BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Online First version

GENERAL TOPOLOGY

Unions of chainable continua with the fixed point property

by

C. L. HAGOPIAN and M. M. MARSH

Presented by the Editors

Summary. We investigate the fixed point property for continua that are unions of chainable continua. We answer Question 2 of our paper [Fund. Math. 231 (2013)] by showing the fixed point property is additive for chainable continua. We additionally show that (i) various finite unions of chainable continua have the fixed point property, and (ii) infinite unions of chainable continua that are 1-dimensional, upper semicontinuous clumps as defined by H. Cook have the fixed point property.

A *continuum* is a nonempty, compact, connected metric space. A *map* or *mapping* is a continuous function between topological spaces. A continuum X has the *fixed point property* (fpp) if each self mapping on X has a fixed point. Let X, Y, and $X \cap Y$ be continua with the fpp. We say that *the fpp is additive for X and Y* if $X \cup Y$ has the fpp. If $\mathcal{G}$ is some class of continua, we say that *the fpp is additive for the class $\mathcal{G}$* provided that whenever X and Y are in $\mathcal{G}$, the fpp is additive for X and Y.

The general question, "for which classes of continua is the fpp additive?", has, until now, had only one positive answer. K. Borsuk's theory of absolute retracts from the 1940s establishes that the fpp is additive for the class of absolute retracts. We show the fpp is additive for the class of chainable continua, as well as establishing the fpp for various other unions of chainable continua. Over the last 55 years, it has been shown the fpp is not additive for the following classes of continua:

(1) (W. Lopez [8], 1967) polyhedra;

2020 *Mathematics Subject Classification*: Primary 54F15, 54H25; Secondary 54G20.
Key words and phrases: chainable continua, unions of chainable continua, clumps, fixed point property.
Received 29 July 2021.
Published online *.

DOI: 10.4064/ba210729-30-7

(2) (A. L. Yandl [16], 1968) one-dimensional, planar continua;
(3) (R. Mańka [11], 1987) uniquely arcwise connected curves;
(4) (R. Mańka [12], 1990) rational, arcwise connected continua;
(5) (C. L. Hagopian and M. M. Marsh [4], 2015) tree-like continua.

Furthermore, the fpp is not additive for the classes of n-cell-like continua for $n \geq 2$, since R. L. Russo [14] and Hagopian [13, Theorem 6] independently proved each tree-like continuum can be realized as an n-cell-like continuum. There are various positive and negative results for the fpp of unions of continua X and Y with the fpp when additional conditions are added, particularly when strong conditions are placed on the continuum $X \cap Y$. For example, it is easy to prove the wedge of two continua with the fpp must have the fpp. Further discussion and more references can be found in [4, Section 1].

Surprisingly, the question of additivity of the fpp for the class of chainable continua has remained unanswered, and consequently presented an obvious gap that needs filling in this area of study. Using results of T. Maćkowiak, who continued work begun by Mańka [10], H. Bell [1], and Sieklucki [15], we establish a number of positive results for unions of chainable continua. Two questions of interest remain open.

Question 1 (Mańka [12, Remark, p. 35]). If X and Y are one-dimensional, planar continua with the fpp, and $X \cap Y$ is arcwise connected, must $X \cup Y$ have the fpp?

Question 2 (Hagopian and Marsh [4, Question 1, p. 218]). If X and Y are tree-like continua with the fpp, and $X \cap Y$ is a dendroid, must $X \cup Y$ have the fpp?

A continuum is *hereditarily unicoherent* if each pair of its intersecting subcontinua has a connected intersection. A *dendroid* is an arcwise connected, hereditarily unicoherent continuum. A continuum X is *chainable* if for each $\epsilon > 0$, there is a finite collection $U_1, \ldots, U_n$ of open sets covering X such that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$, and $\operatorname{diam}(U_i) < \epsilon$ for $1 \leq i \leq n$. In 1951, O. H. Hamilton [5] proved chainable continua have the fpp.

Given $\epsilon > 0$, a mapping $f\colon X \to Y$ is an *ϵ-mapping* if for each $y \in Y$, $\operatorname{diam}(f^{-1}(y)) < \epsilon$. A continuum X is *arc-like* if for each $\epsilon > 0$, there exists an ϵ-mapping from X onto $[0, 1]$. A continuum X is *tree-like* if for each $\epsilon > 0$, there exists a tree T and an ϵ-mapping from X onto T. It is well known that a continuum is chainable if and only if it is arc-like.

A continuum is *indecomposable* if it is not the union of two proper subcontinua. Let x be a point of a continuum X. The *x-composant* of X is the union of all proper subcontinua of X that contain x. If X is indecomposable, then X is the union of uncountably many dense, pairwise disjoint composants.

The following four theorems are of fundamental importance in establishing the results in this paper.

THEOREM 1 (Cook, [2, Theorem 1]). *Suppose X is a hereditarily unicoherent continuum such that each indecomposable subcontinuum of X is tree-like. Then X is tree-like.*

THEOREM 2 (Cook, [2, Lemma 2]). *If a continuum M is the union of two hereditarily unicoherent continua H and K whose intersection is a continuum, then M is hereditarily unicoherent.*

THEOREM 3 (Cook, [2, Theorem 2]). *If a continuum M is the union of two tree-like continua H and K whose intersection is connected, then M is tree-like.*

As defined by Maćkowiak [9], a composant C of an indecomposable subcontinuum Q of a continuum X is a *K-composant in X* if there is a subcontinuum L of X such that $\emptyset \neq L \cap Q \subset C$ and $L \setminus Q \neq \emptyset$.

THEOREM 4 (Maćkowiak, [9, Corollary 1(iv)]). *Suppose X is a hereditarily unicoherent continuum, and each nondegenerate indecomposable subcontinuum of X has the fpp and contains a composant that is not a K-composant. Then X has the fpp.*

DEFINITION 1. Let J be a subset of $\mathbb{N}$ with at least two elements, and suppose for each $i \in J$, H_i is a continuum. Let $X = \bigcup_{i\in J} H_i$ be a continuum, and suppose that

(i) $\bigcap_{i\in J} H_i \neq \emptyset$,
(ii) for $i \in J$, $H_i \setminus \bigcup_{j\neq i} H_j \neq \emptyset$, and
(iii) for $i, j \in J$, $H_i \cap H_j$ is a continuum.

Then we call X a *fan of continua*. More specifically, if J is a finite [infinite] subset of $\mathbb{N}$, we call X a *finite* [*countable*] *fan of continua*. If H_i is hereditarily unicoherent [treelike, chainable] for each $i \in J$, then we call X a *fan of hereditarily unicoherent* [*tree-like, chainable*] *continua*.

We establish some lemmas about fans of hereditarily unicoherent continua before we consider fans and other unions of chainable continua.

LEMMA 1. *Suppose $X = \bigcup_{i\in J} H_i$ is a fan of hereditarily unicoherent continua. Then $\bigcap_{i\in J'} H_i$ is connected for each nonempty $J' \subset J$.*

Proof. Suppose $\bigcap_{i\in J'} H_i$ is not connected for some $J' \subset J$. Assume, without loss of generality, that $1 \in J'$. Let a and b be points in different components of $\bigcap_{i\in J'} H_i$. Let A be a subcontinuum of H_1 that is irreducible between a and b. For some $j \in J'$, H_j does not contain some point p in A. By

Definition 1(iii), $H_1 \cap H_j$ is a subcontinuum of H_1. We note that $a, b \in H_1 \cap H_j \cap A$. It follows from the hereditary unicoherence of H_1 that $(H_1 \cap H_j) \cap A$ is a continuum. By the irreducibility of A, we know that $H_1 \cap H_j \cap A = A$. So, $p \in H_j$, contradicting our choice of j. Hence, $\bigcap_{i\in J'} H_i$ is connected. ∎

Lemma 2. *Suppose $X = \bigcup_{i=1}^{n} H_i$ is a finite fan of hereditarily unicoherent continua. Then X is a hereditarily unicoherent continuum.*

Proof. We use induction on n. For $n = 2$, the lemma follows from Theorem 2. Assume the lemma holds for some $n - 1 \geq 2$. So, $\bigcup_{i=1}^{n-1} H_i$ is a hereditarily unicoherent continuum. We consider the two hereditarily unicoherent continua $\bigcup_{i=1}^{n-1} H_i$ and H_n, and we show their intersection is a continuum.

Note that $(\bigcup_{i=1}^{n-1} H_i) \cap H_n = \bigcup_{i=1}^{n-1}(H_i \cap H_n)$. For $1 \leq i \leq n-1$, by Definition 1, $H_i \cap H_n$ is a hereditarily unicoherent continuum. By Lemma 1, for $1 \leq i < j \leq n-1$, $(H_i \cap H_n) \cap (H_j \cap H_n) = H_i \cap H_j \cap H_n$ is a continuum.

So, by inductive assumption, $(\bigcup_{i=1}^{n-1} H_i) \cap H_n$ is a hereditarily unicoherent continuum. So, by Theorem 2, $\bigcup_{i=1}^{n} H_i$ is a hereditarily unicoherent continuum. ∎

Lemma 3. *Suppose $X = \bigcup_{i=1}^{n} H_i$ is a finite fan of tree-like continua. Then X is a tree-like continuum.*

Proof. By Lemma 2, X is a hereditarily unicoherent continuum. We use induction on n to show that X is tree-like.

For $n = 2$, the lemma follows from Theorem 3. Assume the lemma holds for some $n-1 \geq 2$. So, $M = \bigcup_{i=1}^{n-1} H_i$ is tree-like. Let Q be an indecomposable subcontinuum of $X = \bigcup_{i=1}^{n} H_i$. Suppose Q meets both $M \setminus H_n$ and $H_n \setminus M$. Then Q is the union of two nonempty proper subcontinua, namely $Q = (Q \cap M) \cup (Q \cap H_n)$, which contradicts the indecomposability of Q. So, either $Q \subset M$ or $Q \subset H_n$, in which case Q is tree-like. By Theorem 1, X is tree-like. ∎

A continuum T is a *triod* if there exist subcontinua A_1, A_2, A_3, and K of T such that $T = A_1 \cup A_2 \cup A_3$, K is a proper subcontinuum of A_i for $i \in \{1, 2, 3\}$, and $K = A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3$.

Lemma 4. *Suppose X is a chainable continuum, and Q is an indecomposable proper subcontinuum of X. Then Q has at most two composants that are K-composants in X.*

Proof. Assume Q has three distinct composants C_1, C_2, and C_3 that are K-composants in X. For $i = 1, 2, 3$, let L_i be a subcontinuum of X such that $\emptyset \neq L_i \cap Q \subset C_i$ and $L_i \setminus Q \neq \emptyset$. Note that $L_i \cap Q = L_i \cap C_i$ for each $i = 1, 2, 3$. So, by hereditary unicoherence, L_1, L_2, and L_3 are pairwise disjoint. It follows that $(L_1 \cup Q) \cup (L_2 \cup Q) \cup (L_3 \cup Q)$ is a triod in X, which contradicts the assumption that X is chainable. ∎

Hereafter, we consider various unions of chainable continua.

LEMMA 5. *Suppose $X = \bigcup_{i=1}^{n} H_i$ is a finite fan of chainable continua. Then each indecomposable subcontinuum of X is contained in H_i for some $1 \le i \le n$.*

Proof. We use induction on n. If $n = 2$, and Q is an indecomposable subcontinuum of X, then either $Q \subset H_1$ or $Q \subset H_2$ as we saw in the inductive step of the proof of Lemma 3.

Assume the lemma is true for some $n-1 \ge 2$. Let Q be an indecomposable subcontinuum of X, and assume $Q \not\subset H_i$ for all $1 \le i \le n$. Let $L = \bigcup_{i=1}^{n-1} H_i$. If $Q \subset L$, then by inductive assumption, there exists $1 \le i \le n-1$ such that $Q \subset H_i$, contradicting our assumption. So, we have $Q \cap (H_n \setminus L) \ne \emptyset$. Also, $Q \cap (L \setminus H_n) \ne \emptyset$ since $Q \not\subset H_n$. By Lemma 2, X is hereditarily unicoherent. So, it follows that Q is the union of two of its nonempty proper subcontinua, specifically $Q = (Q \cap L) \cup (Q \cap H_n)$, contradicting the indecomposability of Q. ■

THEOREM 5. *Suppose $X = \bigcup_{i=1}^{n} H_i$ is a finite fan of chainable continua. Then X is a tree-like continuum with the fpp.*

Proof. By Lemma 3, X is a tree-like continuum, and hence hereditarily unicoherent. Since chainable continua have the fpp, it follows from Lemma 4, Lemma 5, and Theorem 4 that X has the fpp. ■

COROLLARY 1. *The fixed point property is additive for the class of chainable continua.*

Example 2, at the end of the paper, shows Theorem 5 cannot be generalized to a countable fan of chainable continua.

DEFINITION 2. A continuum X is a *tree of chainable continua* if X is hereditarily unicoherent, and X is a finite union of finite fans of chainable continua.

THEOREM 6. *Suppose X is a tree of chainable continua. Then each indecomposable subcontinuum of X is contained in some chainable subcontinuum of X.*

Proof. Let $X = \bigcup_{i=1}^{n} X_i$, where each $X_i = \bigcup_{j=1}^{m_i} H_{ij}$ is a fan of chainable continua. We show, in particular, that each indecomposable subcontinuum of X is contained in H_{ij} for some $1 \le i \le n$ and $1 \le j \le m_i$. We use induction on n.

If $n = 1$, the result follows from Lemma 5. Assume the lemma holds for some $n - 1 \ge 1$.

Let $L = \bigcup_{i=1}^{n-1} X_i$, and let Q be an indecomposable subcontinuum of X. If either $Q \subset L$, or $Q \subset X_n$, then the theorem follows, respectively, either from the inductive assumption or from Lemma 5. So, we assume $Q \cap (L \setminus X_n) \neq \emptyset \neq Q \cap (X_n \setminus L)$. It follows that $Q = (Q \cap L) \cup (Q \cap X_n)$ is not indecomposable, a contradiction. ∎

THEOREM 7. *Suppose X is a tree of chainable continua. Then X is a tree-like continuum with the fpp.*

Proof. It follows from Theorems 6 and 1 that X is tree-like. It follows from Theorem 6, Lemma 4, and Theorem 4 that X has the fpp. ∎

Considering infinite unions of hereditarily unicoherent, tree-like, or chainable continua, it is useful to recall Cook's clumps of continua. In [3], Cook defines a nondegenerate collection G of continua to be a *clump* provided that $\bigcup_{g \in G} g$ is a continuum, there exists a continuum C that is a proper subcontinuum of each $g \in G$, and C is the intersection of any two members of G. Cook calls C the *center* of G. As Cook's clumps relate to notions and results in this paper, for convenience we also say that $X = \bigcup_{g \in G} g$ is a *clump of continua*. Thus, saying a continuum is either a clump or a fan of continua is a structural statement about realizing X as a certain union of subcontinua.

For subsets $J \subset \mathbb{N}$ as in Definition 1, it is easy to see that if $X = \bigcup_{i \in J} H_i$ is a clump of continua, then $X = \bigcup_{i \in J} H_i$ is a fan of continua. However, a fan $X = \bigcup_{i \in J} H_i$ of continua may not be a clump of continua, expressed as a union of the same H_is, since the intersection of some two H_j and H_k may not equal $\bigcap_{i \in J} H_i$. Nevertheless, for finite J, we see, from Lemmas 2 and 3, that each fan $X = \bigcup_{i \in J} H_i$ of hereditarily unicoherent (tree-like) continua is always a clump of two hereditarily unicoherent (tree-like) continua. The same cannot be said for finite fans of chainable continua, as can be seen by viewing a tree T that has exactly three branchpoints, each of order 3, as a fan of three arcs. In fact, T cannot be realized as a clump of chainable continua. In general, for finite J with more than two elements, a fan union is a finer structure than a clump union.

A clump of continua G with center C is said to be *upper semicontinuous* provided that whenever $\{p_i\}_{i \geq 1}$ and $\{q_i\}_{i \geq 1}$ are two sequences of points in $X = \bigcup_{g \in G} g$ converging, respectively, to p and q in $X \setminus C$, where for each $i \geq 1$, p_i and q_i are in the same member of G, it follows that p and q are in the same member of G.

THEOREM 8. *Suppose $X = \bigcup_{g \in G} g$ is an upper semicontinuous clump of chainable continua, and* $\dim X = 1$. *Then X is a tree-like continuum with the fpp.*

Proof. It follows from [3, Theorem 12] that X is tree-like, and hence hereditarily unicoherent. It follows from [3, Theorem 10] that each indecomposable subcontinuum of X is contained in g for some $g \in G$. By Lemma 4 and Theorem 4, X has the fpp. ∎

In [6, Theorem 3.4], W. T. Ingram defines an inverse limit X on $[0,1]$ with a single surjective set-valued bonding function f, where the graph of f is the union of the graphs of two mappings f_1 and f_2 on $[0,1]$, the graphs of f_1 and f_2 intersect only at a common fixed point x of f_1 and f_2, and $f_1^{-1}(x) = \{x\} = f_2^{-1}(x)$. In the proof of Theorem 3.4, Ingram shows that $\dim X = 1$, and that X is the union of an upper semicontinuous clump of chainable continua with a degenerate center. In [7, Theorem 5.6], Ingram extends this result, allowing f to be the union of finitely many mappings or continuum-valued functions. In the case of f being a union of mappings, we have the following theorem that follows immediately from Ingram's Theorem 5.6 and our Theorem 8.

THEOREM 9. *Suppose $\mathcal{F} = \{f_1, \ldots, f_n\}$ is a finite collection of mappings on $[0,1]$ such that the union of the graphs of the members of $\mathcal{F}$ is the graph of a surjective upper semicontinuous set-valued function f. Suppose also that there is a point $x \in [0,1]$ such that (1) if $1 \le i \le n$, then $f_i(x) = x$ and $f_i^{-1}(x) = \{x\}$, and (2) if f_i and f_j are two members of $\mathcal{F}$, then x is the only coincidence point of f_i and f_j. Then $X = \varprojlim f$ is a tree-like continuum with the fpp.*

THEOREM 10. *Suppose $X = \bigcup_{i=1}^{\infty} H_i$ is a countable clump of chainable continua. Then X is a tree-like continuum with the fpp.*

Proof. It follows from [3, Theorem 15] that X is tree-like. As in the proof of Theorem 8, by [3, Theorem 10], each indecomposable subcontinuum of X is contained in H_i for some $i \ge 1$. By Lemma 4 and Theorem 4, X has the fpp. ∎

If X is a countable clump of continua with a degenerate center, then X is called a *countable wedge of continua.*

COROLLARY 2. *If X is a countable wedge of chainable continua, then X is a tree-like continuum with the fpp.*

The following example shows Corollary 2 cannot be generalized to wedges of tree-like continua.

EXAMPLE 1 (Hagopian and Marsh, [4, Example 1]). *There exists a countable wedge of tree-like continua, each having the fpp, that is a tree-like continuum without the fpp. Furthermore, all but one of the tree-like continua are arcs.*

Example 2 below shows the structure of countable fans of chainable continua is too general to necessitate the fpp.

EXAMPLE 2. *There is a countable fan of chainable continua, in fact of arcs, that is not tree-like and does not have the fpp.*

Proof. Let X be the continuum in Figure 1. We note that X is a countable fan of arcs. Let H_1 be the arc in X with endpoints q and p_1. For $i \geq 2$, let H_i be the arc in X with endpoints q and p_i that contains the point w. It is clear that $X = \bigcup_{i\geq 1} H_i$ is a countable fan of arcs that is not tree-like, and does not have the fpp. ■

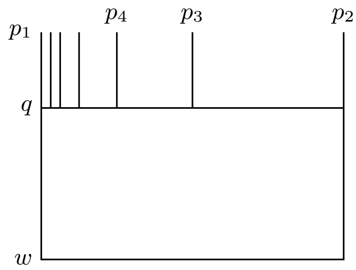

Fig. 1. A countable fan of arcs

References

[1] H. Bell, *On fixed point properties of plane continua*, Trans. Amer. Math. Soc. 128 (1967), 529–548.
[2] H. Cook, *Tree-likeness of dendroids and λ-dendroids*, Fund. Math. 68 (1970), 19–22.
[3] H. Cook, *Clumps of continua*, Fund. Math. 86 (1974), 91–100.
[4] C. L. Hagopian and M. M. Marsh, *Non-additivity of the fixed point property for tree-like continua*, Fund. Math. 231 (2015), 113–137.
[5] O. H. Hamilton, *A fixed point theorem for pseudo-arcs and certain other metric continua*, Proc. Amer. Math. Soc. 2 (1951), 173–174.
[6] W. T. Ingram, *Tree-likeness of certain inverse limits with set-valued functions*, Topology Proc. 42 (2013), 17–24.
[7] W. T. Ingram, *One-dimensional inverse limits with set-valued functions*, Topology Proc. 46 (2015), 243–253.
[8] W. Lopez, *An example in the fixed point theory of polyhedra*, Bull. Amer. Math. Soc. 73 (1967), 922–924.
[9] T. Maćkowiak, *Indecomposable continua and the fixed point property*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 27 (1979), 903–912.
[10] R. Mańka, *Association and fixed points*, Fund. Math. 91 (1976), 105–121.
[11] R. Mańka, *On uniquely arcwise connected curves*, Colloq. Math. 51 (1987), 227–238.
[12] R. Mańka, *On the additivity of the fixed point property for 1-dimensional continua*, Fund. Math. 136 (1990), 27–36.
[13] M. M. Marsh, *Covering spaces, inverse limits, and induced coincidence producing mappings*, Houston J. Math. 29 (2003), 983–992.
[14] R. L. Russo, *Universal continua*, Fund. Math. 105 (1979), 41–60.
[15] K. Sieklucki, *On a class of plane acyclic continua with the fixed point property*, Fund. Math. 63 (1968), 257–278.
[16] A. L. Yandl, *On a question concerning fixed points*, Amer. Math. Monthly 75 (1968), 152–156.

C. L. Hagopian, M. M. Marsh
Department of Mathematics & Statistics
California State University, Sacramento
Sacramento, CA 95819-6051, USA
ORCID (M. M. Marsh): 0000-0002-7472-3389
E-mail: hagopian@csus.edu
mmarsh@csus.edu