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CHAINABILITY OF INVERSE LIMITS ON $[0,1]$ WITH INTERVAL-VALUED FUNCTIONS

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Abstract. We provide sufficient conditions for the inverse limit of an inverse sequence on $[0,1]$ with upper semi-continuous set-valued bonding functions to be chainable. The conditions are placed on the bonding functions. Our results answer several questions of W.T. Ingram. We also show that analogous conditions placed on the inverses of the bonding functions produce a chainable inverse limit.

1. Introduction

In the setting of inverse sequences on $[0,1]$ with upper semi-continuous set-valued bonding functions, we provide sufficient conditions for chainability of the inverse limit space. The conditions are properties placed on the bonding functions. Our main theorems (Theorems 4 and 7) give two related solutions to a generalized version of W.T. Ingram's Problem 1.1 in [2], and Corollary 2 gives an answer to Ingram's Problem 4.4 in [4]. Our results generalize Ingram's Theorem 4.3 in [4], and the results, related to chainability of inverse limits on $[0,1]$, of W.J. Charatonik, F. Mena, and R.P. Roe, which were announced at the Spring Topology and Dynamics Conference held at the University of Alabama, Birmingham in March of 2019.

Additionally, in the more general setting of upper semi-continuous set-valued functions on compacta, we prove a number of lemmas and theorems

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in Section 3 that may have utility to others working in this area, and that are essential tools in the proof of our main theorem.

A *compactum* is a compact metric space. All spaces considered in this paper are compacta. A *continuum* is a connected compactum. A *mapping* is a continuous function. A continuum X is *chainable* if for each $\epsilon > 0$, X admits a finite ϵ-chain of open sets covering X. A continuum X is *arclike* if for each $\epsilon > 0$, X admits an ϵ-mapping onto $[0, 1]$. It is well-known that for a continuum X, the following are equivalent.

- X is chainable.
- X is arclike.
- X is representable as an inverse limit of an inverse sequence on $[0, 1]$ with mappings for bonding functions.

See the end of Section 2 in [2] for specific definitions and discussion of these equivalences.

As with many investigations, those that led to this article began in the study of an example. The main theorem in this paper can be traced to an example of an inverse limit of an upper semi-continuous interval-valued function from $[0, 1]$ into $[0, 1]$ having a $\sin(1/x)$-curve for its graph. In the setting of set-valued inverse limits, the first published study of this example came in 2011 in a paper by Scott Varagona where he isolated a theorem, one consequence of which is that the inverse limit on $[0, 1]$ with a single bonding function having a graph that is a $\sin(1/x)$-curve is an indecomposable continuum [12, Theorem 3.2]. In 2013, in an early study of chainability of inverse limits on $[0, 1]$ with set-valued functions it was shown that a specific function having a $\sin(1/x)$-curve for its graph produces a chainable continuum [2, Example 5.4]. Then, in 2019, it was shown that any inverse limit on $[0, 1]$ using a sequence of functions having graphs that are sinusiods produces a chainable continuum [4]. In this paper we generalize that result.

Our approach to finding properties of the bonding functions that will ensure chainability of the inverse limit and extend the result in [4] is an obvious one. Namely, we ask "What properties of set-valued functions whose graphs are sinusoids, when placed on other set-valued functions from $[0, 1]$ to $[0, 1]$, are sufficient for chainability of the inverse limit?".

Let $\{[0, 1], f_i\}_{i \geq 1}$ be an inverse sequence on $[0, 1]$ with upper semi-continuous surjective set-valued bonding functions. The bonding function properties that we consider are the four listed below. Definitions of the first three properties and the notation used in the fourth property follow this discussion.

(1) Each f_i is interval-valued.
(2) Each f_i is C-set-valued.

(3) For each $i \geq 2$, no flat spot of f_i composes to a non-degenerate value of f_j for $j < i$.
(4) $G(f_i)$ is chainable for each $i \geq 1$.

Taken together, properties (1), (2), and (3) are sufficient assumptions on either the bonding functions or their inverses for chainability of the inverse limit. As we will see, property (4) follows from properties (1), (2), and (3). When property (1) is assumed, property (3) is also a necessary property for chainability of the inverse limit in our setting. This follows from Theorem 2 in [10]. Property (1) is not a necessary condition for chainability of the inverse limit, see [5, Example 4.8] along with [6, Example 5.3]. Also, property (2) is not a necessary property, see Examples 2.2 and 2.6 in [1]. It is not known if chainability of the graphs of the bonding functions is necessary.

Lastly, we mention that, among the inverse sequences on $[0, 1]$ with interval-valued bonding functions, both conditions (2) and (3) are necessary to ensure that the inverse limit is chainable. For an example of a non-chainable inverse limit satisfying (2), but not (3), see Example 4.5 in [4]; and for an example of a non-chainable inverse limit satisfying (3), but not (2), see Examples 2.4 and 2.14 in [1]. It is also of interest to note that in these last two examples, the graphs of the bonding functions are arcs.

2. Definitions

Let X and Y be compacta. We refer to functions $f : X \to 2^Y$ as *set-valued functions* from X to Y and we write $f : X \to Y$ is a set-valued function. Note that throughout, we are assuming that, for $x \in X$, the value $f(x)$ of a set-valued function is a closed set. The *graph* of f, which we denote by $G(f)$, is the set in $X \times Y$ consisting of all points (x, y) with $y \in f(x)$.

A set-valued function $f : X \to Y$ is *upper semi-continuous at the point* $x \in X$ if for each open set V in Y containing the closed set $f(x)$, there is an open set U in X such that $x \in U$ and $f(p) \subset V$ for each $p \in U$. If $f : X \to Y$ is upper semi-continuous at each point of X, then f is said to be *upper semi-continuous.*

The set-valued function $f : X \to Y$ is *surjective* if for each $y \in Y$, there exists $x \in X$ such that $y \in f(x)$. If the set-valued function $f : X \to Y$ is surjective, we let $f^{-1} : Y \to X$ be the set-valued function such that $x \in f^{-1}(y)$ if and only if $y \in f(x)$. Clearly, $G(f^{-1})$ is homeomorphic to $G(f)$; so, it follows from [1, Theorem 1.2] that f^{-1} is upper semi-continuous if f is upper semi-continuous.

A set-valued function $f\colon X \to Y$ is *continuum-valued* if for each $x \in X$, the set $f(x)$ is a subcontinuum of Y. A subset H of a continuum M is a *C-set in M* provided that whenever K is a subcontinuum of M that meets both H and $M - H$, we have that $H \subset K$. We note that some authors have analogously defined the phrase H is *terminal in M*. A set-valued function $f\colon X \to Y$ is said to be *C-set-valued* if for each $x \in X$, $\{x\} \times f(x)$ is a C-set in the graph of f. Because $G(f)$ and $G(f^{-1})$ are homeomorphic, $\{x\} \times f(x)$ is a C-set in $G(f)$ if and only if $f(x) \times \{x\}$ is a C-set in $G(f^{-1})$.

For $f\colon X \to Y$ a set-valued function, and $A \subset X$, we let $f|_A$ be the set-valued function whose domain is A, and $f|_A(x) = f(x)$ for $x \in A$. If $x \in X$ and $f(x)$ is degenerate, we will sometimes treat $f(x)$ as a point of Y.

For $i \geq 1$, let X_i be a compactum, and let $f_i\colon X_{i+1} \to X_i$ be a surjective upper semi-continuous set-valued function. Throughout, we let $\{X_i, f_i\}_{i\geq 1}$ denote an inverse sequence, and its inverse limit is given by

$$\varprojlim\{X_i, f_i\} = \{\boldsymbol{x} = (x_1, x_2, \dots) \in \prod_{i\geq 1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } i \geq 1\}.$$

For $n \in \mathbb{N}$, we define the set below.

$$G'_n = G'(f_1, \dots, f_n) = \{\boldsymbol{x} \in \prod_{i=1}^{n+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}.$$

We refer to these sets as *partial graphs* in the inverse sequence. For consistency of notation, we let $G'_0 = X_1$. The notation $X \overset{T}{\approx} Y$ will indicate that X is homeomorphic to Y.

A set-valued function $f\colon X \to Y$ has a *flat spot* if there exists a point $p \in Y$ and a nondegenerate continuum $X' \subset X$ such that $X' \times \{p\} \subset G(f)$. We say that p is a *flat spot for f*.

For $1 \leq i < j$, we denote the set-valued composition function $f_i \circ f_{i+1} \circ \dots \circ f_j\colon X_{j+1} \to X_i$ by $f_{i,j+1}$. A *flat spot at x_j for f_j composes to a nondegenerate value of f_i* in the composition $f_i \circ f_{i+1} \circ \dots \circ f_j$ if $f_i(x_j)$ is nondegenerate for $i = j - 1$, or if there exists a point x_{i+1} in $f_{i+1,j}(x_j)$ such that $f_i(x_{i+1})$ is nondegenerate for some $i < j - 1$. Although, here the functions f_i compose right to left, as in an inverse sequence, it is clear that if we have a sequence of compacta and set-valued functions that compose left to right, as in $X_1 \overset{g_1}{\to} X_2 \overset{g_2}{\to} \dots \overset{g_n}{\to} X_{n+1}$, there is analogous terminology for flat spots of g_i composing to non-degenerate values of g_j for $i < j$.

Let $\{X_i, f_i\}_{i\geq 1}$ be an inverse sequence with upper semi-continuous surjective set-valued bonding functions. For $n \geq 1$, we define the set-valued function $F_n\colon X_{n+1} \to G'_{n-1}$, where $(x_1, x_2, \ldots, x_n) \in F_n(t)$ if and only if $(x_1, x_2, \ldots, x_n, t)$ is in G'_n. V. Nall introduced this function in [11], and showed that F_n is upper semi-continuous. If f_i is continuum-valued for each $1 \leq i \leq n$, M.M. Marsh showed in [9] that F_n is continuum-valued.

3. Preliminary lemmas and theorems

This section contains lemmas and theorems pertinent to our main theorem. Most of the results in this section only involve two set-valued functions $f : X \to Y$ and $g : Y \to Z$, where X, Y and Z are compacta. So, we adopt the following notation and employ it often throughout the section. Define the set-valued function $F : X \to G(g^{-1})$ by $F(x) = \{(z, y) \in G(g^{-1}) \mid y \text{ is in } f(x)\}$. We refer to F as the function *induced* by f and g. The function F is simply the Nall function F_2 mentioned at the end of Section 2. Note that for x in X, $F(x) = G((g|_{f(x)})^{-1})$.

We let π_1, π_2, π_3, $\pi_{\{1,2\}}$, and $\pi_{\{2,3\}}$ be the natural projection mappings from $Z \times Y \times X$ onto Z, Y, X, $Z \times Y$, and $Y \times X$ respectively. Our first lemma is little more than an observation.

Lemma 1. *Suppose X and Y are compacta and $f : X \to Y$ and $g : Y \to Z$ are surjective upper semi-continuous set-valued functions, and F is the function induced by f and g. If x is a point of X and y is a point of Y such that $f(x) = \{y\}$ and $g(y)$ is a singleton, then the only point of $G'(g, f)$ with third coordinate x is $(g(y), y, x)$, i.e., $F(x) = \{(g(f(x)), f(x))\}$.*

Suppose $\{X_i, f_i\}_{i=1}^{n}$ is a finite inverse sequence with surjective upper semi-continuous set-valued bonding functions. If p is a point of X_{n+1}, by the *orbit* of p we mean the sequence of sets $f_n(p), f_{n-1,n+1}(p), \ldots, f_{1,n+1}(p)$. If each set in the orbit of p is a singleton, we consider the orbit of p to be a sequence of points. Our next lemma follows by induction from Lemma 1.

Lemma 2. *Suppose $n \geq 2$, and $\{X_i, f_i\}_{i=1}^{n}$ is a finite inverse sequence with upper semi-continuous surjective set-valued bonding functions. If p is a point of X_{n+1} such that the orbit of p is a sequence of singletons, then $F_n(p)$ is a singleton.*

Lemma 3. *Suppose X, Y, and Z are continua, $f : X \to Y$ and $g : Y \to Z$ are surjective upper semi-continuous continuum-valued functions, and if p is a flat spot for f, then $g(p)$ is a singleton. Then, if s is a point of Y such that $g(s)$ and $f^{-1}(s)$ are nondegenerate, then $f^{-1}(s)$ is closed and totally disconnected. Moreover, $\{(z, y, x) \in G'(g, f) \mid y = s\} = g(s) \times \{s\} \times f^{-1}(s)$, and if M is a continuum in $G'(g, f)$ that intersects two components of $g(s) \times \{s\} \times f^{-1}(s)$, then $\pi_2(M)$ is a nondegenerate continuum.*

Proof. If s is a point of Y, then $f^{-1}(s)$ is closed because f^{-1} is upper semi-continuous. Because $g(s)$ is nondegenerate, $f^{-1}(s)$ does not contain a nondegenerate continuum. Thus, $f^{-1}(s)$ is totally disconnected.

If (z, y, x) is in $G'(g, f)$ and $y = s$, then z is in $g(s)$ and s is in $f(x)$, so x is in $f^{-1}(s)$. Thus, (z, y, x) is in $g(s) \times \{s\} \times f^{-1}(s)$.

On the other hand, if (z, y, x) is in $g(s) \times \{s\} \times f^{-1}(s)$ then $y = s, z$ is in $g(s)$, and x is in $f^{-1}(s)$. Thus y is in $f(x)$, so (z, y, x) is a point of $G'(g, f)$.

Suppose M is a continuum in $G'(g, f)$ intersecting two components of $g(s) \times \{s\} \times f^{-1}(s)$. For t in $f^{-1}(s), g(s) \times \{s\} \times \{t\}$ is homeomorphic to $g(s) \times \{s\}$. Therefore, there exist two points t_1 and t_2 of $f^{-1}(s)$ and points z_1 and z_2 of $g(s)$ such that (z_1, s, t_1) and (z_2, s, t_2) are points of M. Because t_1 and t_2 are two points of $f^{-1}(s)$ in the continuum $\pi_3(M)$, there is a point w of $\pi_3(M)$ not in $f^{-1}(s)$. It follows that $\pi_2(M)$ contains a point v of $f(w)$. Because w is not in $f^{-1}(s)$, v is not s, so $\pi_2(M)$ is nondegenerate. □

Lemma 4. *Suppose X, Y, and Z are continua and $f : X \to Y$ and $g : Y \to Z$ are surjective upper semi-continuous continuum-valued functions, and J is a subcontinuum of Y such that if y is in J then $g(y)$ is a singleton. Then, $M = \{(z, y, x) \in G'(g, f) \mid y \text{ is in } J\}$ is homeomorphic to $G(f^{-1}|_J)$ under $\pi_{\{2,3\}}$.*

Proof. Because J is compact, it follows that M is compact. Suppose (z, y, x) is a point of M. Then, y is in $f(x)$, so (y, x) is in $G(f^{-1})$. Because y is a point of $J, (y, x)$ is in $G(f^{-1}|_J)$. Thus, $\pi_{\{2,3\}}$ maps M into $G(f^{-1}|_J)$.

Let each of (z_1, y_1, x_1) and (z_2, y_2, x_2) be a point of M, and suppose that $\pi_{\{2,3\}}(z_1, y_1, x_1) = \pi_{\{2,3\}}(z_2, y_2, x_2)$. Then, $y_1 = y_2$ and $x_1 = x_2$. Because (z_1, y_1, x_1) is in M, y_1 is in J, so $g(y_1)$ is a single point. Thus, $z_1 = z_2$. So, $\pi_{\{2,3\}}$ is one-to-one.

If (b, a) is in $G(f^{-1}|_J)$, then b is in $f(a) \cap J$. Thus, $g(b)$ is a singleton and we have $(g(b), b, a)$ is a point of M. Therefore, $\pi_{\{2,3\}}$ is surjective. Because M is compact and $\pi_{\{2,3\}}$ is one-to-one and continuous, $\pi_{\{2,3\}}$ is a surjective homeomorphism. □

Lemma 5. *Suppose X, Y, and Z are continua, and $f : X \to Y$ and $g : Y \to Z$ are surjective upper semi-continuous continuum-valued functions. If q is a point of Y such that $g(q)$ is a singleton and p is a separating point of X such that $f(p) = \{q\}$ then $(g(q), q, p)$ is a separating point of $G'(g, f)$. Moreover, if u and v are in different components of $X - \{p\}$ and $\boldsymbol{r}$ and $\boldsymbol{s}$ are points of $G'(g, f)$ with third coordinates u and v, respectively, then $\boldsymbol{r}$ and $\boldsymbol{s}$ are in different components of $G'(g, f) - \{(g(q), q, p)\}$.*

Proof. Suppose U and V are mutually exclusive open subsets of X such that $X - \{p\} = U \cup V$. Let $U' = \{(z, y, x) \in G'(g, f) \mid x \text{ is in } U\}$ and $V' = \{(z, y, x) \in G'(g, f) \mid x \text{ is in } V\}$. Then, U' and V' are mutually exclusive open subsets of $G'(g, f)$, and $G'(g, f) - \{(g(q), q, p)\} = U' \cup V'$. □

3.1. **C-sets and hereditarily decomposable chainable continua.** Without altering the proof, a stronger version of Theorem 3.8 in [4] could have been stated. We state this stronger version as Theorem 1 below.

Theorem 1. *Suppose M is a continuum, N is an hereditarily decomposable chainable continuum, and $f : M \to N$ is a surjective monotone mapping such that $f^{-1}(x)$ is an hereditarily decomposble chainable C-set in M for each point x of N. Then, M is an hereditarily decomposable chainable continuum.*

We make use of the following theorem in the proofs of Theorems 3 and 4.

Theorem 2. *Suppose H is an hereditarily decomposable chainable continuum, K is a continuum, and $f : H \to K$ is a surjective upper semi-continuous, continuum-valued function. Suppose also that f is C-set-valued, and that $f(x)$ is an hereditarily decomposable chainable subcontinuum of K for each x in H. Then $G(f)$ is an hereditarily decomposable chainable continuum.*

Proof. Because H and K are continua and f is continuum-valued, $G(f)$ is a continuum. Let p denote the restriction to $G(f)$ of the projection of $H \times K$ onto H. Then, p is a monotone map from $G(f)$ onto the hereditarily decomposable chainable continuum H such that $p^{-1}(x)$ is an hereditarily decomposable chainable continuum for each x in H. By Theorem 1, $G(f)$ is an hereditarily decomposable chainable continuum. □

3.2. **C-set-valued functions.**

Lemma 6. *Suppose X, Y, and Z are continua, $f : X \to Y$ and $g : Y \to Z$ are surjective upper semi-continuous continuum-valued functions, and f is C-set valued. If x is a point of X and K is a subcontinuum of $G'(g, f)$ that intersects $F(x) \times \{x\}$ and contains a point not in $F(x) \times \{x\}$, then $f(x) \times \{x\}$ is a subset of $\pi_{\{2,3\}}(K)$ and, thus, $f(x)$ is a subset of $\pi_2(K)$.*

Proof. Note that $\pi_{\{2,3\}}(K)$ is a continuum lying in $G(f^{-1})$. Suppose (c, b, x) is a point of K in $F(x) \times \{x\}$. Then, (c, b) is in $F(x)$. Thus, b is in $f(x)$, so $\pi_{\{2,3\}}(K)$ contains a point of $f(x) \times \{x\}$.

Suppose (c, b, a) is a point of K not in $F(x) \times \{x\}$. Because K is a subset of $G'(g, f)$, we have c is in $g(b)$ and b is in $f(a)$. Further, (c, b) is not in $F(x)$ or $a \neq x$. If $a \neq x$, then (b, a) is not in $f(x) \times \{x\}$, so

$\pi_{\{2,3\}}(K)$ contains a point not in $f(x) \times \{x\}$. If $a = x$, then (c, b) is not in $F(x)$. Because c is a point of $g(b)$, we have b is not in $f(x)$, so (b, a) is not in $f(x) \times \{x\}$ and, again, $\pi_{\{2,3\}}(K)$ contains a point not in $f(x) \times \{x\}$. Thus, because $\{x\} \times f(x)$ is a C-set in $G(f)$, it follows that $\pi_{\{2,3\}}(K)$ contains $f(x) \times \{x\}$ and $\pi_2(K)$ contains $f(x)$. □

Lemma 7. *Suppose X, Y, and Z are continua, $f : X \to Y$ and $g : Y \to Z$ are surjective upper semi-continuous continuum-valued functions, and f is C-set-valued. Suppose t is a point of X such that $f(t)$ is nondegenerate and $g(y)$ is a singleton for each y in $f(t)$. If K is a subcontinuum of $G'(g, f)$ that contains a point of $F(t) \times \{t\}$ and a point not in $F(t) \times \{t\}$, then $F(t) \times \{t\}$ is a subset of K.*

Proof. Let $J = f(t)$. By Lemma 6, $f(t) \times \{t\}$ is a subset of $\pi_{\{2,3\}}(K)$. By Lemma 4, $\pi_{\{2,3\}}$ maps $\{(p_1, p_2, p_3) \in G'(g, f) \mid p_2 \in J\}$ homeomorphically into $G(f^{-1}|_J)$. Thus, $\pi^{-1}_{\{2,3\}}(f(t) \times \{t\})$ is a subset of K. But, $\pi^{-1}_{\{2,3\}}(f(t) \times \{t\}) = F(t) \times \{t\}$. □

Lemma 8. *Suppose X, Y, and Z are continua, and $f : X \to Y$ and $g : Y \to Z$ are surjective upper semi-continuous continuum-valued functions with the property that if p is a flat spot for f, then $g(p)$ is a singleton. Suppose x is a point of X and y is a point of Y such that $f(x) = \{y\}$ and $g(y)$ is nondegenerate. If K is a subcontinuum of $G'(g, f)$ that contains a point of $F(x) \times \{x\}$ and a point not in $F(x) \times \{x\}$, then $\pi_2(K)$ is a nondegenerate continuum.*

Proof. Because K is a continuum containing a point of $F(x) \times \{x\}$, $\pi_2(K)$ is a continuum and y is in $\pi_2(K)$. Suppose (c, b, a) is a point of K not in $F(x) \times \{x\}$. Then, (c, b) is not in $F(x)$ or $a \neq x$. If (c, b) is not a point of $F(x)$, then it follows that $b \neq y$. For if we assume that $b = y$, then b is in $f(x)$. However, because (c, b, a) is in $G'(g, f)$, c is in $g(b)$. So, (c, b) is in $F(x)$, which is a contradiction. Thus, in case (c, b) is not in $F(x)$, $\pi_2(K)$ is a continuum that contains the two points b and y. Suppose (c, b) is in $F(x)$. Then, $a \neq x$ and b is in $f(x)$. So, $b = y$. By Lemma 3, $\{(p_1, p_2, p_3) \in G'(g, f) \mid p_2 = b\} = g(b) \times \{b\} \times f^{-1}(b)$ with $f^{-1}(b)$ totally disconnected, and, because K is a continuum that intersects two components of $g(b) \times \{b\} \times f^{-1}(b)$, we have that $\pi_2(K)$ is a nondegenerate continuum. □

Lemma 9. *Suppose X, Y, and Z are continua, and $f : X \to Y$ and $g : Y \to Z$ are surjective upper semi-continuous continuum-valued functions. Suppose also that f and g are C-set-valued functions with the property that if p is a flat spot for f, then $g(p)$ is a singleton. Suppose t is a point of X and y is a point of $f(t)$. If K is a subcontinuum of $G'(g, f)$ that* *contains a point of* $F(t)\times\{t\}$ *and a point not in* $F(t)\times\{t\}$*, then* $g(y)\times\{y\}$ *is a subset of* $\pi_{\{1,2\}}(K)$*.*

Proof. We consider two cases, $f(t)$ is a singleton and $f(t)$ is nondegenerate.

First, suppose $f(t)=\{y\}$. Let (c,b,t) be a point of K in $F(t)\times\{t\}$. Then, $b=y$ and (c,b) is in $F(t)$, so c is a point of $g(y)$. Thus, $\pi_{\{1,2\}}(K)$ contains a point of $g(y)\times\{y\}$. If $g(y)=\{c\}$, then (c,y,t) is a point of K and $g(y)\times\{y\}=\{(c,y)\}$. So $g(y)\times\{y\}$ is a subset of $\pi_{\{1,2\}}(K)$. In case $g(y)$ is nondegenerate, $\pi_2(K)$ is nondegenerate by Lemma 8. Let d be a point of $\pi_2(K)$ such that $d\neq y$. There are points x in X and z in Z such that (z,d,x) is in K. Thus, $\pi_{\{1,2\}}(K)$ contains a point having second coordinate d, so $\pi_{\{1,2\}}(K)$ contains a point not in $g(y)\times\{y\}$. Because $\pi_{\{1,2\}}(K)$ is a continuum and g is C-set-valued, it follows that $g(y)\times\{y\}$ is a subset of $\pi_{\{1,2\}}(K)$.

On the other hand, if $f(t)$ is a nondegenerate continuum J, Lemma 6 yields that $f(t)\times\{t\}$ is a subset of $\pi_{\{2,3\}}(K)$. Because y is a point of $f(t)$, we have (y,t) is a point of $\pi_{\{2,3\}}(K)$, so there is a point c of Z such that (c,y,t) is a point of K. Therefore, $\pi_{\{1,2\}}(K)$ contains the point (c,y) of $g(y)\times\{y\}$. There is a point w of J such that $w\neq y$. Because w is in $f(t)$, (w,t) is in $\pi_{\{2,3\}}(K)$. There is a point d of Z such that (d,w,t) is a point of K, so (d,w) is a point of $\pi_{\{1,2\}}(K)$. But, (d,w) is not in $g(y)\times\{y\}$. Again, because $\pi_{\{1,2\}}(K)$ is a continuum and $\{y\}\times g(y)$ is a C-set in $G(g)$, it follows that $g(y)\times\{y\}$ is a subset of $\pi_{\{1,2\}}(K)$. □

Lemma 10. *Suppose* X,Y*, and* Z *are continua, and* $f:X\to Y$ *and* $g:Y\to Z$ *are surjective upper semi-continuous continuum-valued functions. Suppose* x *is a point of* X *and* y *is a point of* $f(x)$ *such that* $g(y)$ *is nondegenerate. If* $\{y\}\times g(y)$ *is a* C*-set in* $G(g)$*,* K *is a subcontinuum of* $G'(g,f)$ *containing a point of* $g(y)\times\{y\}\times\{x\}$ *and a point not in* $g(y)\times\{y\}\times\{x\}$*, and* x *is the only point of* $f^{-1}(y)$ *in* $\pi_3(K)$*, then* $g(y)\times\{y\}\subseteq\pi_{\{1,2\}}(K)$ *and* $g(y)\times\{y\}\times\{x\}\subseteq K$*.*

Proof. Suppose (p_1,p_2,p_3) is a point of K in $g(y)\times\{y\}\times\{x\}$. Then, (p_1,p_2) is a point of $\pi_{\{1,2\}}(K)$, so $\pi_{\{1,2\}}(K)$ contains a point of $g(y)\times\{y\}$.

If (q_1,q_2,q_3) is a point of K and $q_2=y$, then $q_3=x$ because $f^{-1}(y)\cap\pi_3(K)=\{x\}$. Thus, because q_1 is in $g(y)$, (q_1,q_2,q_3) is in $g(y)\times\{y\}\times\{x\}$. Therefore, if (q_1,q_2,q_3) is a point of K not in $g(y)\times\{y\}\times\{x\}$, then $q_2\neq y$ and we have that $\pi_{\{1,2\}}(K)$ contains a point not in $g(y)\times\{y\}$.

Because $\pi_{\{1,2\}}(K)$ is a continuum containing a point of $g(y)\times\{y\}$ and a point not in $g(y)\times\{y\}$, it follows that $g(y)\times\{y\}$ is a subset of $\pi_{\{1,2\}}(K)$ and, consequently, $g(y)\times\{y\}\times\{x\}$ is a subset of K. □

Theorem 3. *Suppose $X = Y = [0,1]$, Z is a continuum, and $f : X \to Y$ and $g : Y \to Z$ are surjective upper semi-continuous continuum-valued functions with the property that if p is a flat spot for f, then $g(p)$ is a singleton. Suppose further that each of f and g is C-set-valued, the set $\{x \in X \mid g(y) \text{ is nondegenerate for some } y \in f(x)\}$ is a 1^{st} category set, and $G'(g,f)$ is hereditarily unicoherent. If F is the function induced by f and g, then F is C-set-valued.*

Proof. It is sufficient to show that $F(x) \times \{x\}$ is a C-set in $G'(g,f)$ for each x in X. To that end, let t be a point of X and suppose K is a subcontinuum of $G'(g,f)$ that contains a point of $F(t) \times \{t\}$ and a point not in $F(t) \times \{t\}$.

If $f(t)$ is a singleton and $g(f(t))$ is a singleton, then by Lemma 1, the only point of $G'(g,f)$ having third coordinate t is $(g(f(t)), f(t), t)$. Because K contains a point of $F(t) \times \{t\}$, $(g(f(t)), f(t), t)$ is a point of K. But, $F(t) \times \{t\} = \{(g(f(t)), f(t), t)\}$, so $F(t) \times \{t\}$ is a subset of K.

Thus, we may assume $f(t)$ is nondegenerate or $f(t)$ contains a point y such that $g(y)$ is nondegenerate. If $f(t)$ is nondegenerate and $g(y)$ is a singleton for each y in $f(t)$, then by Lemma 7, $F(t) \times \{t\}$ is a subset of K.

Two possibilities remain: (a) $f(t)$ is nondegenerate and $g(y)$ is nondegenerate for some y in $f(t)$, and (b) $f(t)$ is a singleton and $g(f(t))$ is nondegenerate. In either case, in order to show that $F(t) \times \{t\}$ is a subset of K, it is sufficient to show that $g(y) \times \{y\} \times \{t\}$ is a subset of K for each y in $f(t)$. To that end, in either of these two remaining cases, we first note that by Lemma 9, if y is a point of $f(t)$, then $g(y) \times \{y\}$ is a subset of $\pi_{\{1,2\}}(K)$. Moreover, if s is a point of $f(t)$ such that t is the only point of $f^{-1}(s)$ in $\pi_3(K)$, then Lemma 10 yields that $g(s) \times \{s\} \times \{t\}$ is a subset of K.

Suppose y is a point of $f(t)$ such that $T = f^{-1}(y) \cap \pi_3(K)$ is nondegenerate. By Lemma 3, $f^{-1}(y)$ is closed and totally disconnected, so T is a closed and totally disconnected subset of $[0,1]$.

Suppose (u,v) is an open interval lying in $X - T$ with both u and v in T. Because u and v are points of $\pi_3(K)$, K contains a point of $F(u) \times \{u\}$ and a point of $F(v) \times \{v\}$. However, because u and v are different points, it follows that K contains a point not in $F(u) \times \{u\}$ and a point not in $F(v) \times \{v\}$. By Lemma 6, both $f(u) \times \{u\}$ and $f(v) \times \{v\}$ are subsets of $\pi_{\{2,3\}}(K)$. It follows that there are points c and d of Z such that (c, y, u) and (d, y, v) belong to K, and because $\pi_3(K)$ contains no point between u and v, we have that $\pi_{\{2,3\}}(K) \cap (\{y\} \times [u,v]) = \{(y,u),(y,v)\}$. Because F is continuum-valued, $F([u,v])$ is a continuum. Then, $G(F|_{[u,v]}^{-1})$ and K

are subcontinua of the hereditarily unicoherent continuum $G'(g,f)$ with a point in common, so $M = G(F|_{[u,v]}^{-1}) \cap K$ is a continuum.

It follows from Theorem 2 that $G(f)$ is one-dimensional. So, by Corollary 3.3 in [3], the set $A = \{x \in [0,1] \mid f(x) \text{ is nondegenerate}\}$ is a 1^{st} category set. The set $B = \{x \in X \mid g(y) \text{ is nondegenerate for some } y \in f(x)\}$ is a 1^{st} category set by assumption. Since $A \cup B$ is a 1^{st} category set, no open interval is a subset of $A \cup B$, so there is a point w of the open interval (u,v) such that $f(w)$ and $g(f(w))$ are singletons. The only point of $G'(g,f)$ having third coordinate w is $(g(f(w)), f(w), w)$. By Lemma 5, $(g(f(w)), f(w), w)$ is a separating point of $G'(g|_{f([u,v])}, f|_{[u,v]}) = G(F|_{[u,v]}^{-1})$. Because $\pi_3(M)$ is a continuum containing u and v, we have that w is a point of $\pi_3(M)$ and $(g(f(w)), f(w), w)$ is a point of M that separates M. Let D be the closure of the component of $M - \{(g(f(w)), f(w), w)\}$ containing (c,y,u), and let E be the closure of the component of $M - \{(g(f(w)), f(w), w)\}$ containing (d,y,v). Then, D and E are continua containing $(g(f(w)), f(w), w)$, and the only point of $f^{-1}(y)$ in $\pi_3(D)$ is u while the only point of $f^{-1}(y)$ in $\pi_3(E)$ is v. The point $(g(f(w)), f(w))$ is a point of both $\pi_{\{1,2\}}(D)$ and $\pi_{\{1,2\}}(E)$ not in $g(y) \times \{y\}$ while (c,y) is a point of $\pi_{\{1,2\}}(D)$ in $g(y) \times \{y\}$ and (d,y) is a point of $\pi_{\{1,2\}}(E)$ in $g(y) \times \{y\}$. It follows from Lemma 10, that $g(y) \times \{y\} \times \{u\}$ is a subset of D and $g(y) \times \{y\} \times \{v\}$ is a subset of E. Because D and E are both subsets of K, we see that $g(y) \times \{y\} \times \{u\}$ is a subset of K as is $g(y) \times \{y\} \times \{v\}$. Therefore, if t is either endpoint of a complementary open interval of T having both endpoints in T, $g(y) \times \{y\} \times \{t\}$ is a subset of K. Because T is closed and totally disconnected, if t is not an endpoint of such a complementary open interval, then t is a limit point of a sequence $p_1, p_2, p_3, \ldots$ of such points. Thus, $g(y) \times \{y\} \times \{t\}$ is a limit of a sequence of subsets of K and again $g(y) \times \{y\} \times \{t\}$ is a subset of K. Consequently, $F(t) \times \{t\}$ is a subset of K.

Therefore, in each case, if K is a subcontinuum of $G'(g,f)$ that contains a point of $F(t) \times \{t\}$ and a point not in $F(t) \times \{t\}$, we see that $F(t) \times \{t\}$ is a subset of K, so $F(t) \times \{t\}$ is a C-set in $G'(g,f)$. □

4. Main theorem and corollaries

We note that an upper semi-continuous set-valued funtion $f\colon [0,1] \to [0,1]$ whose graph is a sinusoid, as defined by Ingram in [4], will satisfy the conditions imposed on the bonding functions in Theorem 4 below. Hence, Theorem 4 and Corollary 1 generalize Ingram's Theorems 4.2 and 4.3 in [4].

Theorem 4. *Let $\{[0,1], f_i\}_{i\geq 1}$ be an inverse sequence, where for each $i \geq 1$, $f_i\colon [0,1] \to [0,1]$ is an upper semi-continuous, surjective interval-valued function. Suppose that for each $i \geq 1$, f_i is C-set-valued and f_i has no flat spot that composes to a non-degenerate value of f_j for $1 \leq j < i$. Then for each $n \geq 1$, the partial graph G'_n is an hereditarily decomposable chainable continuum.*

Proof. First we note that by Theorem 2, it follows that $G(f_i)$ is an hereditarily decomposable chainable continuum for each $i \geq 1$.

To establish the result, we show that, for each $n \geq 1$, $F_n\colon [0,1] \to G'_{n-1}$ is an upper semi-continuous continuum-valued function that is also C-set-valued, and $G(F_n)$ is an hereditarily decomposable chainable continuum.

It follows from [10, Corollary 4] that, for each $n \geq 1$, G'_n is a λ-dendroid. Since G'_n is homeomorphic to $G(F_n)$, we have that $G(F_n)$ is 1-dimensional, hereditarily unicoherent, and hereditarily decomposable for each $n \geq 1$. We recall from the last paragraph of Section 2 that for each $n \geq 2$, F_n is an upper semi-continuous continuum-valued function.

We use proof by induction to show that for each $n \geq 1$, F_n is C-set-valued, and $G(F_n)$ is a chainable continuum. Let $n = 1$. Since $F_1 = f_1$, it follows that F_1 has the desired properties.

Assume the statement is true for all positive integers less than or equal to some $n-1 \geq 1$. So, we assume that F_{n-1} is C-set-valued, and $G(F_{n-1}) \overset{T}{\approx} G'_{n-1}$ is a chainable continuum.

The continuum-valued functions $f_n\colon [0,1] \to [0,1]$ and $F_{n-1}\colon [0,1] \to G'_{n-2}$ satisfy the conditions of Theorem 3. We verify this claim below.

Our assumption that no flat spot for f_n composes to a nondegenerate value of f_j for $j < n$ gives us that if p is a flat spot for f_n, the orbit of p in $X_{n-1}, X_{n-2}, \ldots, X_1$ is a sequence of singletons. It follows from Lemma 2 that $F_{n-1}(p)$ is a singleton.

Let $S = \{x \in [0,1] \mid F_n(x) \text{ is nondegenerate}\}$. By Lemma 15 and Corollary 16 in [9], S is a 1^{st} category set. By noting that the set $S' = \{x \in [0,1] \mid F_{n-1}(y) \text{ is nondegenerate for some } y \in f_n(x)\}$ is a subset of S, we have that S' is a 1^{st} category set.

Lastly, we note that F_n is the function induced by f_n and F_{n-1}, and that $G'(F_{n-1}, f_n) \overset{T}{\approx} G(F_n) \overset{T}{\approx} G'_n$. Hence, $G'(F_{n-1}, f_n)$ is hereditarily unicoherent.

Hence, by Theorem 3, F_n is C-set-valued. Since, by inductive assumption, G'_{n-1} is a chainable continuum, and we noted earlier that G'_{n-1} is hereditarily decomposable, we have that, for each $t \in [0,1]$, $F_n(t)$ is an hereditarily decomposable chainable continuum in G'_{n-1}. It follows from

Theorem 2 that $G(F_n) \overset{T}{\approx} G'_n$ is an hereditarily decomposable chainable continuum. □

Corollary 1. *Let $\{[0,1], f_i\}_{i\geq 1}$ be an inverse sequence, where for each $i \geq 1$, $f_i\colon [0,1] \to [0,1]$ is an upper semi-continuous, surjective interval-valued function satisfying the conditions of Theorem 4. Then $\varprojlim\{X_i, f_i\}$ is a chainable continuum.*

Proof. The corollary follows from Theorem 4, and from Theorem 2.4 in [4]. □

Next, we provide an answer to Ingram's Problem 4.4 in [4].

Corollary 2. *Let $\{[0,1], f_i\}_{i\geq 1}$ be an inverse sequence, where for each $i \geq 1$, $f_i\colon [0,1] \to [0,1]$ is an upper semi-continuous, surjective interval-valued function. Suppose also that, for each $i \geq 1$, f_i is C-set-valued. If $X = \varprojlim\{X_i, f_i\}$ is tree-like, then X is chainable.*

Proof. We note that since the inverse limit X is tree-like, we have, by [10, Corollary 3], that no flat spot composes to a non-degenerate value in the inverse sequence. So, by Corollary 1, X is chainable. □

5. Reverse sequences of inverse functions

Let $\{X_i, f_i\}_{i\geq 1}$ be an inverse sequence of upper semi-continuous, surjective set-valued functions between compacta. In this section we consider conditions on the sequence $f_1^{-1}, f_2^{-1}, f_3^{-1}, \ldots$ of inverses that are sufficient for the chainability of $\varprojlim\{X, f_i\}$. To that end, let n be a positive integer and consider the finite inverse sequence below.

$$X_{n+1} \overset{f_n^{-1}}{\leftarrow} X_n \overset{f_{n-1}^{-1}}{\leftarrow} \ldots\ldots \overset{f_2^{-1}}{\leftarrow} X_2 \overset{f_1^{-1}}{\leftarrow} X_1$$

Although the indexing does not match the usual definition, we may consider the above sequence as a finite inverse sequence. We could make the indexing formally consistent with our earlier definition by letting $Y_i = X_{n+2-i}$ for $1 \leq i \leq n+1$, and $g_i = f_{n+1-i}^{-1}$ for $1 \leq i \leq n$. However, this would add a layer of unnecessary notation, obscuring the idea in our use of such sequences. So, we do not alter the indexing. We call this finite inverse sequence the *reverse sequence of inverse functions* associated with the finite inverse sequence

$$X_1 \overset{f_1}{\leftarrow} X_2 \overset{f_2}{\leftarrow} \ldots\ldots \overset{f_{n-1}}{\leftarrow} X_n \overset{f_n}{\leftarrow} X_{n+1}.$$

If there is no confusion, we simply say "the reverse sequence of inverse functions."

Recall the definition of the partial graph

$$G'_n = G'(f_1, \ldots, f_n) = \{(x_1, \ldots, x_{n+1}) \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \le i \le n\}.$$

Analogously, we let

$$G'(f_n^{-1}, \ldots, f_1^{-1}) = \{(x_{n+1}, \ldots, x_1) \mid x_{i+1} \in f_i^{-1}(x_i) \text{ for } 1 \le i \le n\}$$

be the partial graph of the reverse sequence of inverse functions. Note that, although the coordinates of the points in the two partial graphs are in reverse order, the conditions for membership in the two are equivalent. So, the mapping $h\colon G'_n \to G'(f_n^{-1}, \ldots, f_1^{-1})$ defined by $h(x_1, \ldots, x_{n+1}) = (x_{n+1}, \ldots, x_1)$ is clearly a homeomorphism from the partial graph G'_n onto the partial graph $G'(f_n^{-1}, \ldots, f_1^{-1})$ of the reverse sequence of inverse functions.

It follows that results that establish topological properties of a partial graph G'_n, based on conditions satisfied by the bonding functions $\{f_i \mid 1 \le i \le n\}$, may be used to establish topological properties of G'_n if the inverse functions $\{f_i^{-1} \mid 1 \le i \le n\}$ satisfy equivalent conditions in the reverse sequence of inverse functions.

We illustrate a simple example of this procedure using the two well-known theorems below.

Theorem 5. [7, Theorem 4.3] *For $1 \le i \le n+1$, let X_i be a continuum, and let $f_i\colon X_{i+1} \to X_i$ be an upper semi-continuous, surjective, continuum-valued function. Then the partial graph G'_n is a continuum.*

Theorem 6. [7, Theorem 4.5] *For $1 \le i \le n+1$, let X_i be a continuum, and let $f_i\colon X_{i+1} \to X_i$ be an upper semi-continuous, surjective, set-valued function whose inverse f_i^{-1} is continuum-valued. Then the partial graph G'_n is a continuum.*

Proof. We observe, from the discussion and definitions above, that the reverse sequence of inverse functions satisfies the hypothesis of Theorem 5, but with reverse indexing. Nevertheless, by Theorem 5, the partial graph $G'(f_n^{-1}, \ldots, f_1^{-1})$ is a continuum. Since $G'(f_n^{-1}, \ldots, f_1^{-1})$ is homeomorphic to G'_n, it follows that G'_n is a continuum. □

Suppose X is a compactum, and for all $i \ge 1$, $X_i = X$. In this case, we emphasize that, for a given $n \ge 1$, the reverse sequence of inverse functions $X \overset{f_n^{-1}}{\leftarrow} X \overset{f_{n-1}^{-1}}{\leftarrow} \ldots \overset{f_1^{-1}}{\leftarrow} X$ is not the same as the inverse sequence $X \overset{f_1^{-1}}{\leftarrow} X \overset{f_2^{-1}}{\leftarrow} \ldots \overset{f_n^{-1}}{\leftarrow} X$, and the two are unlikely to have homeomorphic partial graphs. Example 2.3 in [8] is an example where, for surjective upper semi-continuous functions $f_1, f_2\colon [0,1] \to [0,1]$, we have $G(f_2^{-1}, f_1^{-1}) \overset{T}{\approx} G(f_1, f_2)$ is connected, but $G(f_1^{-1}, f_2^{-1})$ is not connected.

Theorem 7. *Let $\{[0,1], f_i\}_{i\geq 1}$ be an inverse sequence, where for each $i \geq 1$, $f_i\colon [0,1] \to [0,1]$ is an upper semi-continuous, surjective set-valued function. Suppose that for each $i \geq 1$, f_i^{-1} is interval-valued, f_i^{-1} is C-set-valued, and f_i^{-1} has no flat spot that composes to a non-degenerate value of f_j^{-1} for $1 \leq i < j$. Then for each $n \geq 1$, the partial graph G'_n is an hereditarily decomposable chainable continuum.*

Proof. We simply observe that for each $n \geq 1$, the reverse sequence of inverse funtions satisfies the hypothesis of Theorem 4. It follows that $G'(f_n^{-1}, \ldots, f_1^{-1}) \overset{T}{\approx} G'_n$ is an hereditarily decomposable chainable continuum. □

We note that the "flat spot" condition in Theorem 7 is equivalent to "f_i has no flat spot that composes to a non-degenerate value of f_j for $1 \leq j < i$." This observation leads us to Corollary 3.

Corollary 3. *Let $\{[0,1], f_i\}_{i\geq 1}$ be an inverse sequence, where for each $i \geq 1$, $f_i\colon [0,1] \to [0,1]$ is an upper semi-continuous, surjective set-valued function satisfying the conditions of Theorem 7. Then $\varprojlim\{X_i, f_i\}$ is a chainable continuum.*

Proof. The corollary follows from Theorem 7, and from Theorem 2.4 in [4]. □

Corollary 4. *Let $\{[0,1], f_i\}_{i\geq 1}$ be an inverse sequence, where for each $i \geq 1$, $f_i\colon [0,1] \to [0,1]$ is an upper semi-continuous, surjective set-valued function, and $G(f_i^{-1})$ is a sinusoid. Then $\varprojlim\{X_i, f_i\}$ is a chainable continuum.*

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