Connectedness of inverse limits with functions f_i where either f_i or f_i^{-1} is a union of continuum-valued functions

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Abstract

We establish a general theorem for connectedness of the inverse limit X of an inverse sequence $\{X_i, f_i\}_{i\geq 1}$ on metric continua with surjective upper semi-continuous set-valued bonding functions, where for each $i \geq 1$, f_i has a connected graph, and either f_i or f_i^{-1} is a union of continuum-valued functions. Properties of certain set-valued functions from the factor spaces onto partial graphs in the inverse sequence imply connectedness of X.

Keywords: connected, inverse limit, partial graph, set-valued function, union of continuum-valued functions
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1. Introduction

In the setting of inverse sequences $\{X_i, f_i\}_{i\geq 1}$ on metric continua with surjective upper semi-continuous set-valued bonding functions, where, for each $i \geq 1$, f_i has a connected graph, and either f_i or f_i^{-1} is a union of continuum-valued functions, we prove several theorems for connectedness of the partial graphs $G'(f_1, \ldots, f_n)$, and of the inverse limit $\varprojlim\{X_i, f_i\}$.

Properties related to the partial graphs that run from bonding functions f_i that are not unions of continuum-valued functions to bonding functions f_j $(i < j)$, where f_j^{-1} is not a union of continuum-valued functions, will be critical in determining connectedness of the inverse limit. If there are no such

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changes in the inverse sequence, as in Corollary 1 or Corollary 3, then the inverse limit is connected.

Our most general results are Theorem 5 and Corollary 4, but other theorems and corollaries herein may be useful for particular applications. We give examples in Section 4 which show that, in general, inverse limits of these types will not be connected. It is known (see Example 2) that the inverse limit may not be connected, even if for each $i \geq 1$, either f_i or f_i^{-1} is itself continuum-valued. So, additional conditions are necessary. We give such conditions in Theorem 5, and in Corollaries 4, 5, 6, and 7.

A general introduction to results and questions related to connectedness of an inverse limit with set-valued functions can be found in Section 2 of [7], in Sections 2.2, 2.3, 2.4, 2.6, and 2.7 of [8], and in Section 4 of [10].

Since the inception of "generalized" inverse limits, there has been considerable interest in conditions that will ensure connectedness of the inverse limit. Our interest is specialized to those inverse sequences on continua where for each $i \geq 1$, the graph of the bonding function f_i is connected, and either f_i or its inverse is a union of continuum-valued functions. Although not specific to this area, we would be remiss to not mention the very nice results on connectedness/nonconnectedness of generalized inverse limits due to Sina Greenwood, Judy Kennedy, and Michael Lockyer, see [1, 2, 3, 4, 5]. The results in the first, second, third, and fifth references are for inverse limits on intervals. The results in the fourth reference generalize to inverse limits on continua. Their results include several characterizations of nonconnectedness of generalized inverse limits. These characterizations are related to the existence of a certain finite sequence in the infinite product of the factor spaces, called variously C-sequences, CC-sequences, and HC-sequences in the cited papers. The definitions are somewhat technical, but simply stated the sequence required for nonconnectedness is related to the existence of a proper basic open set U and a closed set $K \subset U$ in the infinite product of the factor spaces, where the inverse limit meets K and U in the same set, and the inverse limit is not contained in U. In the fifth reference, components of the inverse limit are discussed and a characterization of nonconnectedness is given in terms of the system admitting a component base. See the cited papers for details and precise defintions.

For inverse limits on $[0, 1]$ with a single upper semi-continuous bonding function f, in [6, Theorems 4.2 and 4.3], Ingram established some connectedness results if f is a union of mappings. In [7], with f a union of two intervalvalued functions, one of which is surjective, Ingram shows that $\varprojlim\{[0,1], f\}$ is connected.

In the Hausdorff setting, Ingram and Mahavier [10, Theorems 4.7 and 4.8] proved that inverse limits on continua, where all bonding functions are continuum-valued or the inverses of all bonding functions are continuum-valued, are connected. Also in the Hausdorff setting, Ingram proved [7, Theorem 2.12] that if X is a continuum and $f\colon X \to 2^X$ is a set-valued function with a closed graph that is the union of continuum-valued functions, one of which is surjective and universal with respect to the others, then $\varprojlim\{X, f\}$ is connected.

In [14], Nall proved three theorems related to an inverse limit on a single continuum with a single surjective upper semi-continuous bonding function. His results are also established in the Hausdorff setting. They are

(1) (Theorem 3.1) If X is a continuum and $f\colon X \to 2^X$ is a union of continuum-valued functions, then $\varprojlim\{X, f\}$ is connected.

(2) (Theorem 3.3) Suppose X is a continuum and $f\colon X \to 2^X$ is surjective. Then $\varprojlim\{X, f\}$ is connected if and only if $\varprojlim\{X, f^{-1}\}$ is connected.

(3) (Theorem 3.5) Suppose X is a continuum, $f\colon X \to 2^X$ is surjective, and $\varprojlim\{X, f\}$ is connected. If $g\colon X \to X$ is a mapping that commutes with f, and the graphs of f and g are not disjoint, then $\varprojlim\{X, f \cup g\}$ is connected.

In [12], the author generalized Nall's Theorem 3.1 above, in the metric setting, to inverse limits on inverse sequences of continua X_i, and bonding functions where either all f_i or all f_i^{-1} are unions of continuum-valued functions. Since the author's notation in that paper was a bit different than in this paper, we include, in Section 5, the results related to connectedness in Section 1 of [12]. Also, it will be convenient for the reader to have all results that are used in this paper readily at hand. These results and a few other early results about connectedness of both partial graphs and inverse limits in this setting are also given in Section 5. First, we introduce the relevant definitions and notation.

2. Definitions and remarks

A *compactum* is a compact metric space. All spaces considered in this paper will be compacta. A *continuum* is a connected compactum. A continuous function with be referred to as a *mapping*.

Let X and Y be compacta. A function $f\colon X \to 2^Y$ is *upper semi-continuous at the point* $x \in X$ if for each open set V in Y containing the set $f(x)$, there is an open set U in X such that $x \in U$ and $f(p) \subset V$ for each $p \in U$. If $f\colon X \to 2^Y$ is upper semi-continuous at each point of X, then f is said to be *upper semi-continuous*. Ingram shows in [8, Theorem 1.2], that the function $f\colon X \to 2^Y$ being upper semi-continuous is equivalent to having the graph of f be closed. We will occasionally apply this theorem without comment.

We refer to functions $f\colon X \to 2^Y$ as *set-valued functions* from X to Y and we write $f\colon X \to Y$ is a set-valued function. The *graph* of f, which we denote by $G(f)$, is the subset of $X \times Y$ consisting of all points (x, y) with $y \in f(x)$. For each product $X \times Y$ of compacta X and Y, let $c_2\colon X \times Y \to Y$ denote coordinate projection. The *range* of the set-valued function $f\colon X \to Y$ is defined as $R(f) = c_2(G(f))$. The set-valued function $f\colon X \to Y$ is *surjective* if $R(f) = Y$. If $A \subset X$, let $f|_A$ be the set-valued function whose domain is A and such that $f|_A(x) = f(x)$ for $x \in A$.

A set-valued function $f\colon X \to Y$ is *continuum-valued* if for each $x \in X$, the set $f(x)$ is a subcontinuum of Y. Suppose that $f\colon X \to Y$ is a set-valued function, and $g'\colon X \to Y$ is a continuum-valued function with $G(g') \subset G(f)$. We say that $g'\colon X \to Y$ is *max continuum-valued* if whenever $g\colon X \to Y$ is continuum-valued with $G(g') \subset G(g) \subset G(f)$, we have that $G(g) = G(g')$. We note that the graph of each continuum-valued function g such that $G(g) \subset G(f)$ is contained in the graph of a max continuum-valued function g' such that $G(g') \subset G(f)$. Specifically, for each $x \in X$, simply define $g'(x)$ to be $c_2(L)$, where L is the component of $(\{x\} \times Y) \cap G(f)$ that contains $\{x\} \times g(x)$. If there exists $x \in X$ such that $f(x) = Y$, we say that f is *full-valued at* x.

We say that a set-valued function $f\colon X \to Y$ is a *union of continuum-valued functions* if for each $x \in X$ and each $y \in f(x)$, there exists a continuum-valued function $g\colon X \to Y$ such that $y \in g(x)$ and $G(g) \subset G(f)$. If the set-valued function $f\colon X \to Y$ is surjective, we say that $f^{-1}\colon Y \to X$ is a *union of continuum-valued functions* if for each $y \in Y$ and $x \in f^{-1}(y)$, there exists a continuum-valued function $g\colon Y \to X$ such that $x \in g(y)$ and $G(g) \subset G(f^{-1})$.

Suppose $f\colon X \to Y$ is a union of continuum-valued functions. Let $\mathcal{C}(f)$ be the collection of all max continuum-valued functions $g\colon X \to Y$ such that $G(g) \subset G(f)$. If there exists $h \in \mathcal{C}(f)$ such that whenever $g \in \mathcal{C}(f)$, we have that $G(g) \cap G(h) \neq \emptyset$, then we say that h is *universal with respect to each* $g \in \mathcal{C}(f)$. Under these conditions, we say that $\mathcal{C}(f)$ *has a universal member.*

A minor variation of this definition was introduced by Ingram in [7]. We could define $\mathcal{C}(f)$ to be the collection of all continuum-valued functions $g\colon X \to Y$ such that $G(g) \subset G(f)$. The existence, or non-existence, of a universal member of $\mathcal{C}(f)$ would not be changed. We prefer the given definition as it is typically easier, for specific examples, to identify the collection of max continuum-valued functions. We note that if $\mathcal{C}(f)$ has a universal member, then $G(f)$ is connected.

Let $X_1, X_2, \ldots$ be a sequence of compacta. Throughout, we let $\{X_i, f_i\}_{i\geq 1}$ denote an inverse sequence with upper semi-continuous set-valued bonding functions $f_i\colon X_{i+1} \to X_i$, and its inverse limit is given by

$$\varprojlim\{X_i, f_i\} = \{\mathbf{x} = (x_1, x_2, \ldots) \in \prod_{i\geq 1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } i \geq 1\}.$$

For $j, n \in \mathbb{N}$ with $j \leq n$, we define the set below.

$$G_j^{n+1} = G'(f_j, \ldots, f_n) = \{\mathbf{x} \in \prod_{i=j}^{n+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } j \leq i \leq n\}.$$

We refer to these sets as *partial graphs* in the inverse sequence. These sets have also been called Mahavier products and Ingram-Mahavier products. We note that $G_1^2 = G'(f_1) = G(f_1^{-1})$. For consistency of notation, for $i \geq 1$, we let $G_i^i = X_i$. We emphasize that hereafter all set-valued functions are assumed to be upper semi-continuous, and to have values that are closed sets. The notation $X \overset{T}{\approx} Y$ will indicate that X is homeomorphic to Y.

Suppose that $\{X_i\}_{i\geq 1}$ is a sequence of compacta, and for each $i \geq 1$, $f_i\colon X_{i+1} \to X_i$ is a surjective, set-valued function. We make the following definitions of functions that will be useful throughout the paper. Fix $n \geq 1$. For $1 \leq i \leq j \leq n+1$, we let

(i) π_j be the projection mapping from $\prod_{k=i}^{n+1} X_k$ onto the j^{th} coordinate,

(ii) $\pi_{(i)}$ be the projection mapping from $\prod_{k=1}^{n+1} X_k$ onto all coordinates except the i^{th} coordinate,

(iii) $F_{i,j}\colon X_{j+1} \to G_i^j$ be the function where $(x_i, \ldots, x_j)$ is in $F_{i,j}(x_{j+1})$ if and only if $(x_i, x_{i+1}, \ldots, x_{j+1}) \in G_i^{j+1}$, and

(iv) $L_j\colon G_{j+1}^{n+1} \to X_j$ be the function where x_j is in $L_j(x_{j+1}, \ldots, x_{n+1})$ if and only if $(x_j, x_{j+1}, \ldots, x_{n+1}) \in G_j^{n+1}$

Note that we must specify some $n \geq 1$ before the domains of the functions in (i), (ii), and (iv) above are clear. We use this convention throughout rather than making the notation more cumbersome.

Remarks.

1. The set-valued function $F_{1,j}: X_{j+1} \to G_1^j$ defined in (iii) above is the function F_j considered by Nall in [13]. Nall proves in Lemma 5.1 that F_j is upper semi-continuous. We will also need the function L_j, defined in (iv) above, in the proofs of Lemma 5 and Theorem 4. That $F_{i,j}$ and L_j are upper semi-continuous follows analogously as in Nall's proof.

2. In Corollary 4.2 of [9], Ingram establishes that the inverse limit X of an inverse sequence $\{X_i, f_i\}_{i\geq 1}$ with set-valued functions is homeomorphic to an inverse limit on the sequence of partial graphs $\{G_1^i\}_{i\geq 1}$ with bonding functions that are surjective mappings. We will use this result in the proof of several corollaries. Also, we observe that since G_1^i is a continuous image of G_1^{i+1} for each $i \geq 1$, it follows that if G_1^n is connected for some $n \geq 1$, then G_1^i is connected for each $1 \leq i \leq n$.

The next remark shows that certain amalgamations of portions of a finite inverse sequence produce a shorter inverse sequence whose partial graph is homeomorphic to the partial graph of the original sequence.

3. Suppose $\{X_i, f_i\}_{i=1}^n$ is a finite inverse sequence on compacta, where for each $1 \leq i \leq n$, f_i is a surjective set-valued function. Let $1 \leq j < k \leq n$. Then G_1^{n+1} is homeomorphic to the partial graph of each inverse sequence below.

(1) $G_1^j \xleftarrow{F_{1,j}} X_{j+1} \xleftarrow{f_{j+1}} \ldots \xleftarrow{f_n} X_{n+1}$, and

(2) $X_1 \xleftarrow{f_1} \ldots \xleftarrow{f_{j-1}} X_j \xleftarrow{\pi_j|_{G_j^k}} G_j^k \xleftarrow{F_{j,k}} X_{k+1} \xleftarrow{f_{k+1}} \ldots \xleftarrow{f_n} X_{n+1}$.

It is clear that G_1^{n+1} is homeomorphic to the partial graph in (1). Let G denote the partial graph of the sequence in (2). That $G_1^{n+1} \overset{T}{\approx} G$ is easily seen by noting that the point $(x_1, \ldots, x_j, x_j, \ldots, x_k, \ldots, x_{n+1})$ is in G if and only if the point $(x_1, \ldots, x_j, x_{j+1}, \ldots, x_k, \ldots, x_{n+1})$ is in G_1^{n+1}, and that the mapping $h: G \to G_1^{n+1}$ that drops one of the repeated j^{th}-coordinates is a homeomorphism.

Our objective is to find reasons why some inverse limits on continua are connected, and others are not, when the bonding functions or their inverses are unions of continuum-valued functions. In this setting, we introduce terminology that should provide immediate clarity and replace the use of long, cumbersome statements. The terminology will also benefit discussion of the examples in Section 4. Although the definitions below are for infinite inverse sequences, we will use analogous terminology for finite inverse sequences.

Let $\{X_i, f_i\}_{i\geq 1}$ be an inverse sequence, where for each $i \geq 1$, X_i is a continuum, $f_i\colon X_{i+1} \to X_i$ is a surjective set-valued function with a connected graph, and either f_i or f_i^{-1} is a union of continuum-valued functions.

For a given $i \geq 1$, if f_i is a union of continuum-valued functions, we say that f_i *is a union*; if f_i^{-1} is a union of continuum-valued functions, we say that f_i *is an inverse union*. We note that under our general assumption, if f_i is not a union, then f_i is an inverse union, and vice versa.

If, in the inverse sequence $\{X_i, f_i\}_{i\geq 1}$, there exist i and j where f_i is not a union and f_j is not an inverse union, then we call $\{X_i, f_i\}_{i\geq 1}$ a *mixed inverse sequence*. If $i < j$, we say that there is *a change from an inverse union (at i) to a union (at j)*; and if $j < i$, we say that there is *a change from a union (at j) to an inverse union (at i)*. As we will see, a change from a union (at j) to an inverse union (at i), that has no other changes from j to i, will have its associated partial graph G_j^{i+1} be connected. This, in general, is not the case for changes from an inverse union to a union. So, additional conditions must be assumed to ensure connectedness of partial graphs of inverse sequences that contain such changes. We provide one such condition, but, of course, there may be others.

It would be of interest to know if any condition related to the composition function $f_{i,j} = f_i \circ f_{i+1} \circ \cdots \circ f_{j-1}$ when $i < j-1$, through a change from an inverse union (at i) to a union (at j) could ensure connectedness of G_i^{j+1}. In the first submission of this paper, the author claimed to have such a condition, but a careful and thorough referee observed that a critical theorem related to the composition property was incorrect. It seems that any condition on the composition function $f_{i,j}$ that is related to our general assumption of being either a union or an inverse union of continuum-valued functions is not sufficient. In fact, even if $j = i+2$ and the composition function $f_{i,j}$ is both a union and an inverse union, the partial graph G_i^j may not be connected. Too much structural information about the partial graph G_i^j is lost in the graph of $f_{i,j}$. The author is indepted to the referee for finding this error.

3. Determining sequences

Let $\{X_i, f_i\}_{i\geq 1}$ be a mixed inverse sequence containing a change from an inverse union (at j) to a union (at k). We say that the sequence $f_j, \ldots, f_k$ is a *determining sequence* if either $j = 1$ or f_{j-1} is a union, and f_i is an inverse union for all $j \leq i < k$. A *max determining sequence* is a determining sequence that is not a "proper" subsequence of any other determining sequence.

A mixed inverse sequence $\{X_i, f_i\}_{i\geq 1}$ is *eventually alternating* if there exists $k \geq 1$ such that for all $i \geq 0$, f_{k+2i} is not a union, and f_{k+2i+1} is not an inverse union. If $k = 1$, we say that $\{X_i, f_i\}_{i\geq 1}$ is *alternating*. We note that in an eventually alternating inverse sequence each successive pair of bonding functions starting with f_{k+2} and f_{k+3} is a max determining sequence. If $k = 1$, then f_1, f_2 is also a max determining sequence.

Since conditions on determining sequences will be a central part of most of our main results, we make some observations related to determining sequences.

Observation 1. *If $\{X_i, f_i\}_{i\geq 1}$ is a mixed inverse sequence containing a change from an inverse union (at ℓ) to a union (at m), then there exists a determining sequence in $f_1, \ldots, f_m$.*

Proof. Let k be the least integer in $\{\ell + 1, \ldots, m\}$ such that f_k is not an inverse union. Since f_m is not an inverse union, k must exist. So, we have that for $\ell \leq i < k$, f_i is an inverse union. If $\ell = 1$, or if, for all $1 \leq i < \ell$, f_i is not a union, then by definition, $f_1, \ldots, f_k$ is a determining sequence. Otherwise, pick the largest j, where $1 < j \leq \ell$ and f_{j-1} is a union. Then $f_j, \ldots, f_k$ is a determining sequence. □

Observation 2. *If $\{X_i, f_i\}_{i\geq 1}$ is a mixed inverse sequence containing two determining sequences $f_j, \ldots, f_k$ and $f_\ell, \ldots, f_m$ with $j \leq \ell$, then either $k < \ell$ or $k = m$.*

Proof. Note that $k \neq \ell$ since f_k is not an inverse union and f_ℓ is an inverse union. So, we assume that $\ell < k$.

If $j = \ell$, assume, without loss of generality, that $k \leq m$. If $k = m$, the proof is complete. If $k < m$, then the determining sequence $f_\ell, \ldots, f_m$ contains f_k, which is not an inverse union, contradicting that $f_\ell, \ldots, f_m$ is a determining sequence.

So, we have that $j < \ell < k$. If either $m < k$ or $k < m$, we violate the definition of determining sequence, as in the previous paragraph, for one of the two sequences. So, $k = m$ and the proof is complete. □

It follows from Observation 2 that each determining sequence is contained in a max determining sequence. Also, if $f_j, \ldots, f_k$ and $f_\ell, \ldots, f_m$ are max determining sequences with $j < \ell$, then $k < \ell$. We say, in this case, that the two max determining sequences are *disjoint*, and we note that each two distinct max determining sequences are disjoint. If there is no determining sequence in $f_{k+1}, \ldots, f_{\ell-1}$, we say that the two max determining sequences are *consecutive*. We say that $f_j, \ldots, f_k$ and $f_\ell, \ldots, f_m$ are *adjacent* if $\ell = k+1$.

Observation 3. *Suppose that $\{X_i, f_i\}_{i=1}^n$ is a finite mixed inverse sequence on continua. Suppose also that $j > 1$, and $f_j, \ldots, f_k$ is the first max determining sequence in $f_1, \ldots, f_n$, in the sense that there is no determining sequence in $f_1, \ldots, f_{j-1}$. Then for all $1 \leq i \leq j-1$, f_i is a union.*

Proof. Suppose there exists m with $1 \leq m \leq j-1$ and f_m is not a union. We assume that m is the least such integer. So, either $m = 1$, or for each $1 \leq i < m$, f_i is a union. Since $f_j, \ldots, f_k$ is a max determining sequence, it follows that there exists $m < r \leq j-1$ where f_r is not an inverse union; for otherwise, $f_m, \ldots, f_k$ is a determining sequence "properly" containing $f_j, \ldots, f_k$. But now we have that $f_m, \ldots, f_r$ is a determining sequence in $f_1, \ldots, f_{j-1}$, which contradicts our hypothesis. □

Observation 4. *Suppose that $\{X_i, f_i\}_{i=1}^n$ is a finite mixed inverse sequence on continua containing two consecutive max determining sequences $f_j, \ldots, f_k$ and $f_\ell, \ldots, f_m$ with $k < \ell$. Then f_i is a union for all $k+1 \leq i \leq \ell-1$.*

Proof. The proof is analogous to the proof of Observation 3. □

Observation 5. *Suppose $\{X_i, f_i\}_{i\geq 1}$ is an inverse sequence on continua, where for each $i \geq 1$, f_i is either a union or an inverse union. Suppose also that $\{X_i, f_i\}_{i\geq 1}$ contains no determining sequence. Then either*

(i) f_i is an inverse union for all $i \geq 1$,

(ii) f_i is a union for all $i \geq 1$, or

(iii) there exists $k > 1$ such that f_i is a union for all $1 \leq i \leq k-1$, and f_i is an inverse union for all $i \geq k$.

Proof. Suppose that (i) and (ii) are not the case. Since (i) is not the case, for some $j \geq 1$, f_j is not an inverse union. So, f_j is a union, and for $1 \leq i < j$, f_i is a union. For otherwise, by Observation 1 there would be a determining sequence in $f_1, \ldots, f_j$. Since (ii) is not the case, for some $k > j$, f_k is not a union. We assume that k is the least such integer. So, we have that for all $1 \leq i < k$, f_i is a union. Suppose there exists $m > k$ such that f_m is not an inverse union. Then, again by Observation 1, there would be a determining sequence in $f_1, \ldots, f_m$, contradicting the hypothesis. Hence, we have that (iii) holds. □

Observation 6. *Suppose $\{X_i, f_i\}_{i\geq 1}$ is an inverse sequence on continua, where for each $i \geq 1$, f_i is surjective with a connected graph, and f_i is either a union or an inverse union. Suppose also that $\{X_i, f_i\}_{i\geq 1}$ contains no determining sequence. Then $\varprojlim\{X_i, f_i\}$ is a continuum.*

Proof. See Corollaries 1 and 3 in Sections 4 and 5 respectively. □

4. Examples

As we will see in the examples below, the determining sequences may destroy the connectedness of a partial graph, and in turn, of the inverse limit. The additional condition in Theorem 5 will ensure that the partial graphs will remain connected through max determining sequences, and the condition will also ensure connectedness of the partial graphs G_1^n and of the inverse limit. So, it is the determining sequences that will, indeed, determine connectedness of inverse limits of mixed inverse sequences if certain properties are satisfied. Results in Section 6 will provide sufficient conditions on max determining sequences to establish connectedness of the partial graphs G_1^n and of the inverse limit space.

For readers familiar with techniques introduced in [4] and [5], connectedness (or non-connectedness) of the partial graphs associated with the max determining sequences could also be determined by applying methods from [4] in the case of inverse sequences on continua, and from [5] in the case of inverse sequences on intervals. However, Example 3 illustrates that connectedness alone of the partial graphs associated with the max determining sequences in $f_1, f_2, \ldots, f_n$ is not enough to determine connectedness of G_1^n. So, the additional contition in Theorem 5 must be checked for the max determining sequences to establish connectedness of the inverse limit space. We discuss these ideas in the examples below. In each connected example in this section, we reference results in later sections that establish connectedness of the example.

Example 1. (Example 2.3 in [11]) In this example, set-valued functions $f_1, f_2\colon [0,1] \to [0,1]$ are defined by Ingram and Marsh (see Figure 1) in such a way that f_1 is a union of two mappings (and not an inverse union), and f_2 is an inverse union of two mappings (and not a union). For $i \geq 3$, let f_i be the identity mapping on $[0,1]$.

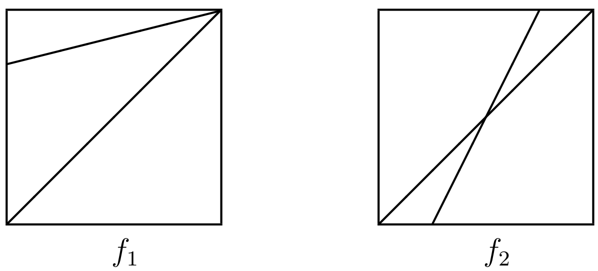

Figure 1. Graphs of the functions in Example 1

Let X_1, X_2, and X_3 be, respectively, the limits of the following three inverse sequences,

$$[0,1] \xleftarrow{f_1} [0,1] \xleftarrow{f_2} [0,1] \xleftarrow{f_3} \ldots\ldots, \qquad [0,1] \xleftarrow{f_2} [0,1] \xleftarrow{f_1} [0,1] \xleftarrow{f_3} \ldots\ldots,$$

$$\text{and} \quad [0,1] \xleftarrow{f_1^{-1}} [0,1] \xleftarrow{f_2^{-1}} [0,1] \xleftarrow{f_3} \ldots\ldots$$

Ingram and Marsh show that X_1 is connected, while X_2 and X_3 are not connected. We note that, in the first inverse sequence, there is a change from f_1 a union to f_2 an inverse union, and there are no other changes. This type of change preserves the connectedness of the partial graph G_1^3 (see Corollary 3, Section 6). For $n \geq 3$, $G_1^n \overset{T}{\approx} G_1^3$, so X_1 is connected.

In the second and third inverse sequences, there is a change from an inverse union to a union, respectively f_2, f_1 and f_1^{-1}, f_2^{-1}, and there are no other changes. The partial graph G_1^3 in each of these two inverse sequences is not connected. Again, for $n \geq 3$, $G_1^n \overset{T}{\approx} G_1^3$, so X_2 and X_3 are not connected.

Even when the bonding functions, or their inverses, are continuum-valued, a determining sequence can destroy connectedness of the associated partial graph. The next example illustrates this, but we indicate how an additional condition added to the determining sequence can ensure connectedness of the partial graph. For this example, the condition will be that $F_{1,2}\colon [0,1] \to G_1^2$ is a union, and $\mathcal{C}(F_{1,2})$ has a universal member (see Corollaries 6 and 7, Section 6).

Example 2. Let $f_1\colon [0,1] \to [0,1]$ be the set-valued function defined by $f_1(t) = \{\frac{1}{2}\}$ for $0 \leq t < \frac{3}{4}$, and $f_1(t) = \{\frac{1}{2}, 4t-3\}$ for $\frac{3}{4} \leq t \leq 1$. Note that f_1^{-1} is given by $f_1^{-1}(\frac{1}{2}) = [0,1]$, and otherwise $f_1^{-1}(t) = \{\frac{t+3}{4}\}$. So, f_1 is not a union, but f_1^{-1} is, in fact, continuum-valued. Let $f_2\colon [0,1] \to [0,1]$ be defined by $f_2(\frac{1}{2}) = [0,1]$, and otherwise, $f_2(t) = \{\frac{1}{2}t + \frac{1}{4}\}$. We see that f_2 is continuum-valued, and f_2^{-1} is not a union (see Figure 2). For $i \geq 3$, let f_i be the identity mapping on $[0,1]$.

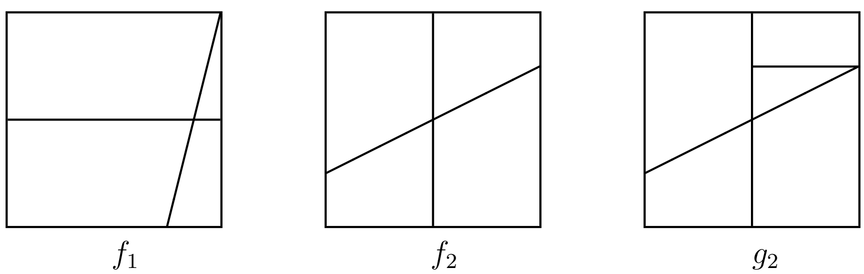

Figure 2. Graphs of the functions in Example 2

As in the second and third inverse sequences from Example 1, we have a max determining sequence f_1, f_2, and, indeed, the partial graph G_1^3 and the inverse limit are not connected. If we add identity mappings between f_1 and f_2 , for example, as in the inverse sequence

$$[0,1] \xleftarrow{f_1} [0,1] \xleftarrow{\text{id}} [0,1] \xleftarrow{\text{id}} [0,1] \xleftarrow{f_2} [0,1] \xleftarrow{f_3} \ldots\ldots,$$

the change from inverse union to union has been "spread out". Now, the max determining sequence runs from the first bonding function through the fourth. All 3-length partial graphs are connected, but the partial graph G_1^5, which contains the change, is not connected. So, if we wish to add conditions to ensure connectedness of the inverse limit, we must account for changes of this type that happen "far apart" in the inverse sequence.

In the original sequence, if we replace f_2 with g_2 defined below, we still have a max determining sequence from 1 to 2, but $G'(f_1, g_2)$ will be connected.

Define $f_2'\colon [0,1] \to [0,1]$ by $f_2'(t) = \{\frac{1}{2}t + \frac{1}{4}\}$ for $0 \le t < \frac{1}{2}$, $f_2'(\frac{1}{2}) = [\frac{1}{2}, \frac{3}{4}]$, and $f_2'(t) = \{\frac{3}{4}\}$ for $\frac{1}{2} < t \le 1$. Note that f_2' is continuum-valued. Let g_2 be the set-valued function whose graph is $G(f_2) \cup G(f_2')$ (see Figure 2). By definition, g_2 is a union of continuum-valued functions. We note that $F_{1,2}\colon [0,1] \to G_1^2$, for the finite inverse sequence $[0,1] \xleftarrow{f_1} [0,1] \xleftarrow{g_2} [0,1]$, is a union, and $\mathcal{C}(F_{1,2})$ has a universal member.

Specifically, $F_{1,2}$ is a union of the three max continuum-valued functions h_1, h_2, and h_3 defined as follows.

Let $h_1(t) = \{(\frac{1}{2}, \frac{1}{2}t + \frac{1}{4})\}$ for $0 \le t < \frac{1}{2}$ or $\frac{1}{2} < t \le 1$, and let $h_1(\frac{1}{2}) = G_1^2 = G(f_1^{-1})$.

Let $h_2(t) = \{(\frac{1}{2}, \frac{1}{2}t + \frac{1}{4})\}$ for $0 \le t < \frac{1}{2}$, $h_2(\frac{1}{2}) = G_1^2$, and $h_2(t) = \{(\frac{1}{2}, \frac{3}{4})\}$ for $\frac{1}{2} < t \le 1$.

Let $h_3(t) = \{(\frac{1}{2}, \frac{1}{2}t + \frac{1}{4})\}$ for $0 \le t < \frac{1}{2}$, $h_3(\frac{1}{2}) = G_1^2$, and $h_3(t) = \{(0, \frac{3}{4})\}$ for $\frac{1}{2} < t \le 1$.

Clearly, each pair in $\{G(h_1), G(h_2), G(h_3)\}$ has the set $\{\frac{1}{2}\} \times G_1^2$ in common. So, each h_i is universal with respect to the other two. It follows that $G'(f_1, g_2)$ is connected, and since $G_1^n \overset{T}{\approx} G_1^3$ for $n \ge 3$, the limit of the inverse sequence

$$[0,1] \xleftarrow{f_1} [0,1] \xleftarrow{g_2} [0,1] \xleftarrow{f_3} [0,1] \xleftarrow{f_4} \ldots\ldots$$

is connected (See Corollaries 6 and 7, Section 6).

A somewhat simpler revision can be made to f_2 (see $\hat{f}_2$ defined below) so that $F_{1,2}$ will itself be continuum-valued. Hence, $G'(f_1, \hat{f}_2) \overset{T}{\approx} G(F_{1,2})$ will be connected by Lemma 1. Let $\hat{f}_2(t) = f_2(t)$ for $0 \le t < 1$, and let $\hat{f}_2(1) = [\frac{3}{4}, 1]$.

One might ask if only having the partial graphs G_j^{k+1} connected for each max determining sequence $f_j, \ldots, f_k$ in a finite mixed inverse sequence $\{X_i, f_i\}_{i=1}^n$ is enough to ensure connectedness of G_1^{n+1}. We show in Example 3 that this is not the case.

Example 3. Let $k_1\colon [0,1] \to [0,1]$ be the set-valued function defined by $k_1(t) = \{\frac{1}{2}\}$ for $0 \le t < \frac{3}{4}$, $k_1(t) = \{\frac{1}{2}, 4t - 3\}$ for $\frac{3}{4} \le t \le \frac{7}{8}$, and $k_1(t) = \{4t - 3\}$ for $\frac{7}{8} < t \le 1$. We see that k_1 is not a union, but k_1^{-1} is, in fact, continuum-valued. Let $k_2\colon [0,1] \to [0,1]$ be defined by $k_2(t) = \{t\}$ for $0 \le t < \frac{7}{8}$, $k_2(\frac{7}{8}) = [\frac{7}{8}, 1]$, and $k_2(t) = \{\frac{7}{4} - t\}$ for $\frac{7}{8} < t \le 1$. We see that f_2 is continuum-valued, and f_2 is not an inverse union. Let $k_3\colon [0,1] \to [0,1]$ be defined by $k_3(t) = \{t\}$ for $0 \le t < \frac{1}{4}$ or $\frac{1}{2} < t \le 1$, and $k_3(t) = \{t, \frac{1}{2}\}$ for $\frac{1}{4} \le t \le \frac{1}{2}$. Finally, let k_4 be the continuum-valued function f_2 defined in Example 2.

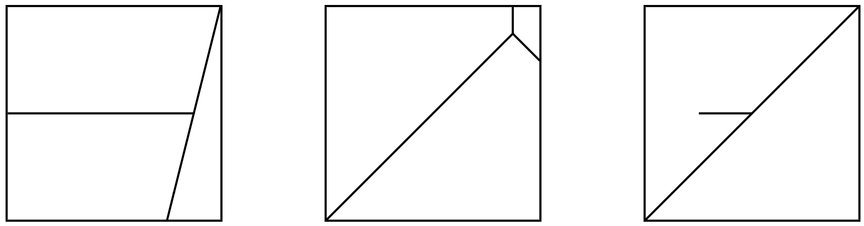

Figure 3. Graphs of the functions in Example 3

Suppose we have the finite alternating inverse sequence below.

$$[0,1] \xleftarrow{k_1} [0,1] \xleftarrow{k_2} [0,1] \xleftarrow{k_3} [0,1] \xleftarrow{k_4} [0,1]$$

It is straightforward to check that k_1, k_2 and k_3, k_4 are adjacent max determining sequences, and that G_1^3 and G_3^5 are connected. We leave to the reader to check that $(0, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, 1)$ is an isolated point in G_1^5; or, using techniques from [5], one may note that $\{\langle \frac{3}{4}, 0\rangle, \langle \frac{3}{4}, \frac{3}{4}\rangle, \langle \frac{3}{4}, \frac{3}{4}\rangle, \langle \frac{3}{4}, 1\rangle\}$ is a component base for the functions k_1, k_2, k_3, and k_4. So, G_1^5 is not connected. It is also interesting to note that $F_{3,4}\colon [0,1] \to G_3^4$ is a union, and $\mathcal{C}(F_{3,4})$ has a universal member, but $F_{1,2}\colon [0,1] \to G_1^2$ is not a union.

5. Results related to our setting

Lemma 1 below is Theorem 4.1 in [10].

Lemma 1. *If $f\colon X_2 \to X_1$ is a continuum-valued function and X_2 is connected, then $G(f)$ is connected.*

The next Lemma is clear.

Lemma 2. *If the set-valued function $f\colon X_2 \to X_1$ is a union of continuum-valued functions, then for each closed subset K of X_2, $f|_K\colon K \to X_1$ is a union of continuum-valued functions.*

Lemma 3 below follows from Theorems 4.3 and 4.5 in [10].

Lemma 3. *Suppose $X_1, X_2, \ldots, X_{k+1}$ are continua and for each $1 \leq i \leq k$, $f_i\colon X_{i+1} \to X_i$ is a surjective set-valued function. If each f_i is continuum-valued or if each f_i^{-1} is continuum-valued, then G_1^{k+1} is connected.*

Lemma 4. *Suppose that $X_1, X_2, \dots, X_{n+1}$ are continua and for each $1 \le i \le n$, $f_i\colon X_{i+1} \to X_i$ is a surjective set-valued function whose graph is connected. Suppose also that for each $1 \le i \le n$, $f_i\colon X_{i+1} \to X_i$ is a union of continuum-valued functions. Then the set-valued function $F_{1,n}\colon X_{n+1} \to G_1^n$ is a union of continuum-valued functions.*

Proof. We use induction on the number of bonding functions in a sequence where each bonding function is a union of continuum-valued functions. If $n = 1$, then $F_{1,1} = f_1$ and $f_1\colon X_2 \to X_1$ is a union of continuum-valued functions by assumption.

Assume that $F_{1,n-1}\colon X_n \to G_1^{n-1}$ is a union of continuum-valued functions for some $n \ge 2$.

Let $(z_1, z_2, \dots, z_n) \in F_{1,n}(z_{n+1})$. Since f_n is a union of continuum-valued functions, there exists a continuum-valued function $h\colon X_{n+1} \to X_n$ such that $z_n \in h(z_{n+1})$ and $G(h) \subset G(f_n)$. Also, by inductive assumption and Lemma 2, there exists a continuum-valued function $g\colon R(h) \to G_1^{n-1}$ such that $(z_1, \dots, z_{n-1}) \in g(z_n)$ and $G(g) \subset G(F_{1,n-1})$. Since $R(h) = c_2(G(h))$, it follows that $R(h)$ is closed and connected in X_n.

Consider the set-valued function $\ell\colon X_{n+1} \to G_1^n$ defined by $\ell(x) = G(g|_{h(x)}^{-1})$. Since both g and h are continuum-valued, it follows by Lemma 1 that $G(g|_{h(x)})$ is connected. So, $G(g|_{h(x)}^{-1})$ is connected. Hence, ℓ is continuum-valued. Also, $(z_1, z_2, \dots, z_n) \in \ell(z_{n+1})$, and $G(\ell) \subset G(F_{1,n})$. We have that $F_{1,n}\colon X_{n+1} \to G_1^n$ is a union of continuum-valued functions. □

Lemma 5. *Suppose that $X_1, X_2, \dots, X_{n+1}$ are continua and for each $1 \le i \le n$, $f_i\colon X_{i+1} \to X_i$ is a surjective set-valued function whose graph is connected. Suppose also that for each $1 \le i \le n$, $f_i^{-1}\colon X_i \to X_{i+1}$ is a union of continuum-valued functions. Then the set-valued function $L_1^{-1}\colon X_1 \to G_2^{n+1}$ is a union of continuum-valued functions.*

Proof. We use induction on the number of bonding functions in a sequence where the inverse of each bonding function is a union of continuum-valued functions. If $n = 1$, then we have $L_1^{-1}\colon X_1 \to X_2$, and by definition, $L_1^{-1} = f_1^{-1}$, which is a union of continuum-valued functions by assumption.

For some $n \ge 2$, in the sequence $X_2, \dots, X_{n+1}$ which has $n - 1$ bonding functions, assume that $L_2^{-1}\colon X_2 \to G_3^{n+1}$ is a union of continuum-valued functions.

Let $(z_2, \dots, z_{n+1}) \in L_1^{-1}(z_1)$. Since f_1^{-1} is a union of continuum-valued functions, there exists a continuum-valued function $h\colon X_1 \to X_2$ such that

$z_2 \in h(z_1)$ and $G(h) \subset G(f_1^{-1})$. Also, by inductive assumption and Lemma 2, there is a continuum-valued function $g\colon R(h) \to G_3^{n+1}$ such that $(z_3, \ldots, z_{n+1})$ is in $g(z_2)$ and $G(g) \subset G(L_2^{-1})$. As in the proof of Lemma 4, it follows that $R(h)$ is closed and connected in X_2.

Consider the set-valued function $\ell\colon X_1 \to G_2^{n+1}$ defined by $\ell(x) = G(g|_{h(x)})$. Since both g and h are continuum-valued, it follows by Lemma 1 that $G(g|_{h(x)})$ is connected. Analogously, as in the proof of Lemma 4, we get that ℓ is continuum-valued, $(z_2, \ldots, z_{n+1}) \in \ell(z_1)$, and $G(\ell) \subset G(L_1^{-1})$. Hence, L_1^{-1} is a union of continuum-valued functions. □

Theorem 1. *Suppose that $X_1, X_2, \ldots, X_{n+1}$ are continua and for each $1 \leq i \leq n$, $f_i\colon X_{i+1} \to X_i$ is a surjective set-valued function whose graph is connected. Suppose also that for each $1 \leq i \leq n$, $f_i\colon X_{i+1} \to X_i$ is a union of continuum-valued functions. Then G_1^{n+1} is connected.*

Proof. We use induction on the number of bonding functions. For $n = 1$, $G_1^2 = G(f_1^{-1})$ is connected by assumption.

Assume that $G_1^n = G'(f_1, \ldots, f_{n-1})$ is connected. Let A and B be closed sets whose union is G_1^{n+1}. Let $\rho\colon G_1^{n+1} \to G_1^n$ be the mapping given by $\rho = \pi_{(n+1)}|_{G_1^{n+1}}$. Since $\rho(A) \cup \rho(B) = G_1^n$, there exists a point $p = (p_1, \ldots, p_n) \in \rho(A) \cap \rho(B)$. Assume $a = (p_1, \ldots, p_n, a_{n+1}) \in A$ and $b = (p_1, \ldots, p_n, b_{n+1}) \in B$. By inductive assumption, G_1^n is connected. Also, by Lemma 4, $F_{1,n-1}\colon X_n \to G_1^{n-1}$ is a union of continuum-valued functions. Let $g\colon X_n \to G_1^{n-1}$ be a continuum-valued function such that $(p_1, \ldots, p_{n-1}) \in g(p_n)$ and $G(g) \subset G(F_{1,n-1})$.

Let $\ell\colon G_n^{n+1} \to G_1^{n-1}$ be defined by $\ell(x_n, x_{n+1}) = g(x_n)$. Note that G_n^{n+1} is connected by assumption, and ℓ is continuum-valued since g is continuum-valued. It follows from Lemma 1 that $G(\ell)$ is connected. So, $G(\ell^{-1})$ is connected. Also, we see that a and b are in $G(\ell^{-1}) \subset G_1^{n+1}$. So, $G(\ell^{-1})$ meets both A and B. It follows that A and B are not mutually separated. Hence, G_1^{n+1} is connected. □

Theorem 2. *Suppose that $X_1, X_2, \ldots, X_{n+1}$ are continua and for each $1 \leq i \leq n$, $f_i\colon X_{i+1} \to X_i$ is a surjective set-valued function whose graph is connected. Suppose also that for each $1 \leq i \leq n$, $f_i^{-1}\colon X_i \to X_{i+1}$ is a union of continuum-valued functions. Then G_1^{n+1} is connected.*

Proof. The proof of this theorem is similar to the proof of Theorem 1, using Lemma 5 analogously as Lemma 4 was used in Theorem 1. □

In Theorem 3.1 of [14], Nall has a condition on a surjective relation $F\colon X \to X$ that ensures $\varprojlim\{X, F\}$ is a continuum. His condition that F be the union of a collection of closed subsets $\{F_\alpha\}_{\alpha\in\Gamma}$ with certain properties is equivalent to F being a union of continuum-valued functions. Via Nall's Theorems 3.1 and 3.3 (see items (1) and (2) in the next to last paragraph of the introduction), having F^{-1} be a union of continuum-valued functions also leads to the connectedness of $\varprojlim\{X, F\}$. We are able to generalize his Theorem 3.1(see Corollary 1 below) by additionally showing that for different factor spaces and different set-valued bonding functions, the inverse limit space $\varprojlim\{X_i, f_i\}$ will be a continuum. The inverse limits X_1 and X_3 in Example 1 show that Nall's Theorem 3.3 cannot be generalized to each sequence of set-valued functions.

Corollary 1. *Let $X_1, X_2, \ldots$ be a sequence of continua and suppose that for each $i \geq 1$, $f_i\colon X_{i+1} \to X_i$ is a surjective set-valued function whose graph is connected. Suppose also that for each $i \geq 1$, $f_i\colon X_{i+1} \to X_i$ is a union of continuum-valued functions, or for each $i \geq 1$, $f_i^{-1}\colon X_i \to X_{i+1}$ is a union of continuum-valued functions. Then $\varprojlim\{X_i, f_i\}$ is a continuum.*

Proof. The corollary follows from Theorems 1 and 2, and from Ingram's result noted in Remark 2. □

6. Main theorems

Corollaries 4 and 7 of this section are our most general results for connectedness of an inverse limit in our setting, but other theorems and corollaries in this section provide conditions for connectedness of both partial graphs and inverse limits that could be useful in specific cases.

The proofs of Theorems 3 and 4 are similar to the proofs of Lemmas 4 and 5, and Theorems 1 and 2 in Section 5. This typically signals some general theorem that includes them all as special cases. The author was unable to find any such easily stated theorem. Perhaps the repetitiveness of the proofs will be helpful rather than bothersome.

Theorem 3 establishes connectedness of the partial graph G_i^{n+1}, when f_n^{-1} is a union of continuum-valued functions, and a partial graph to the left, namely G_i^n, is connected. Theorem 4 establishes connectedness of the partial graph G_k^{n+1}, when f_k is a union of continuum-valued functions, and a partial graph to the right, namely G_{k+1}^{n+1}, is connected.

Theorem 3. *Suppose that X_1, X_2, $\ldots$, X_{n+1} are continua, and for each $1 \le i \le n$, $f_i\colon X_{i+1} \to X_i$ is a surjective set-valued function whose graph is connected. Suppose also that $f_n^{-1}\colon X_n \to X_{n+1}$ is a union of continuum-valued functions. Then, for $1 \le i < n$,*

(1) $F_{i,n}^{-1}\colon G_i^n \to X_{n+1}$ is a union of continuum-valued functions,

(2) if G_i^n is connected, then $G(F_{i,n}^{-1}) \overset{T}{\approx} G_i^{n+1}$ is connected, and

(3) if $F_{i,n}$ is a union of continuum-valued functions, then f_n is a union of continuum-valued functions.

Proof. (1) Let $z_{n+1} \in F_{i,n}^{-1}(z_i, \ldots, z_n)$. Since $z_{n+1} \in f_n^{-1}(z_n)$ and f_n^{-1} is a union of continuum-valued functions, there exists a continuum-valued function $g\colon X_n \to X_{n+1}$ such that $z_{n+1} \in g(z_n)$ and $G(g) \subset G(f_n^{-1})$.

Let $\ell\colon G_i^n \to X_{n+1}$ be the set-valued function defined by $\ell(x_i, \ldots, x_n) = g(x_n)$. We see that $z_{n+1} \in \ell(z_i, \ldots, z_n)$, and $G(\ell) \subset G(F_{i,n}^{-1})$. Since g is continuum-valued, it follows that ℓ is continuum-valued. So, $F_{i,n}^{-1}\colon G_i^n \to X_{n+1}$ is a union of continuum-valued functions.

(2) Let A and B be closed sets such that $A \cup B = G_i^{n+1}$. Let $\rho\colon G_i^{n+1} \to G_n^{n+1}$ be projection onto the n^{th} and $(n+1)^{\text{th}}$ coordinates. Since $G(f_n)$ is connected and $\rho(A) \cup \rho(B) = G_n^{n+1} = G(f_n^{-1})$, there is a point $(p_n, p_{n+1}) \in \rho(A) \cap \rho(B)$. Let $(a_i, \ldots, a_{n-1}, p_n, p_{n+1}) \in A$, and $(b_i, \ldots, b_{n-1}, p_n, p_{n+1}) \in B$. We define the continuum-valued functions $g\colon X_n \to X_{n+1}$ and $\ell\colon G_i^n \to X_{n+1}$ with $p_{n+1} \in g(p_n)$ analogously as in (1). Recall that $G(\ell) \subset G(F_{i,n}^{-1}) = G_i^{n+1}$. Since G_i^n is connected by hypothesis, it follows from Lemma 1 that $G(\ell)$ is connected. Also, $G(\ell)$ intersects both A and B; so, A and B are not mutually separated. Hence, G_i^{n+1} is connected.

(3) Let $z_n \in f_n(z_{n+1})$. Let $(z_i, \ldots, z_n) \in G_i^n$. Then $(z_i, \ldots, z_n) \in F_{i,n}(z_{n+1})$, and, by assumption, there exists a continuum-valued $g\colon X_{n+1} \to G_i^n$ such that $(z_i, \ldots, z_n) \in g(z_{n+1})$ and $G(g) \subset G(F_{i,n})$. Let $g'\colon X_{n+1} \to X_n$ be defined by $g' = \pi_n \circ g$. We note that g' is continuum-valued, since g is continuum-valued and π_n is a mapping. Also, $z_n \in g'(z_{n+1})$ and $G(g') \subset G(f_n)$. So, f_n is a union of continuum-valued functions. □

Corollary 2. *Suppose that $\{X_i, f_i\}_{i \ge 1}$ is an inverse sequence on continua, and for each $i \ge 1$, $f_i\colon X_{i+1} \to X_i$ is a surjective set-valued function whose graph is connected. Suppose also that there exists $k \ge 1$ where G_1^k is connected, and for all $i \ge k$, $f_i^{-1}\colon X_i \to X_{i+1}$ is a union of continuum-valued* *functions. Then for each $n > k$, G_1^n is connected, and hence, $\varprojlim\{X_i, f_i\}$ is a continuum.*

Proof. Fix $n > k$. By Theorem 3, $F_{1,k}\colon X_{k+1} \to G_1^k$ is an inverse union with a connected graph. So, we have that

$$G_1^k \xleftarrow{F_{1,k}} X_{k+1} \xleftarrow{f_{k+1}} \dots \xleftarrow{f_n} X_{n+1}$$

is a finite inverse sequence, where each bonding function is an inverse union with a connected graph. By Theorem 2 and Remark 3, G_1^n is connected.

By Remark 2 and the fact that ordinary inverse limits on continua are continua, it follows that $\varprojlim\{X_i, f_i\}$ is a continuum. □

Corollary 3. *Suppose that $\{X_i, f_i\}_{i\geq 1}$ is an inverse sequence on continua, and for each $i \geq 1$, $f_i\colon X_{i+1} \to X_i$ is a surjective set-valued function whose graph is connected. Suppose also that there exists $k \in \mathbb{N}$ with $k \geq 2$ such that*

for each $1 \leq i < k$, $f_i\colon X_{i+1} \to X_i$ is a union of continuum-valued functions, and

for each $i \geq k$, $f_i^{-1}\colon X_i \to X_{i+1}$ is a union of continuum-valued functions.

Then, for each $n > k$, G_1^n is connected, and hence, $\varprojlim\{X_i, f_i\}$ is a continuum.

Proof. By Theorem 1, G_1^k is connected. By Corollary 2, the result follows. □

Theorem 4 below is not needed to prove any of the theorems that follow it, but we include it and its proof since it could be useful in establishing connectness of some partial graphs in certain applications. Also, it is a nice companion theorem to Theorem 3.

Theorem 4. *Suppose that X_k, X_{k+1}, …, X_{n+1} are continua, and for each $k \leq i \leq n$, $f_i\colon X_{i+1} \to X_i$ is a surjective set-valued function whose graph is connected. Suppose also that $f_k\colon X_{k+1} \to X_k$ is a union of continuum-valued functions. Then*

(1) $L_k\colon G_{k+1}^{n+1} \to X_k$ is a union of continuum-valued functions,

(2) if G_{k+1}^{n+1} is connected, then $G(L_k) \overset{T}{\approx} G_k^{n+1}$ is connected, and

(3) if L_k^{-1} is a union of continuum-valued functions, then f_k^{-1} is a union of continuum-valued functions.

Proof. (1) Let $z_k \in L_k(z_{k+1}, \ldots, z_{n+1})$. Since $z_k \in f_k(z_{k+1})$ and f_k is a union of continuum-valued functions, there exists a continuum-valued function $g\colon X_{k+1} \to X_k$ such that $z_k \in g(z_{k+1})$ and $G(g) \subset G(f_k)$.

Let $\ell\colon G_{k+1}^{n+1} \to X_k$ be the set-valued function defined by $\ell(x_{k+1}, \ldots, x_{n+1}) = g(x_{k+1})$. We see that $z_k \in \ell(z_{k+1}, \ldots, z_{n+1})$, and $G(\ell) \subset G(L_k)$. Since g is continuum-valued, it follows that ℓ is continuum-valued. So, $L_k\colon G_{k+1}^{n+1} \to X_k$ is a union of continuum-valued functions.

(2) Let A and B be closed sets such that $A \cup B = G_k^{n+1}$. Let $\rho\colon G_k^{n+1} \to G_k^{k+1}$ be projection. Since $G(f_k)$ is connected and $\rho(A) \cup \rho(B) = G_k^{k+1} = G(f_k^{-1})$, there is a point $(p_k, p_{k+1}) \in \rho(A) \cap \rho(B)$. Let $(p_k, p_{k+1}, a_{k+2}, \ldots, a_{n+1}) \in A$, and $(p_k, p_{k+1}, b_{k+2}, \ldots, b_{n+1}) \in B$. We define the continuum-valued functions $g\colon X_{k+1} \to X_k$ and $\ell\colon G_{k+1}^{n+1} \to X_k$ with $p_k \in g(p_{k+1})$ analogously as in (1). Recall that $G(\ell) \subset G(L_k)$. Since G_{k+1}^{n+1} is connected by hypothesis, it follows from Lemma 1 that $G(\ell)$ is connected. Also, $G(\ell)$ intersects both A and B; so, A and B are not mutually separated. Hence, G_k^{n+1} is connected.

(3) Let $z_k \in f_k(z_{k+1})$. Let $(z_{k+1}, \ldots, z_{n+1}) \in G_{k+1}^{n+1}$. Then $(z_{k+1}, \ldots, z_{n+1}) \in L_k^{-1}(z_k)$, and, by assumption, there exists a continuum-valued $g\colon X_k \to G_{k+1}^{n+1}$ such that $(z_{k+1}, \ldots, z_{n+1}) \in g(z_k)$ and $G(g) \subset G(L_k^{-1})$. Let $g'\colon X_k \to X_{k+1}$ be defined by $g' = \pi_{k+1} \circ g$. We note that g' is continuum-valued, since g is continuum-valued and π_{k+1} is a mapping. Also, $z_{k+1} \in g'(z_k)$ and $G(g') \subset G(f_k^{-1})$. So, f_k^{-1} is a union of continuum-valued functions. □

Our remaining lemma, theorem, and corollaries give additional conditions that will ensure connectedness of partial graphs and inverse limits that have determining sequences in their associated inverse sequences.

Our most general results, in this setting, are those that follow.

Theorem 5. *Suppose that $\{X_i, f_i\}_{i=1}^n$ is a finite mixed inverse sequence on continua containing a sequence of max determining sequences $\{f_{j_i}, \ldots, f_{k_i}\}_{i=1}^m$, where $f_{j_1}, \ldots, f_{k_1}$ is the first max determining sequence in $f_1, \ldots, f_n$, for each $1 \le i < m$, $f_{j_i}, \ldots, f_{k_i}$ and $f_{j_{i+1}}, \ldots, f_{k_{i+1}}$ are consecutive, and if $k_m < n$, there are no determining sequences in $f_{k_m+1}, \ldots, f_n$. Suppose also that for each $1 \le i \le m$,*

$$F_{j_i,k_i}\colon X_{k_i+1} \to G_{j_i}^{k_i} \text{ is a union, and } G(F_{j_i,k_i}) \text{ is connected.}$$

Then G_1^{n+1} is connected.

Proof. In order to simplify notation, we prove this theorem for $m = 2$, and we denote the two max determining sequences by $f_j, \ldots, f_k$ and $f_\ell, \ldots, f_m$ with $k < \ell$. The idea of the proof is straightforward using the tools we have set up, and it will be clear that the proof will work analogously for m max determining sequences when $m > 2$.

We consider the finite inverse sequence in our hypothesis and an amalgamated form of it below. By Remark 3, the two inverse sequences have homeomorphic partial graphs.

$$(1) \quad X_1 \xleftarrow{f_1} \ldots \xleftarrow{f_{j-1}} X_j \xleftarrow{f_j} \ldots \xleftarrow{f_k} X_{k+1} \xleftarrow{f_{k+1}} \ldots \xleftarrow{f_{\ell-1}} X_\ell \xleftarrow{f_\ell} \ldots \xleftarrow{f_m} X_{m+1} \xleftarrow{f_{m+1}} \ldots \xleftarrow{f_n} X_{n+1}$$

In sequence (1), by Observations 3 and 4, for $1 \leq i < j$ and $k+1 \leq i < \ell$, f_i is a union. Amalgamating the max determining sequences, we get the inverse sequence below.

$$(2) \quad X_1 \xleftarrow{f_1} \ldots \xleftarrow{f_{j-1}} X_j \xleftarrow{\pi_j|_{G_j^k}} G_j^k \xleftarrow{F_{j,k}} X_{k+1} \xleftarrow{f_{k+1}} \ldots \xleftarrow{f_{\ell-1}} X_\ell \xleftarrow{\pi_\ell|_{G_\ell^m}} G_\ell^m \xleftarrow{F_{\ell,m}} X_{m+1} \xleftarrow{f_{m+1}} \ldots \xleftarrow{f_n} X_{n+1}$$

In sequence (2), since π_j and π_ℓ are mappings, we view them as degenerate continuum-valued functions. By hypothesis, the set-valued functions $F_{j,k}$ and $F_{\ell,m}$ are unions of continuum-valued functions, and have connected graphs. In sequence (2), we have that all bonding functions through $F_{\ell,m}$ are unions with connected graphs. If $m = n$, then, by Corollary 1, the partial graph of inverse sequence (2) is connected. Hence, G_1^{n+1} is connected. If $m < n$, then there are no determining sequences in $f_{m+1}, \ldots, f_n$, so we have that either

(i) f_i is an inverse union for all $m + 1 \leq i \leq n$,

(ii) f_i is a union for all $m + 1 \leq i \leq n$, or

(iii) there exists $m+1 < n' < n$ such that f_i a union for all $m+1 \leq i < n'$, and f_i is an inverse union for all $n' \leq i \leq n$.

If (ii) is the case, it follows from Corollary 1 that the partial graph of inverse sequence (2) is connected. Hence, G_1^{n+1} is connected.

If either (i) or (iii) is the case, it follows from Corollary 3 that the partial graph of inverse sequence (2) is connected. Hence, G_1^{n+1} is connected. □

Corollary 4. *Suppose that $\{X_i, f_i\}_{i \geq 1}$ is a mixed inverse sequence on continua. Suppose also that, for each max determining sequence $f_j, \ldots, f_k$,*

$F_{j,k}: X_{k+1} \to G_j^k$ is a union, and $G(F_{j,k})$ is connected. Then $\varprojlim\{X_i, f_i\}$ is a continuum.

Proof. **Case 1.** Suppose that $\{X_i, f_i\}_{i\geq 1}$ has finitely many max determining sequences. Pick n large enough so that all max determining sequences are contained in the finite inverse sequence $\{X_i, f_i\}_{i=1}^n$. It follows from Theorem 5 that G_1^{i+1} is connected for each $i \geq n$. By Remark 2, G_1^i is also connected for each $1 \leq i \leq n$. As we saw in the proof of Corollary 2, it follows from Remark 2 that $\varprojlim\{X_i, f_i\}$ is a continuum.

Case 2. Suppose that $\{X_i, f_i\}_{i\geq 1}$ has infinitely many max determining sequences. Pick the sequence $\{n_i\}_{i\geq 1}$ of integers, where for each $i \geq 1$, f_{n_i} is the last member of the n_i^{th} max determining sequence. Then it follows from Theorem 5 that $G_1^{n_i+1}$ is connected for each $i \geq 1$, and as in Case 1, it follows that $\varprojlim\{X_i, f_i\}$ is a continuum. □

Corollary 5. *Suppose that $\{X_i, f_i\}_{i\geq 1}$ is an eventually alternating (beginning at k) inverse sequence on continua that contains no determining sequences in $f_1, \ldots, f_{k-1}$. Suppose also that, for each $i \geq k$, $F_{i,i+1}: X_{i+2} \to G_i^{i+1}$ is a union, and $G(F_{i,i+1})$ is connected. Then $\varprojlim\{X_i, f_i\}$ is a continuum.*

Although Corollaries 4 and 5 reduce the determination of connectedness of an inverse limit of a mixed inverse sequence from checking that all partial graphs G_1^n are connected to checking if, in max determining sequences $f_j, \ldots, f_k$, two conditions are present, it would be helpful to know of simple, observable properties of either $F_{j,k}$ or some of the functions f_i in the max determining sequence that would ensure that $F_{j,k}$ is a union with a connected graph. Even though f_k is a union, no simple property is apparent to the author that would make $F_{j,k}$ a union. However, if $F_{j,k}$ is found to be a union, there are a few relatively easy properties that will make $G(F_{j,k}) \overset{T}{\approx} G_j^{k+1}$ connected. They are

(A) $F_{j,k}$ is continuum-valued,

(B) $\mathcal{C}(F_{j,k})$ has a universal member,

(C) there exists a continuum K in $G(F_{j,k})$ such that whenever $g \in \mathcal{C}(F_{j,k})$, $K \cap G(g) \neq \emptyset$, and

(D) $f_k: X_{k+1} \to X_k$ is full-valued at some point.

It is clear that each of properties (A), (B), and (C) make $G(F_{j,k})$ connected. In general, property (D) would be the easiest to check. Lemma 6 below shows that property (D) ensures that $G(F_{j,k})$ is connected through a max determining sequence.

Lemma 6. *Suppose that $\{X_i, f_i\}_{i=1}^n$ is a finite mixed inverse sequence on continua containing a max determining sequence $\{f_j, \ldots, f_k\}$. Suppose also that $F_{j,k}$ is a union, and f_k is full-valued at some point of X_{k+1}. Then $G(F_{j,k})$ is connected.*

Proof. Since f_k is full-valued at some point of X_{k+1}, there exists a point z_{k+1} in X_{k+1} where $f_k(z_{k+1}) = X_k$. Let $K = \{z_{k+1}\} \times G_j^k$. We claim that K is a subcontinuum of $G(F_{j,k})$ that meets the graph of each $g \in \mathcal{C}(F_{j,k})$. Since, for each $j \le i \le k-1$, f_i is an inverse union, we have that G_j^k is connected. So, K is connected.

If $(z_{k+1}, x_j, \ldots, x_k) \in K$, we have that $x_k \in f_k(z_{k+1})$, and since $(x_j, \ldots, x_k)$ is in G_j^k, it follows that $(x_j, \ldots, x_k, z_{k+1}) \in G_j^{k+1}$. So, by definition, $(x_j, \ldots, x_k)$ is in $F_{j,k}(z_{k+1})$. Hence, K is a subcontinuum of $G(F_{j,k})$.

Lastly, let $g \in \mathcal{C}(F_{j,k})$. Since g is max continuum-valued, it follows that $g(z_{k+1}) = G_j^k$. So, $K \subset G(g)$. Hence, property (C) above is satisfied, and it follows that $G(F_{j,k})$ is connected. □

Corollary 6. *Suppose that $\{X_i, f_i\}_{i=1}^n$ is a finite mixed inverse sequence on continua containing a sequence of consecutive max determining sequences $\{f_{j_i}, \ldots, f_{k_i}\}_{i=1}^m$ as in Theorem 5. Suppose also that for each $1 \le i \le m$,*

$$F_{j_i,k_i}\colon X_{k_i+1} \to G_{j_i}^{k_i} \text{ is a union, and one of properties (A)} - \text{(D) holds.}$$

Then G_1^{n+1} is connected.

Corollary 7. *Suppose that $\{X_i, f_i\}_{i\ge 1}$ is a mixed inverse sequence on continua. Suppose also that, for each max determining sequence $f_j, \ldots, f_k$, $F_{j,k}\colon X_{k+1} \to G_j^k$ is a union, and one of properties (A)-(D) holds. Then $\varprojlim\{X_i, f_i\}$ is a continuum.*

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