A fixed point theorem for conical shells

A. Domokos and M. M. Marsh

Abstract. We prove a fixed point theorem for mappings f defined on conical shells F in $\mathbb{R}^n$, where the image of f need not be a subset of F, nor even a subset of the cone that contains F. In this sense, our results extend, in $\mathbb{R}^n$, Krasnosel'skiĭ's well-known fixed point result on cones in Banach spaces [12]. Sufficiency for fixed points of f is dependent only on the behavior of f on the boundary of F. This behavior is related to notions of compressing or extending the conical shell F. We also discuss possible extensions of our theorem to infinite dimensional Banach spaces.

Mathematics Subject Classification (2010). Primary 54H25; Secondary 55M20, 54F15.

Keywords. convex cone, conical shell, fixed point.

1. Introduction

In 1960, Krasnosel'skiĭ [12, Theorem 2] proved a fixed point theorem for a completely continuous operator A on a real Banach space E that leaves a convex cone K invariant. If A either compresses or extends K, then A has a nonzero fixed point in K. The notions of compressing and extending the cone K are related to the order generated by K, and the fixed point result requires that the compressing or extending conditions are satisfied for all x in K that are outside the interior of a conical shell in K. Furthermore, the fixed point is in the conical shell.

Others have proved variations of Krasnosel'skiĭ's result, see for example [15, Theorems 1.2 & 1.3], [8, Theorems 2.5 & 2.6], and [7, Proposition 4.3]. In [9, Section 4], Górniewicz and Granas proved a version of Krasnosel'skiĭ's result for mappings f of compact attraction on a closed cone K, with $f(K) \subset K$, that only requires extension of K on the "top" and "bottom" boundaries of the conical shell in K. They use relative Lefschetz theory for mappings of pairs of ANRs as introduced by Bowszyc [1], and the Lefschetz theory extended to arbitrary ANRs as developed by Granas [10].

We will prove a fixed point theorem, of Krasnosel'skiǐ type, for mappings defined on a conical shell, where the image of the mapping is not necessarily a subset of the cone, and behavior of the mapping only on the boundary of the conical shell determines if the mapping has a fixed point. There are other fixed point results for mappings f on n-cells or compact convex sets, where the image of f is not required to be a subset of the domain of f, see for example [3], [11], [13], [14], and [16]. However, these results require additional geometric properties that are not needed in our fixed point theorem. For example, in [3] and [16], it is assumed that for every x, $f(x) - x$ belongs to the tangent cone of the weakly compact, convex domain of f at x (see also Remark 3 in section 4). Our theorem is not implied by any of these results. In section 4, we discuss examples that demonstrate this.

Our proof will use the relative Lefschetz theory applied to compact ANRs. To do this, we apply some basic notions from algebraic topology such as homotopy of mappings, degree theory for mappings of spheres, and Lefschetz theory for compact ANRs. Fundamental properties of these notions can be found in [17]. R. F. Brown's book [2] gives a more thorough discussion of index theory, Lefschetz theory, and Nielsen theory. A good reference for index theory and Lefschetz theory for Euclidean neighborhood retracts is [4].

2. Definitions and Preliminaries

Our setting will be Euclidean n-space $\mathbb{R}^n$. After the proof of our fixed point theorem, we will comment on possible generalizations of the theorem to infinite dimensional Banach and Hilbert spaces.

For a set S in $\mathbb{R}^n$, let $\mathrm{cl}S$, $\mathrm{int}S$, and $\mathrm{bd}S$ denote, respectively, the topological closure, interior, and boundary of S. If spaces S and T are homeomorphic, we write $S \approx T$. We refer to continuous functions as *mappings*. If A and B are subsets of $\mathbb{R}^n$, and $f\colon A \to B$ is a mapping that is not homotopic to a constant mapping from A into B, we say that f is *essential*.

For $r > 0$, let $S_r = \{x \mid \|x\| = r\}$, and let $B_r = \{x \mid \|x\| < r\}$. Let $\rho_r\colon \mathbb{R}^n/\{0\} \to S_r$ be the radial projection given by $\rho_r(x) = r\frac{x}{\|x\|}$. When $1 \le k < n$, we denote the unit k-sphere in $\mathbb{R}^{k+1}$ by S^k.

The set C is a *cone* in $\mathbb{R}^n$ if $\lambda x \in C$ for each $x \in C$ and $\lambda \ge 0$. For a set $B \subset \mathbb{R}^n$, let $\mathrm{cone}(B) = \{\lambda x \mid x \in B \text{ and } \lambda \ge 0\}$. For a cone C in $\mathbb{R}^n$, we recall some standard terminology and introduce notation that we use throughout the paper. If the interior of C is non-empty, C is called a *full cone.* If C is closed under vector addition, C is called a *convex cone.* It is straightforward to check that convex cones are convex. If $-C \cap C = \{0\}$, C is a *pointed cone.* Although we do not need the cone C in our fixed point result (Theorem 3.1) to be pointed, one variation of the result does require pointedness of C (see Remark 1).

For a cone C, and positive numbers r and R with $0 < r < R$, let the *conical shell* between r and R be defined by $F = (\mathrm{cl}B_R \setminus B_r) \cap C$. For t when

$r \le t \le R$, we let $F_t = S_t \cap C$. We refer to F_R as the *top of* F, and to F_r as the *bottom of* F.

For a full, closed, convex cone C that is a proper subset of $\mathbb{R}^n$, we call $L = \mathrm{bd}C \cap F$ the *lateral side of* F or simply the *side of* F. For such conical shells, it follows from results in Chapter 1 of [6] (see specifically Theorem 1 and Sections 4 and 6) that $F \approx$ a closed n-cell, $F_r \approx F_R \approx$ a closed $(n-1)$-cell, $F_t \cap L \approx$ an $(n-2)$-sphere for each $r \le t \le R$, and L is topologically the product of an $(n-2)$-sphere and an interval.

Hereafter, for each pair of full, closed, convex cones C and K, with conical shell F in C between r and R, and $C \cap K = \{0\}$, we define two associated retractions.

Let $P\colon \mathrm{cl}(\mathbb{R}^n \setminus K) \to C$ be a retraction such that for every $a > 0$, $P(S_a \setminus \mathrm{int}(C \cup K)) = S_a \cap \mathrm{bd}C$, and let $\rho\colon C \setminus \{0\} \to F$ be defined by

$$\rho(x) = \begin{cases} x & \text{if } x \in F \\ \rho_R(x) & \text{if } x \in C \setminus B_R \\ \rho_r(x) & \text{if } x \in (C \cap B_r) \setminus \{0\}. \end{cases}$$

Note that for points x in $R^n \setminus \mathrm{int}(C \cup K)$, $P(x) \in \mathrm{bd}C$, and for points x in $C \setminus F$, $\rho(x) \in F_r \cup F_R \subset \mathrm{bd}F$.

Let $f\colon F \to \mathbb{R}^n$ be a mapping, where F is a conical shell between r and R in a full, closed, convex cone C. Suppose also that there exists a full, closed, convex cone K such that $C \cap K = \{0\}$, and $f(F) \subset \mathbb{R}^n \setminus K$. Under these assumptions, we will pick a conical shell F' in $K' = \mathrm{cl}(\mathbb{R}^n \setminus K)$ such that $F \cup f(F) \subset \mathrm{int}F'$. See Figure 1 below.

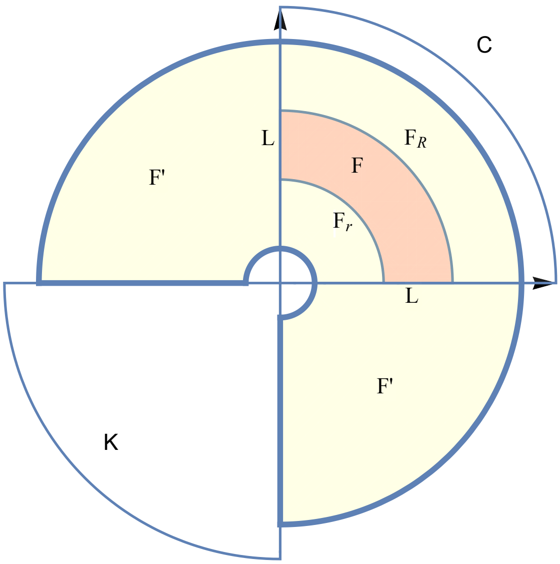

Figure 1. Conical shells F and F'

We adopt the following descriptive terminology to give a sense of the behavior of the mapping f relative to the conical shell F. If $f(L) \subset F' \setminus \mathrm{int}C$, we say that *$f$ maps the side of F outward*. If f maps the side of F outward and there exists a t with $r \le t \le R$ such that $f|_{F_t \cap L} \colon F_t \cap L \to F' \setminus \mathrm{int}C$ is essential, we say that *f maps the side of F outward and around C*, or more specifically that *f maps $F_t \cap L$ outward and around C* . If $f(L) \subset C$, we say that *f maps the side of F inward into C*.

If $f(F_R) \subset F' \setminus B_R$, we say that *$f$ maps the top of F outward*, and if $f(F_r) \subset \mathrm{cl}B_r \cap F'$, we say that *$f$ maps the bottom of F outward*. If $f(F_R) \subset \mathrm{cl}B_R \cap F'$, we say that *$f$ maps the top of F inward*, and if $f(F_r) \subset F' \setminus B_r$, we say that *$f$ maps the bottom of F inward*.

If f either maps both the top and bottom of F outward (extension) or maps both the top and bottom of F inward (compression), then we say that *f maps the top and bottom of F in opposite directions*.

3. A Fixed Point Theorem

Theorem 3.1. *Suppose $f \colon F \to \mathbb{R}^n$ is a mapping defined on a conical shell F in a full, closed, convex cone C. Suppose also that there exists a full, closed, convex cone K such that $C \cap K = \{0\}$, and $f(F) \subset \mathbb{R}^n \setminus K$.*

(1) *If f maps the side of F outward and around C, and f maps the top and bottom of F in opposite directions, then f has a fixed point.*

(2) *If f maps the side of F inward into C, and f maps the top and bottom of F in opposite directions, then f has a fixed point.*

Proof. We assume that the conical shell F' has been chosen as above. There are four cases to consider. The idea underlying the proof of each case, except the last, is to pick a set $A \subset F'$ that contains the points $f(x)$, where $x \in \mathrm{bd}F$ and f maps x outward from F. Then we define a mapping, which is a composition of f and retractions onto F, and is also a mapping of the pair (F', A) to itself. We apply relative Lefschetz theory to this mapping to get a fixed point of f in F. Since the set A differs from case to case, the details of the proofs also differ and necessitate inclusion of all cases.

Case (1a): Suppose that f maps L outward and around C, and maps both F_R and F_r outward. Let $A = F' \setminus \mathrm{int}F$. We note that $A \approx \mathrm{bd}F \times [0,1] \approx S^{n-1} \times [0,1]$. Since f maps L outward and around C, there exists $r \le t \le R$ such that $f|_{F_t \cap L} \colon F_t \cap L \to F' \setminus \mathrm{int}C$ is essential. Since $f|_L$ is itself a homotopy between $f|_{F_r \cap L}$ and $f|_{F_R \cap L}$ (recall that, for each $r \le t \le R$, $F_t \cap L \approx$ an $(n-2)$-sphere), we also have that $f|_{F_R \cap L} \colon F_R \cap L \to F' \setminus \mathrm{int}C$ is essential.

We consider the mapping of ANR pairs $f\rho P|_{F'} : (F', A) \to (F', A)$. It has been observed that F' is an AR, A is an ANR, and ρP maps $A = F' \setminus \mathrm{int}F$ onto the boundary of F. So, $f\rho P(F') \subset F'$. We also need to see that $f\rho P(A) \subset A$. Let $x \in A$. Since $\rho P(x) \in \mathrm{bd}F = L \cup F_R \cup F_r$, by the assumptions of this case, we have that $f\rho P(x) \in A$.

We denote the relative Lefschetz number of the map of pairs $f\rho P|_{F'}$: $(F', A) \to (F', A)$ by $\widetilde{\Lambda}\ (f\rho P|_{F'})$. By Bowszyc [1], $\widetilde{\Lambda}\ (f\rho P|_{F'}) = \Lambda(f\rho P|_{F'}) - \Lambda(f\rho P|_A)$. Since F' is contractible, we have that $\Lambda(f\rho P|_{F'}) = 1$.

In order to calculate $\Lambda(f\rho P|_A)$, consider the mappings $f|_{\mathrm{bd}F}\colon \mathrm{bd}F \to A$ and $\rho P|_A\colon A \to \mathrm{bd}F$. By the commutativity property of the Lefschetz theory (see [2, Exercise 7, page 50] or [4, (1.8) & Theorem 4.1]), we have that $\Lambda(f\rho P|_A) = \Lambda(\rho P f|_{\mathrm{bd}F})$. So, we wish to calculate $\Lambda(\rho P f|_{\mathrm{bd}F})$.

It is clear that the mapping $\rho P\colon F' \setminus \mathrm{int}C \to F' \setminus \mathrm{int}C$ is homotopic to the identity mapping on $F' \setminus \mathrm{int}C$. So, $\rho P f|_{F_R \cap L}\colon F_R \cap L \to F' \setminus \mathrm{int}C$ is homotopic to $f|_{F_R\cap L} = \mathrm{id}|_{F'\setminus \mathrm{int}C} \circ f|_{F_R\cap L}$. Note that $\rho P f(F_R \cap L) = F_R \cap L$. So, $\rho P f|_{F_R\cap L}$ is a map of a topological $(n-2)$-sphere to itself. Since $f|_{F_R\cap L}\colon F_R \cap L \to F' \setminus \mathrm{int}C$ is essential and $F_R \cap L \subset F' \setminus \mathrm{int}C$, it follows that $\rho P f|_{F_R\cap L}\colon F_R \cap L \to F_R \cap L$ is essential. Hence, $\deg(\rho P f|_{F_R\cap L}) \neq 0$. Let $\deg(\rho P f|_{F_R\cap L}) = k \neq 0$.

Let g be the suspension of $\rho P f|_{F_R\cap L}$, see [17, page 27]. That is, using standard notation for suspensions, $g = \sum \rho P f|_{F_R\cap L}\colon \sum(F_R \cap L) \to \sum(F_R \cap L)$. We identify $\sum(F_R \cap L)$ with $\mathrm{bd}F$, and note that $\rho P f$ maps F_R to F_R. Since F_R is contractible, $\rho P f$ is homotopic to g on F_R. Also note that $\rho P f$ maps $F_r \cup L$ to $F_r \cup L$. Since $F_r \cup L$ is contractible, $\rho P f$ is homotopic to g on $F_r \cup L$. Since $g = \rho P f$ on the intersection of F_R and $F_r \cup L$, it follows that $\rho P f|_{\mathrm{bd}F}$ is homotopic to g. Now, $\mathrm{bd}F \approx S^{n-1}$, and by Vick [17, Prop.1.20], $\deg(\rho P f|_{\mathrm{bd}F}) = \deg(g) = \deg(\rho P f|_{F_R\cap L}) = k \neq 0$.

Thus, $\Lambda(\rho P f|_{\mathrm{bd}F}) = 1 + (-1)^{n-1}\deg(\rho P f|_{\mathrm{bd}F}) \neq 1$. Hence, $\widetilde{\Lambda}\ (f\rho P|_{F'})$ is not zero. By the relative Lefschetz theory, $f\rho P$ has a fixed point in $\mathrm{cl}(F' \setminus A) = F$. Let $x \in F$ with $f\rho P(x) = x$. Since $x \in F$, $\rho P(x) = x$. Therefore, $x = f(x)$.

Case (1b): Suppose that f maps L outward and around C, maps F_R inward, and maps F_r inward. Let $A = F' \setminus \mathrm{int}C$. We note that $A \approx S^{n-2} \times [0,1]^2$.

Again, we have that $f\rho P|_{F'}$: $(F', A) \to (F', A)$ is a mapping of ANR pairs. To see that $f\rho P(A) \subset A$. Let $x \in A$. We observe that $\rho P(x) \in L$. By the assumptions of this case, and the choice of A, we have that $f(L) \subset A$. So, we have that $f\rho P(x) \in A$.

Since f maps L outward and around C, $f|_{F_t\cap L}\colon F_t \cap L \to A$ is essential for some $r \leq t \leq R$. Since $\rho P|_{F_t}$ is the identity map on F_t, we have that $f\rho P|_{F_t\cap L} = f|_{F_t\cap L}\colon F_t \cap L \to A$ is essential. It follows that $f\rho P|_A\colon A \to A$ is essential.

Since $F_t \cap L \approx (n-2)$-sphere is a strong deformation retract of A, S^{n-2} and A are homotopy equivalent. So, S^{n-2} and A have the same homology (see [17, Prop.1.11]. That is, $H_i(S^{n-2}, \mathbb{Q})$ is isomorphic to $H_i(A, \mathbb{Q})$ for all $i \geq 0$. Hence, the Lefschetz number of $f\rho P|_A$ can be calculated by $\Lambda(f\rho P|_A) = 1 + (-1)^{n-2}k$, where $(f\rho P|_A)_*\colon H_n(A, \mathbb{Q}) \to H_n(A, \mathbb{Q})$ is multiplication by $k \neq 0$. So, $\Lambda(f\rho P|_A) \neq 1$. Hence, $\widetilde{\Lambda}\ (f\rho P|_{F'}) = \Lambda(f\rho P|_{F'}) - \Lambda(f\rho P|_A) = 1 - \Lambda(f\rho P|_A) \neq 0$. So, $f\rho P$ has a fixed point in $\mathrm{cl}(F' \setminus A) = F' \cap C$. Let

$x \in F' \cap C$ with $x = f\rho P(x)$. If $x \in F$, then $x = f\rho P(x) = f(x)$. So, x is a fixed point of f in F and the proof is complete.

Suppose that $x \in C \setminus \mathrm{cl}B_R$. Then $x = f\rho(x)$ and $\rho(x) \in F_R$. Since f maps F_R inward, $f\rho(x) \in \mathrm{cl}B_R$. So, $x \in \mathrm{cl}B_R$. But this contradicts that x is in $C \setminus \mathrm{cl}B_R$. The remaining case that $x \in C \cap B_r$ is analogous.

Case (2a): Suppose that f maps L inward into C, and maps F_r and F_R outward. Recall, from our definitions, that $f(\mathrm{int}F \cup F_r \cup F_R)$ may intersect the complement of C in this case, but we have that $Pf(F) \subset F' \cap C$. Let $A = \mathrm{cl}(F' \cap C \setminus F)$. We note that A is the disjoint union of two ARs, namely $(C \setminus B_R) \cap F'$ and $C \cap \mathrm{cl}B_r \cap F'$.

We consider the map of ANR pairs $Pf\rho|_{F'\cap C}\colon (F'\cap C, A) \to (F'\cap C, A)$. Since f maps F_r and F_R outward, we see that $Pf\rho(A) \subset A$. We also observe that $\Lambda(Pf\rho|_A) \neq 1$. In fact, $\Lambda(Pf\rho|_A) = 2$, since A has two contractible components as mentioned above.

So, as in the previous cases, by the relative Lefschetz theory, there exists a point x in $\mathrm{cl}((F' \cap C) \setminus A) = F$ such that $x = Pf\rho(x)$. Since $x \in F$, $x = Pf(x)$.

Suppose that $f(x) \notin C$. Since f maps L inward into C, it follows that x must be in $F \setminus L \subset \mathrm{int}C$. But then $x \neq Pf(x)$, since P maps $F' \setminus C$ to $\mathrm{bd}C$. So, $f(x) \in C$. Since P is the identity mapping on C, we have that $x = f(x)$ and the proof is complete.

Case (2b): Suppose that f maps L inward into C, and maps F_r and F_R inward. We will not need the relative Lefschetz theory in this case.

The mapping $Pf\rho|_{F'\cap C}\colon F' \cap C \to F' \cap C$ has a fixed point since $F' \cap C$ is an AR. Let $x \in F' \cap C$ such that $x = Pf\rho(x)$. Since f maps F_r and F_R inward in this case, $Pf\rho|_{F'\cap C}$ has no fixed points in $(F' \cap C) \setminus F$. So, $x \in F$ and $x = Pf(x)$. As in the last paragraph of Case (2a), it follows that $x = f(x)$. □

The two remarks below are immediate, but we include them for thoroughness. Also, as in Remark 1, we use the hyperplane version of our Theorem 3.1(1) in the examples discussed in section 4.

Remark 1. Theorem 3.1 also holds for mappings defined on the portion of a full, pointed, closed, convex cone that lies between two parallel hyperplanes. Specifically, let H be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ such that $H \cap C = \{0\}$, let $x \in C$ with $\|x\| = 1$, and let $0 < r < R$. Define F to be those points of C that lie in $H + \lambda x$ for some $r \leq \lambda \leq R$. Let U be any open convex set containing 0 such that $\mathrm{bd}U \cap C = (H + \{x\}) \cap C$. Let $\mathrm{bd}U$ be the unit sphere for a norm $\|\cdot\|_*$ on $\mathbb{R}^n$. If f is a mapping from F into $\mathbb{R}^n$ having analogous properties as in Theorem 3.1 relative to $\|\cdot\|_*$, then f has a fixed point.

Remark 2. Let C be a closed cone in $\mathbb{R}^n$ whose complement is a full convex cone. Theorem 3.1 also holds for mappings f on conical shells F in C, where f satisfies analogous properties relative to F and C. This can be seen by observing that the proof of Theorem 3.1 is primarily topological and that the associated sets C, F, F', and A in this setting are homeomorphic to their counterparts in Theorem 3.1.

4. Examples in $\mathbb{R}^3$

We describe below, an example in $\mathbb{R}^3$ that satisfies the conditions of Theorem 3.1(1a), and thus, yields a fixed point for the mapping described. We also point out how a slight modification in this example gives an example satisfying the conditions of Theorem 3.1(1b). Additionally, we show that other results of a similar nature in the literature cannot be used to get a fixed point for the mapping since these results require geometric conditions that are not satisfied. We also discuss the necessity of the essential condition in the definition of "mapping outward and around" the conical shell.

Since any example satisfying Theorem 3.1(1a) will map the side of the conical shell outward, Krasnosel'skiĭ's result will not apply since $f(C) \not\subset C$. Since conical shells are not convex, Halpern and Bergman's results in [11] will not apply. From Remark 1, however, one might ask if a convex version of Theorem 3.1 would follow from any of the results in [3], [11], [13], [14], or [16]. The example below illustrates that this is not the case.

Example 1. There is a mapping on a convex portion of a cone in $\mathbb{R}^3$ that has a fixed point since it satisfies the conditions of Theorem 3.1(1a). The mapping does not satisfy the geometric conditions of the theorems in the references above.

Proof. In $\mathbb{R}^3$, let H be the xy-plane, and let $p = (0, 0, 1)$. Let S_1, B_1 and S_2, B_2 be, respectively, the circles and closed disks of radius one and two about $\{p\}$ in $H + \{p\}$. Let $C = \text{cone}(B_1)$ and $C' = \text{cone}(B_2)$ in $\mathbb{R}^3$. Let F be the convex region in C between $H + \{p\}$ and $H + \{2p\}$, and let F' be the convex region in C' between $H + \{\frac{1}{2}p\}$ and $H + \{\frac{5}{2}p\}$. Note that the lateral side of F is $L = \{tx \mid x \in S_1,\ 1 \le t \le 2\}$, and the lateral side of F' is $L' = \{tx \mid x \in S_2,\ \frac{1}{2} \le t \le \frac{5}{2}\}$. So, the circle S_1 is the "bottom edge" of L, and the circle S_2 is contained in L'. Also, note that the bottom and top of F are, respectively, the sets $F_1 = B_1$ and $F_2 = 2B_1$. It is clear that $F \subset F'$. We now define a mapping $f\colon F \to F'$ that satisfies the conditions of our Theorem 3.1(1a).

Let $\sigma_1 \colon S_1 \to S_1$ be the squaring map on the circle S_1, treating $(1, 0, 1)$ as the unit element in S_1. Let $h\colon S_1 \to S_2$ be defined by $h(x, y, 1) = (2x, 2y, 1)$, and let $\sigma\colon S_1 \to S_2$ be given by $\sigma = h \circ \sigma_1$. So, σ is a two-to-one covering map. We note that $\sigma\colon S_1 \to F' \setminus \text{int}C$ is essential. Let $\ell\colon [1, 2] \to [\frac{1}{2}, \frac{5}{2}]$ be linear increasing; that is, $\ell(t) = 2t - \frac{3}{2}$. Let $\hat{\sigma}\colon L \to L'$ be given by $\hat{\sigma}(tx) = \ell(t)\sigma(x)$ for $x \in S_1$ and $1 \le t \le 2$.

We may now extend $\hat{\sigma}$ to any mapping $f\colon F \to F'$ that maps the top and bottom of F in opposite outward directions into F'. That is, f can be chosen to be any mapping such that $f|_L = \hat{\sigma}$, $f(F_1) \subset \{tx \mid x \in B_2,\ \frac{1}{2} \le t \le 1\}$, $f(F_2) \subset \{tx \mid x \in B_2,\ 2 \le t \le \frac{5}{2}\}$, and $f(F) \subset F'$. Clearly, any such mapping

f satisfies the conditions of Theorem 3.1(1a), and hence has a fixed point. We show in Remarks 3 and 4 below that f does not satisfy the additional geometric conditions needed for application of related theorems. □

Remark 3. In order for any of Halpern and Bergman's results in [11] to apply to the mapping f, one of the following three conditions must hold. By Lemma 18.1 in [3], condition (i) below is the same as condition (1) in [16]. See [11] for the precise definitions of inward, outward, and outward normal sets.

(i) Each point x of F is mapped by f into the closure of its inward set I_x.

(ii) Each point x of F is mapped by f into the closure of its outward set O_x.

(iii) No point x of F is mapped by f into its outward normal set N_x.

Since f maps the point $x = (-1, 0, 1)$ of S_1 to the point $(1, 0, \frac{1}{2})$ on the bottom circle of L', $f(x)$ is neither in the closure of I_x nor in the closure of O_x. So, neither (i) nor (ii) is satisfied.

Since $\ell(\frac{9}{8}) = \frac{3}{4}$, f maps the point $y = (\frac{9}{8}, 0, \frac{9}{8})$ in L to the point $(\frac{3}{2}, 0, \frac{3}{4})$ in L'. It is easy to check that the vector $y - f(y)$ is perpendicular to the vector y. So, $f(y)$ is in the outward normal set N_y. Thus, (iii) is not satisfied.

Remark 4. In order to apply either the results of Marsh [14] or the results of Poincaré-Miranda [13], L would need to be topologically partitioned into two pairs of "opposite" 2-cells, and f would need to map these 2-cells, in this case, outward relative to the planes containing them. The double covering behavior of f on L to L' eliminates this possibility. So, as mentioned earlier, our theorem requires none of the geometric conditions found in related results, and is not implied by any these results.

Remark 5. A similar example that satisfies our Theorem 3.1(1b) can be constructed simply by defining the map ℓ in the example above to be linear decreasing from $[1, 2]$ onto $[\frac{1}{2}, \frac{5}{2}]$. The maps σ, $\hat{\sigma}$, and f are defined analogously. The new mapping f now maps the top and bottom of F inward. In a manner similar to that above, it can be seen that this mapping f also does not satisfy the conditions of the other mentioned results.

Remark 6. Although the assumption in Theorem 3.1(1) that the map f be essential on L may seem rather strong, without it there are easy examples of fixed-point-free mappings satisfying the other conditions of our theorem. For example, if we simply change the map σ in Example 1 above to the two-to-one circle map $\alpha\colon S_1 \to S_2$, where $\alpha(1, 0, 1) = (2, 0, 1) = \alpha(-1, 0, 1)$, α maps the top half of S_1 counter-clockwise around all of S_2, and maps the bottom half of S_1 clockwise around all of S_2. The map $\alpha\colon S_1 \to F' \setminus \mathrm{int}C$ is inessential. We extend α to $\hat{\alpha}\colon L \to L'$ analogously as σ was extended to $\hat{\sigma}$ in Example 1. Lastly, we can extend $\hat{\alpha}$ to a map $f\colon F \to F'$ so that our outward conditions on L, and on the top and bottom of F are satisfied, yet $f(F) \cap F = \emptyset$. Thus, f will not have a fixed point.

5. Concerning variations and extensions of Theorem 3.1

There are easy examples of continuous, fixed-point-free mappings on conical shells in infinite dimensional Hilbert spaces that satisfy properties as in Theorem 3.1 (see Example at the end of this section). Since Krasnosel'skiǐ's fixed point result, and other related fixed point results mentioned in the introduction, hold for completely continuous mappings on cones in infinite dimensional Banach spaces, it is natural to ask if any of the four cases in Theorem 3.1 will extend to completely continuous mappings on conical shells in infinite dimensional Banach spaces. A mapping $f\colon X \to Y$ between Banach spaces is *completely continuous* provided that $f(B)$ is contained in a compact set for each bounded set B. We claim that Theorem 3.1(2) has an infinite dimensional analog for completely continuous mappings. It seems unlikely that Theorem 3.1(1) has such an infinite dimensional analog. Discussion follows.

Let X be an infinite dimensional Banach space, and suppose that $f\colon F \to X$ is a completely continuous mapping defined on a conical shell F in a full, closed, convex cone C in X. Also assume that all sets S_r, S_R, B_r, B_R, F_r, F_R, and L are defined analogously as they were in $\mathbb{R}^n$, and that $F \cup f(F) \subset X \setminus K$ for some full, closed, convex cone K with $C \cap K = \{0\}$.

Extension of Theorem 3.1(2) to infinite dimensional Banach spaces. Using the Lefschetz theory extended to arbitrary ANRs in [10], and the general theory of ARs and ANRs as discussed in [5] and applied to cones in [8, page 204], we see that the proof of both Cases (2a) and (2b) remain valid for completely continuous mappings. One should also note that the mapping $Pf\rho P$ used in these two proofs is completely continuous, since in our setting, ρ and P map bounded sets to bounded sets, and f is assumed to be completely continuous.

Non-extension of Theorem 3.1(1) to infinite dimensional Banach spaces. In this setting, the homotopy condition used to define "around C" in $\mathbb{R}^n$ gives no information to distinguish between the behavior of mappings on L. In particular, since closed balls can be retracted to their bounding spheres and spheres are contractible (see [5]), but not compact, there seems to be no "around C" condition on the mapping f that would make f completely continuous, or be strong enough to guarantee fixed points of continuous-only maps satisfying the other conditions of Theorem 3.1(1). Furthermore, the relative Lefschetz theory proofs given in Cases (1a) and (1b) will not work since the set A, in each case, is contractible.

Below, in the setting of an infinite dimensional Hilbert space, we give a simple example of a fixed-point-free continuous, but not completely continuous, mapping that satisfies the remaining conditions of Theorem 3.1(1).

Example 2. There exists, in an infinite dimensional Hilbert space, a fixed-point-free continuous, but not completely continuous, mapping that satisfies the conditions of Theorem 3.1(1).

Proof. Let X be an infinite dimensional Hilbert space, and let H be a subspace of codimension one in X. Fix a point $p \neq 0$ in X. Let S_1, B_1 and S_2, B_2

be, respectively, the spheres and closed balls of radius one and two about $\{p\}$ in $H+\{p\}$. We note that since $H+\{p\}$ is isomorphic to an infinite dimensional Hilbert space, S_1 is not compact. Also, by Theorem 6.2 in [5], S_1 is an AR.

Let $C = \text{cone}(B_1)$ in X, and let $C' = \text{cone}(B_2)$ in X. Let F be the conical shell between $r = 1$ and $R = 2$ in C, and let F' be the conical shell between $r' = \frac{1}{2}$ and $R' = \frac{5}{2}$ in C'. Assume, without loss of generality, that S_1 is the "bottom sphere" on the lateral side L of F, and S_2 lies in the lateral side L' of F'. Note that $L \approx S_1 \times [1,2] \approx S_2 \times [\frac{1}{2}, \frac{5}{2}] \approx L'$. So, let $h\colon L \to L'$ be a homeomorphism that maps S_1 to $\frac{1}{2}S_2$, and maps $2S_1$ to $\frac{5}{2}S_2$.

Now, $F \approx B_1 \times [1,2] \approx$ the 1-ball in X; so, bd$F \approx$ the 1-sphere in X. By [5, Theorem 6.2], there exists a retraction $r_0\colon F \to \text{bd}F$. Let F_1 and F_2 denote, respectively, the bottom and top of F. So, $\text{bd}F = L \cup F_1 \cup F_2$. Note that $F_1 \cap L \approx S_1 \approx F_2 \cap L$. Since S_1 is an AR, there exist retractions $r_1\colon F_1 \to F_1 \cap L$ and $r_2\colon F_2 \to F_2 \cap L$. So, $(r_1 \cup r_2 \cup \text{id}|_L) \circ r_0\colon F \to L$ is a retraction taking F_1 to the bottom sphere S_1 of L, and taking F_2 to the top sphere $2S_1$ of L. It is now easy to see that $h \circ (r_1 \cup r_2 \cup \text{id}|_L) \circ r_0\colon F \to L'$ is a fixed-point-free mapping that satisfies the hypothesis of Theorem 3.1(1a). □

Acknowledgement. The authors wish to thank the referees for suggestions that improved the paper. In particular, we thank the referee who noticed that Theorem 3.1(2) could be strengthened to its present form.

References

[1] C. Bowszyc, *Fixed point theorems for the pairs of spaces*, Bull. Acad. Polon. Sci. **XVI**(1968), #11, 845–850.

[2] R. F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Company, Glenview, Illinois, 1970.

[3] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, 1985.

[4] A. Dold, *Fixed point index and fixed point theorem for Euclidean neighborhood retracts*, Topology **4**(1965), 1–8.

[5] J. Dugundji, *An extension of Tietze's Theorem*, Pacific J. Math. **1**(1951), 353–367.

[6] H. G. Eggleston, *Convexity*, Cambridge University Press, 1969.

[7] G. Fournier, *Fixed point principles for cones of linear normed space*, Can. J. Math. **XXXII**(1980), #6, 1372–1381.

[8] G. Fournier and H. O. Peitgen, *On some fixed point principles for cones in linear normed spaces*, Math. Ann. **225**(1977), 205–218.

[9] L. Górniewicz and A. Granas, *On a theorem of C. Bowszyc concerning the relative version of the Lefschetz fixed point theorem*, Bull. Institute Math. Acad. Sinica **13**(1985), #2, 137–142.

[10] A. Granas, *The Leray-Schauder Index and the fixed point theory for arbitrary ANRs*, Bull. Soc. Math. France **100**(1972), 209–228.

[11] B. R. Halpern and G. M. Bergman, *A fixed point theorem for inward and outward maps*, Trans. Amer. Math. Soc. **130**(1968), #2, 353–358.

[12] M. A. Krasnosel'skiĭ, *Fixed points of cone-compressing or cone-extending operators*, Soviet Math. Dokl. **1**(1960), 1285–1288.

[13] W. Kulpa, *The Poincaré-Miranda Theorem*, Amer. Math. Monthly **104**(1997), #6, 545–550.

[14] M. M. Marsh, *Fixed point theorems for partially outward mappings*, Topology and Its Appl. **153**(2006), 3546–3554.

[15] R. D. Nussbaum, *Periodic solutions of some nonlinear, autonomous functional differential equations* II, J. of Differential Equations **14**(1973), 360–394.

[16] S. Reich, *On fixed point theorems obtained from existence theorems for differential equations*, J. Math. Anal. Appl. **54**(1976), 26–36.

[17] J. W. Vick, *Homology Theory, An Introduction to Algebraic Topology*, Springer-Verlag, New York, 1994.

A. Domokos
Department of Mathematics & Statistics
California State University, Sacramento
Sacramento,CA 95819-6051
e-mail: `domokos@csus.edu`

M. M. Marsh
Department of Mathematics & Statistics
California State University, Sacramento
Sacramento,CA 95819-6051
e-mail: `mmarsh@csus.edu`