone(C')$. We note that $X'$ is a compact metric brush space with core $cone(C)$ and fibers $cone(S_n)$ for $n \geq 0$. Since $cone(C)$ and each $cone(S_n)$ are homeomorphic to planar disks, it follows that $X'$ has the fpp.

For a second example, let $K = \{(1, 0, 1)\} \cup \{(1 + \frac{1}{1+\theta}, \theta, 1) \mid \theta = 2n\pi \text{ for some } n \geq 0\}$ and let $X'' = cone(S \cup \{(1, 0, 1)\})$. We note that $X''$ is a non-compact, metric weak brush space with core $X_0 = cone(K)$ and fibers $X_n = S(2(n-1)\pi, 2n\pi)$ for $n \geq 1$. The core $X_0$ is a harmonic fan, which has the fpp, and each fiber $X_n$ is homeomorphic to a planar disk. Define a retraction $r \colon X'' \to X_0$ by retracting each $X_n$ to the arc $I_{(1+\frac{1}{1+2(n-1)\pi}, 2(n-1)\pi, 1)} \cup I_{(1+\frac{1}{1+2n\pi}, 2n\pi, 1)}$ in its boundary. Each n-fibered brush $Y_n$ in $X''$ is itself a brush space with core $\{v\}$ and with either intervals or topological disks as fibers. So, by Corollary 1, each $Y_n$ in $X''$ has the fpp and by Theorem 1, $X''$ has the fpp.

**Remark.** The cone-with-a-skirt $X$ in Example 2 provides an example of a metric contractible continuum without the fixed point property that is the union of a space $X''$ with the fpp and a disk, namely $cone(C)$, where the intersection of $X''$ and $cone(C)$ is an arc, namely $I_{(1,0,1)}$. Each of $X''$ and $cone(C)$ is 2-dimensional, although as already noted, $X''$ is not compact. A non-planar example is given by Bing in [1, Theorem 15] and a planar example is given by Hagopian and Prajs in [5]. Their examples attach 1-dimensional metric continua with the fpp to a disk, and their examples are not contractible.

We now show that each of properties (1) through (4) in the definition of a brush space is necessary in order for a brush space $X$ to have the fpp when the core $X_0$ and all fibers $X_\alpha$ have the fpp. There are simple examples if any one of property (1), property (3), or the degeneracy of the sets $X_\alpha \cap X_0$ is omitted. We give examples below when property (2) is omitted and when property (4) is omitted.

**Example 3.** *There is a metric continuum $X$ that satisfies properties (1), (3), and (4) of a brush space, but does not have the fpp, even though $X_0$ and all $X_\alpha$ have the fpp.*

*Proof.* Let $B$ be the "can-with-a-skirt" as defined by Knill in [7, Definition 3.2]. Also, see Bing [1, p. 131]. Let $X = B \times [0,1]$. Knill shows in [7, Theorem 3.4] that $B$ has the fpp, but $X$ does not. Let $X_0 = B \times \{0\}$ in $X$, and let $r\colon X \to X_0$ be defined by $r(b,t) = (b,0)$. Then we express $X$ as $X = X_0 \cup \bigcup_{b\in B}(\{b\} \times [0,1])$ and note that $X$ satisfies properties (1), (3), and (4), $X_0$ and all $\{b\} \times [0,1]$ have the fpp, but $X$ does not have the fpp. □

**Example 4.** *There is a metric continuum $X$ that satisfies properties (1), (2), and (3) of a brush space, but does not have the fpp, even though $X_0$ and all $X_\alpha$ have the fpp.*

*Proof.* Let $P$ denote the plane in polar coordinates $(r,\theta)$ with $r \geq 0$. Let $D_0 = \{(r,\theta) \mid 0 < r < 4\} \subseteq P$. Our example will be a continuum lying in $D_0$.

First we introduce some convenient notation. If $A \subseteq P - \{(0,0)\}$ and $t > 0$, we let $A + t = \{(r+t,\theta) \mid (r,\theta) \in A\}$. If $p = (r,\theta)$ is a point, we let $p + t = (r+t,\theta)$. If $p = (r,\theta_1)$ and $q = (r,\theta_2)$ are two points with $0 \leq \theta_1 < \theta_2 \leq 2\pi$, let $C(p,q)[r] = \{(r,\theta) \mid \theta_1 \leq \theta \leq \theta_2\}$.

Let $g\colon D_0 \to D_0$ be defined by $g(r,\theta) = (4-r,\theta+\pi)$. It is easy to see that $g^2$ is the identity map on $D_0$, $g$ is a homeomorphism, and $g$ is fixed-point-free. In particular, $g$ is a $\pi$-rotation followed by a flip through the circle $r = 2$.

We proceed to define our continuum $X$ (see Figure 1). For $0 \leq \theta < \pi$, let $\alpha(\theta) = 2 + \dfrac{1}{2}\sin\dfrac{3\pi^2}{2(\pi-\theta)}$. Let $S = \{(\alpha(\theta),\theta) \mid 0 \leq \theta < \pi\} \cup \{(r,\pi) \mid \frac{3}{2} \leq r \leq \frac{5}{2}\}$. Note that $S$ is a topologist's sine curve lying in the closed annulus bounded by the circles $r = \frac{3}{2}$ and $r = \frac{5}{2}$. We also note that for integers $n \geq 0$, $\alpha(\frac{4n}{4n+3}\pi) = \frac{3}{2}$ and $\alpha(\frac{4n+2}{4n+5}\pi) = \frac{5}{2}$. So, we let $\{v_n\}$ and $\{q_n\}$ be the two sequences of points in $S$ where $v_n = (\frac{3}{2}, \frac{4n}{4n+3}\pi)$ and $q_n = (\frac{5}{2}, \frac{4n+2}{4n+5}\pi)$ for $n \geq 0$. Let $u = (\frac{5}{2}, 0)$ and $q = g(v_0)$.

Let $W = S \cup g(S)$. Note that $g(W) = W$. We have that $W$ is a double Warsaw circle and $g|_W$ is a fixed-point-free homeomorphism on $W$. If $x$ and $y$ are points lying in the same arc-component of $W$, we let $[x,y]$ denote the unique arc in $W$ with endpoints $x$ and $y$. Let $\{B_n \mid n \geq 1\}$ be the collection of arcs lying in $D_0$ defined by $B_n = \{(\alpha(\theta) + \frac{5\theta}{2^{n+1}\pi}, \theta) \mid 0 \leq \theta \leq \frac{2\pi}{5}\}$. It is easy to check that

i) $B_n$ has endpoints $v_0$ and $q_0 + \frac{1}{2^n}$ for each $n \geq 1$,
ii) $\{B_n\}$ converges to $[v_0, q_0]$, and
iii) $B_n \cap B_m = \{v_0\}$ if $n \neq m$.

Let $A_1 = B_1 \cup C(q_0 + \frac{1}{2}, q + \frac{1}{2})[\frac{5}{2} + \frac{1}{2}] \cup ([q, g(v_1)] + \frac{1}{2})$. For $n \geq 2$, let $A_n = B_n \cup ([q_0, q_{n-1}] + \frac{1}{2^n}) \cup C(q_{n-1} + \frac{1}{2^n}, q + \frac{1}{2^n})[\frac{5}{2} + \frac{1}{2^n}] \cup ([q, g(v_n)] + \frac{1}{2^n})$. We observe that $A_n$ is a sequence of arcs, attached to $W$ at the point $v_0$, converging to $W$.

For $n \geq 1$, let $X_n = g(A_n)$. Then $X_n$ is a sequence of arcs, attached to $W$ at the point $q$, also converging to $W$. We also note that each $A_n - \{v_0\}$ lies in the unbounded

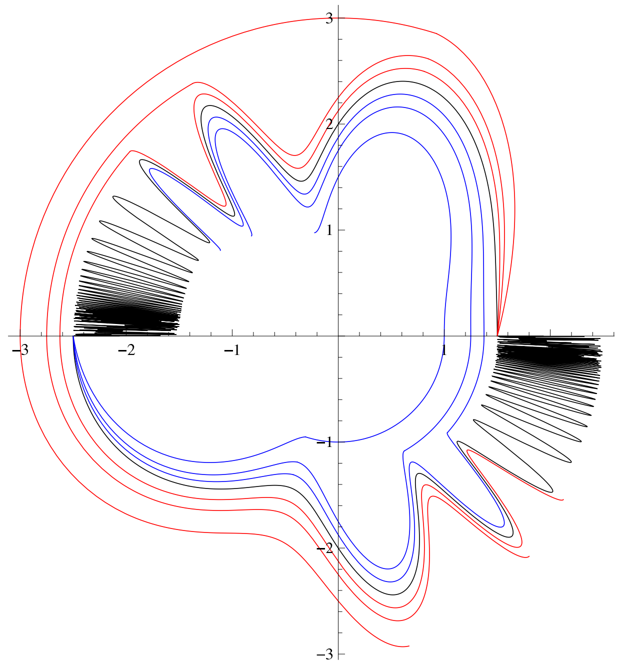


FIGURE 1

complementary domain of $W$ and each $X_n - \{q\}$ lies in the bounded complementary domain of $W$.

Let $X_0 = W \cup \bigcup_{n\geq 1} A_n$ and let $X = X_0 \cup \bigcup_{n\geq 1} X_n$. The continuum $X$ is our example. By construction of $X$ and properties of $g$ previously mentioned, $g(X) = X$ and $g$ is a fixed-point-free homeomorphism. Clearly, $X_0$ and all $X_n$ are continua. For $n \geq 1$, $X_0 \cap X_n = \{q\}$, and each $X_n - \{q\}$ is open in $X$. Also, for $n \neq m$, $X_n \cap X_m = \{q\} \subseteq X_0$. Since each $X_n$ is an arc, it has the fpp. So, we only need to see that $X_0$ has the fpp.

Suppose that $f \colon X_0 \to X_0$ is a fixed-point-free map on $X_0$. Let $K$ be the arc-component of $X_0$ containing $v_0$ and let $L$ be the arc-component of $X_0$ containing $q$. Suppose $f(L) \subseteq L$. Then $f(\overline{L}) \subseteq \overline{L}$. Since $\overline{L}$ is a topologist's sine curve, $f$ has a fixed point in $\overline{L}$, a contradiction. So, $f(L) \subseteq K$.

Suppose $f(K) \subseteq L$. Then $f(\overline{K}) \subseteq \overline{L}$ and since $L \subseteq \overline{K}$, $f(L) \subseteq \overline{L}$. Hence, $f(\overline{L}) \subseteq \overline{L}$ and we have a contradiction as in the previous paragraph. Thus, $f(K) \subseteq K$.

Since $X_0 = K \cup L$, $f(X_0) \subseteq K$. Since each subcontinuum of $K$ is arcwise connected, has empty interior in the plane, and does not separate the plane, it follows that $f(X_0)$ is a dendroid $J$. Thus, we have that $f(J) \subseteq J$, and $f$ has a fixed point in $J$, a contradiction. Hence, $X_0$ has the fpp.

By Corollary 1, it follows that there is no retraction of $X$ onto $X_0$. However, the following simple argument verifies this directly. Suppose $r \colon X \to X_0$ is a retraction. Then $r(L) = L$. So, $r(A_n) \subseteq L$ for each $n \geq 1$. Let $\{x_n\}$ converge to $q_0$ with $x_n \in A_n$ for each $n \geq 1$. Then $r(x_n) \in L$ for each $n \geq 1$, but $r(q_0) = \mathrm{id}(q_0) = q_0$, a violation of continuity of $r$.

□

**Question 1.** *Is every metric dendroid $X$ an inverse limit of an inverse sequence*

$$X_0 \overset{r_1}{\longleftarrow} X_1 \overset{r_2}{\longleftarrow} X_2 \overset{r_3}{\longleftarrow} X_3 \overset{r_4}{\longleftarrow} \ldots\ldots \quad X \ ,$$

*where $X_0$ is a tree, and for each $n \geq 1$,*

i) *$X_n$ is a brush space with core $X_{n-1}$,*
ii) *$X_n$ has only trees for fibers, and*
iii) *$r_n \colon X_n \to X_{n-1}$ is a retraction.*

Note that if we have such an inverse sequence, and for $n \geq 1$ we choose a $k_n$-fibered brush $Y_{k_n}$ in $X_n$, then it follows from Proposition 5 that the inverse limit $Y$ of the inverse sequence

$$X_0 \overset{r_1|_{Y_{k_1}}}{\longleftarrow} Y_{k_1} \overset{r_2|_{Y_{k_2}}}{\longleftarrow} Y_{k_2} \overset{r_3|_{Y_{k_3}}}{\longleftarrow} Y_{k_3} \overset{r_4|_{Y_{k_4}}}{\longleftarrow} \ldots\ldots \quad Y$$

is a tree-like subcontinuum of $X$. Furthermore, since the bonding maps are retractions (and therefore universal), each such tree-like subcontinuum of $X$ will have the fpp.

Also of interest is the analogous question for Hausdorff dendroids as inverse limits of inverse systems taken over directed sets.

**Question 2.** (Unsolved Problem, [8, page 36]; see also [2, Question 5.6]) *If $X$ is a dendroid, does there exists a sequence of retractions $\{r_n\}$ of $X$ onto trees in $X$ such that $\{r_n\}$ converges uniformly to* $\mathrm{id}|_X$*?*

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