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CONNECTEDNESS OF INVERSE LIMITS WITH SET-VALUED FUNCTIONS

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Abstract. We establish general results for determining connectedness of inverse limits on continua with set-valued bonding functions. These results generalize all theorems in the literature where connectedness of the inverse limit can be established by checking easily observable properties of the bonding functions. For inverse limits on $[0, 1]$, we note several useful special cases of our main theorem. The results provide answers to two questions of W. T. Ingram. We give a number of examples to illustrate the utility of the results.

1. Introduction

Unlike ordinary inverse sequences on continua with mappings for bonding functions, an inverse sequence on continua with set-valued bonding functions may not have a connected limit, even when the graphs of the bonding functions are continua. We establish sufficient conditions for connectedness of inverse limits on continua with upper semi-continuous set-valued functions. The conditions are simply-checked properties of the bonding functions. We also provide an additional result for connectedness of inverse limits on $[0, 1]$ with set-valued bonding functions. The results provide answers to Problems 6.3 and 6.4 of Ingram in [2]. The results also generalize all theorems presently in the literature that give connectedness of an inverse limit with set-valued functions, where one only needs

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to observe that the bonding functions satisfy certain conditions. In the last section of the paper, we provide examples that illustrate the utility of the results.

A general introduction to results and questions related to connectedness of an inverse limit with set-valued functions can be found in Section 2 of [1], in Sections 2.2, 2.3, 2.4, 2.6, and 2.7 of [2], and in Section 4 of [3]. A more recent detailed discussion of results in the literature related to connectedness or non-connectedness of inverse limits with set-valued functions can be found in Section 1 of [7].

2. Basic definitions and observations

A *compactum* is a compact metric space. All spaces considered in this paper will be compacta. A *continuum* is a connected compactum. We note that a continuum may be degenerate. A continuous function will be referred to as a *mapping*.

Let X and Y be compacta. We refer to functions $f\colon X \to 2^Y$ as *set-valued functions* from X to Y and we write $f\colon X \to Y$ is a set-valued function. Note that throughout, we are assuming that, for $x \in X$, the value $f(x)$ of a set-valued function is a closed set. The *graph* of f, which we denote by $G(f)$, is the set in $X \times Y$ consisting of all points (x, y) with $y \in f(x)$.

A set-valued function $f\colon X \to Y$ is *upper semi-continuous at the point* $x \in X$ if for each open set V in Y containing the closed set $f(x)$, there is an open set U in X such that $x \in U$ and $f(p) \subset V$ for each $p \in U$. If $f\colon X \to Y$ is upper semi-continuous at each point of X, then f is said to be *upper semi-continuous*.

A set-valued function $f\colon X \to Y$ is *surjective* if for each $y \in Y$, there exists $x \in X$ such that $y \in f(x)$. If the set-valued function $f\colon X \to Y$ is surjective, we let $f^{-1}\colon Y \to X$ be the set-valued function such that $x \in f^{-1}(y)$ if and only if $y \in f(x)$. Clearly, $G(f^{-1})$ is homeomorphic to $G(f)$; so, it follows from [2, Theorem 1.2] that f^{-1} is upper semi-continuous if f is upper semi-continuous. For $f\colon X \to Y$ a set-valued function, and $A \subset X$, we let $f|_A$ be the set-valued function whose domain is A, and $f|_A(x) = f(x)$ for $x \in A$. If $x \in X$ and $f(x)$ is degenerate, we will sometimes treat $f(x)$ as a point of Y.

A set-valued function $f\colon X \to Y$ is *continuum-valued* if for each $x \in X$, the set $f(x)$ is a subcontinuum of Y. If, for each $(x, y) \in G(f)$, there exists a continuum-valued function $g\colon X \to Y$ such that $G(g) \subset G(f)$ and $(x, y) \in G(g)$, we say that $f\colon X \to Y$ (and $G(f)$) is a *union of continuum-valued functions*. Let $A \subset X$ and $B \subset Y$. If, for each point $(x, y) \in G(f) \cap (A \times B)$, there exists a continuum-valued function $g\colon A \to B$ such that $(x, y) \in G(g) \subset G(f)$, then we say that $G(f) \cap (A \times B)$

is a *union of continuum-valued functions*. Letting $f(A,B)\colon A \to B$ be the set-valued function whose graph is $G(f) \cap (A \times B)$, we also say that $f(A,B)$ is a *union of continuum-valued functions*. We note that for some A and B, $G(f) \cap (A \times B)$ may be empty, in which case, $f(A,B)$ does not exist.

For $i \geq 1$, let X_i be a compactum, and let $f_i\colon X_{i+1} \to X_i$ be a surjective upper semi-continuous set-valued function. Throughout, we let $\{X_i, f_i\}$ denote an inverse sequence, and its inverse limit is given by

$$\varprojlim\{X_i, f_i\} = \{\boldsymbol{x} = (x_1, x_2, \ldots) \in \prod_{i \geq 1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } i \geq 1\}.$$

For $n \in \mathbb{N}$, we define the set below.

$$G'_n = G'(f_1, \ldots, f_n) = \{\boldsymbol{x} \in \prod_{i=1}^{n+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}.$$

We refer to these sets as *partial graphs* in the inverse sequence. For consistency of notation, we let $G'_0 = X_1$. The notation $X \overset{T}{\approx} Y$ will indicate that X is homeomorphic to Y.

Let $\{X_i, f_i\}$ be an inverse sequence on compact sets with upper semi-continuous surjective set-valued bonding functions. For $n \geq 1$, we define the set-valued function $F_n\colon X_{n+1} \to G'_{n-1}$, where $(x_1, x_2, \ldots, x_n) \in F_n(z)$ if and only if $(x_1, x_2, \ldots, x_n, z)$ is in G'_n. So, $G(F_n^{-1}) = G'_n$. V. Nall introduced this function in [8], and showed that F_n is upper semi-continuous. If, for each $1 \leq i \leq n$, X_i is a continuum, and f_i is continuum-valued, the author showed in [6, Theorem 5] that F_n is continuum-valued.

The well-known results and observations below will be used throughout the paper, sometimes without reference.

For items 1 and 2, let X and Y be compacta, and $f\colon X \to Y$ be an upper semi-continuous, set-valued function.

1. ([3, Theorem 4.1]) If f is continuum-valued and X is connected, then $G(f)$ is connected.

2. (a) If f is continuum-valued and $K \subset X$, then $f|_K\colon K \to Y$ is continuum-valued.
(b) If f is a union of continuum-valued functions and $K \subset X$, then $f|_K\colon K \to Y$ is a union of continuum-valued functions.
(c) If $x \in X$, $f|_{\{x\}}\colon \{x\} \to Y$ is a union of continuum-valued functions.

For items 3 and 4, let $X = \varprojlim\{X_i, f_i\}$, where, for each $i \geq 1$, X_i is a compactum and f_i is a surjective, upper semi-continuous, set-valued function. For $i \geq 1$, define the mapping $\rho_i\colon G'_i \to G'_{i-1}$ by $\rho_i(x_1 \ldots, x_i, x_{i+1}) = (x_1 \ldots, x_i)$. Note that $\{G'_{i-1}, \rho_i\}$ is an inverse sequence with mappings for bonding functions.

3. ([2, Corollary 4.2]) $X \overset{T}{\approx} \varprojlim\{G'_{i-1}, \rho_i\}$. Also, we observe that since X_i and G'_{i-1} are continuous images of G'_j for each $1 \leq i \leq j$, it follows that if G'_n is connected for some $n \geq 1$, then both X_i and G'_i are connected for each $1 \leq i \leq n$. Furthermore, X is connected if and only if G'_n is connected for each $n \geq 0$.

4. Note that $F_1\colon [0,1] \to G'_0$ is $f_1\colon [0,1] \to [0,1]$. For $n > 1$, a pointwise, inductive definition of the function $F_n\colon X_{n+1} \to G'_{n-1}$ is given by $F_n(z) = G(F_{n-1}|^{-1}_{f_n(z)})$. It is easily seen from this definition and item 1, that if both F_{n-1} and f_n are continuum-valued, then F_n is continuum-valued.

5. ([9, Theorem 3.3] Let M be a continuum, and $f\colon M \to M$ be a surjective, upper semi-continuous, set-valued function. Then $\varprojlim\{M, f\}$ is connected if and only if $\varprojlim\{M, f^{-1}\}$ is connected.

3. Connectedness of partial graphs and inverse limits on continua

For the product $X \times Y$ of two compacta, let $c_1\colon X \times Y \to X$ and $c_2\colon X \times Y \to Y$ denote the coordinate projection mappings.

Lemma 1. *Let X, Y, and Z be continua, and let $f\colon Z \to Y$ and $g\colon Y \to X$ be surjective, upper semi-continuous, set-valued functions whose graphs are connected. Suppose whenever (z_1, y) and (z_2, y) are two points of $G(f)$, there exists a continuum $L \subset G(f)$ containing the points (z_1, y) and (z_2, y) such that $g|_{c_2(L)}\colon c_2(L) \to X$ is a union of continuum-valued functions. Then $G'(g, f)$ is connected.*

Proof. Let A and B be closed sets such that $A \cup B = G'(g, f)$. Let $\rho\colon G'(g, f) \to G(g^{-1})$ be the natural projection mapping. Since $G(g^{-1})$ is connected, there is a point (x, y) in $\rho(A) \cap \rho(B)$. Let $a = (x, y, z_1) \in A$, and $b = (x, y, z_2) \in B$. We assume that $a \neq b$.

By hypothesis, there exists a continuum $L \subset G(f)$ containing the points (z_1, y) and (z_2, y) such that $g|_{c_2(L)}\colon c_2(L) \to X$ is a union of continuum-valued functions. Let $K = c_2(L)$, and let $\hat{g}\colon K \to X$ be a continuum-valued function such that $(y, x) \in G(\hat{g}) \subset G(g|_K)$. Define the set-valued function $\ell\colon L^{-1} \to X$ by $\ell(s, t) = \hat{g}(s)$. Note that ℓ is continuum-valued since $\hat{g}$ is continuum-valued. Since L^{-1} is a continuum, it follows that $G(\ell)$ is connected. So, $G(\ell^{-1})$ is connected. Also, we see that a and b are in $G(\ell^{-1}) \subset G'(g, f)$. So, $G(\ell^{-1})$ meets A and B, and it follows that A and B are not disjoint. Hence, $G'(g, f)$ is connected. □

Corollary 1. *Let X, Y, and Z be continua, and let $f\colon Z \to Y$ and $g\colon Y \to X$ be surjective, upper semi-continuous, set-valued functions whose graphs are connected. If f^{-1} is continuum-valued, then $G'(g, f)$ is connected.*

Proof. Suppose (z_1, y) and (z_2, y) are two points of $G(f)$. Since f^{-1} is continuum-valued, $H = f^{-1}(y)$ is a continuum containing the points z_1 and z_2. By item 2(c) in the previous section, $g|_{\{y\}}$ is a union of continuum-valued functions. So, by Lemma 1, $G'(g, f)$ is connected. □

Let $\{X_i, f_i\}$ be an inverse sequence, where for each $i \geq 1$, X_i is a compactum, and $f_i\colon X_{i+1} \to X_i$ is a surjective, upper semi-continuous, set-valued function. For each $n \geq 1$, let $F_n\colon X_{n+1} \to G'_{n-1}$ be the set-valued function defined in Section 2 and discussed in item 4 of Section 2.

Corollary 2. *Let $\{X_i, f_i\}$ be an inverse sequence, where for each $i \geq 1$, X_i is a continuum, and $f_i\colon X_{i+1} \to X_i$ is a surjective, upper semi-continuous, set-valued function whose graph is connected. Suppose for $i \geq 2$, if (a, x) and (b, x) are two points of $G(f_i)$, then there exists a continuum $L_i \subset G(f_i)$ containing (a, x) and (b, x) such that $F_{i-1}|_{c_2(L_i)}\colon c_2(L_i) \to G'_{i-2}$ is a union of continuum-valued functions. Then G'_n is connected for each $n \geq 1$. Furthermore, $\varprojlim\{X_i, f_i\}$ is a continuum.*

Proof. We use induction on n to see that G'_n is connected for each $n \geq 1$. For $n = 1$, $G'_1 = G(f_1^{-1})$ is connected by hypothesis.

Assume G'_n is connected for some $n \geq 1$. We note, by hypothesis, $f_{n+1}\colon X_{n+2} \to X_{n+1}$ and $F_n\colon X_{n+1} \to G'_{n-1}$ satisfy the conditions of Lemma 1. So, $G'(F_n, f_{n+1}) \overset{T}{\approx} G'_{n+1}$ is connected. By induction, G'_n is connected for each $n \geq 1$.

It follows from item 3 of the previous section that $\varprojlim\{X_i, f_i\}$ is a continuum. □

Although Corollary 2 is a general result for establishing connectedness of inverse limits on continua with set-valued functions, it is not particularly user-friendly. Even for an inverse sequence on a single continuum with a single bonding function, it can be difficult to determine if the functions F_n or restrictions of them are unions of continuum-valued functions. Nevertheless, in Example 6, we show that for a given inverse sequence of functions, it may be possible to inductively show that appropriate restrictions of the functions F_n are unions of continuum-valued functions.

We would like to have easily-checked conditions on each of the bonding functions f_i that would ensure the conditions of Corollary 2 on the functions f_i and F_i. There may be many sets of conditions that could do this. We offer some that are easy to check, generalize well-known connectedness results in this setting, and apply to many examples in the literature.

In the remainder of this section, the results are for inverse limits on continua. In the next section, we consider inverse limits on $[0, 1]$.

Lemma 2. *Let $\{X_i, f_i\}$ be an inverse sequence, where for each $i \geq 1$, X_i is a continuum, and $f_i \colon X_{i+1} \to X_i$ is a surjective, upper semi-continuous, set-valued function whose graph is connected. Let $\{\mathcal{S}_i\}_{i\geq 1}$ be a sequence such that, for each $i \geq 1$, $\mathcal{S}_i$ is a family of subcontinua of X_i. Suppose the following conditions hold for each $i \geq 1$.*

1. *$f_i(Z) \subset \cup\mathcal{S}_i$ for each $Z \in \mathcal{S}_{i+1}$.*
2. *If $G(f_i) \cap (Z \times Y)$ is non-empty for $(Z, Y) \in (\mathcal{S}_{i+1} \times \mathcal{S}_i)$, then it is a union of continuum-valued functions.*

Then $F_n|_Z \colon Z \to G'_{n-1}$ is a union of continuum-valued functions for each $n \geq 1$ and $Z \in \mathcal{S}_{n+1}$.

Proof. For $i \geq 1$ and $(Z, Y) \in (\mathcal{S}_{i+1} \times \mathcal{S}_i)$, let $f_i(Z, Y)$ be the set-valued function, if it exists, whose graph is $G(f_i) \cap (Z \times Y)$. Fix $n \geq 1$, and let $Z_{n+1} \in \mathcal{S}_{n+1}$. Let $(p_{n+1}, p_1, \ldots, p_n) \in G(F_n)$ with $p_{n+1} \in Z_{n+1}$. Let $Y_{n+1} = Z_{n+1}$. By (1), $(p_{n+1}, p_n) \in G(f_n) \cap (Y_{n+1} \times Y_n)$ for some $Y_n \in \mathcal{S}_n$. There is no loss of generality to choose any such Y_n. So, $p_n \in f_n(Y_{n+1}, Y_n)(p_{n+1})$. Similarly applying (1) for $1 \leq i \leq n-1$, $(p_{i+1}, p_i) \in G(f_i) \cap (Y_{i+1} \times Y_i)$ for some $Y_i \in \mathcal{S}_i$; that is, $p_i \in f_i(Y_{i+1}, Y_i)(p_{i+1})$. Hence, $(p_1, \ldots, p_n, p_{n+1}) \in G'(f_1(Y_2, Y_1), \ldots, f_n(Y_{n+1}, Y_n)) \subset G'_n$. By (2), we have that, for each $1 \leq i \leq n$, $f_i(Y_{i+1}, Y_i)$ is a union of continuum-valued functions. So, beginning with $i = n$, one by one, for each $n \geq i \geq 1$, we pick a continuum-valued function $g_i \colon Z_{i+1} \to g_i(Z_{i+1})$ such that $(p_{i+1}, p_i) \in G(g_i) \subset G(f_i(Y_{i+1}, Y_i))$, and we let $Z_i = g_i(Z_{i+1}) \subset Y_i$.

Now, the set-valued function $\hat{F}_n \colon Z_{n+1} \to G'(g_1, \ldots, g_{n-1})$ whose inverse graph is $G'(g_1, \ldots, g_n)$ is a continuum-valued function such that $(p_{n+1}, p_1, \ldots, p_n)$ is in $G(\hat{F}_n)$; recall the last sentence of the paragraph in Section 2 where F_n was defined. We have that $F_n|_{Z_{n+1}} \colon Z_{n+1} \to G'_{n-1}$ is a union of continuum-valued functions. □

Theorem 1 generalizes Theorem 1.6 in [5] and Theorem 1 in [7]. To see this, simply let $\mathcal{S}_i = \{X_i\}$ for each $i \geq 1$. See Example 3 in Section 5 for an example that uses Theorem 1.

Theorem 1. *Let $\{X_i, f_i\}$ be an inverse sequence, where for each $i \geq 1$, X_i is a continuum, and $f_i\colon X_{i+1} \to X_i$ is a surjective, upper semi-continuous, set-valued function whose graph is connected. Let $\{\mathcal{S}_i\}_{i\geq 1}$ be a sequence such that, for each $i \geq 1$, $\mathcal{S}_i$ is a family of subcontinua of X_i. Suppose the following conditions hold for each $i \geq 1$.*

(1) *$f_i(Z) \subset \cup\mathcal{S}_i$ for each $Z \in \mathcal{S}_{i+1}$.*
(2) *If $G(f_i) \cap (Z \times Y)$ is non-empty for $(Z, Y) \in (\mathcal{S}_{i+1} \times \mathcal{S}_i)$, then it is a union of continuum-valued functions.*
(3) *If (a, x) and (b, x) are two points of $G(f_i)$, then there exists a continuum $L_i \subset G(f_i)$ containing (a, x) and (b, x) such that either $c_2(L_i) = \{x\}$ or $c_2(L_i) \subset Y$ for some $Y \in \mathcal{S}_i$.*

Then G'_n is connected for each $n \geq 1$.

Proof. We wish to apply Corollary 2. Fix $n \geq 2$. Suppose (a_{n+1}, x_n) and (b_{n+1}, x_n) are two points of $G(f_n)$. By (3), there exists a continuum $L_n \subset G(f_n)$ containing (a_{n+1}, x_n) and (b_{n+1}, x_n) such that either $c_2(L_n) = \{x_n\}$ or $c_2(L_n) \subset Y_n$ for some $Y_n \in \mathcal{S}_n$. If $c_2(L_n) = \{x_n\}$, then, by item 2(c) of Section 2, $F_{n-1}|_{\{x_n\}}$ is a union of continuum-valued functions. Otherwise, $F_{n-1}|_{Y_n}$ is a union of continuum-valued functions by Lemma 2. In either case, we have $F_{n-1}|_{c_2(L_n)}$ is a union of continuum-valued functions. By Corollary 2, G'_n is connected. □

Corollary 3. *Let $\{X_i, f_i\}$ be an inverse sequence, where for each $i \geq 1$, X_i is a continuum, and $f_i\colon X_{i+1} \to X_i$ is a surjective, upper semi-continuous, set-valued function whose graph is connected. Suppose there exists a sequence $\{\mathcal{S}_i\}_{i\geq 1}$ of families of subcontinua of X_i such that analogous conditions as in Theorem 1 are satisfied for the inverse functions f_i^{-1}. That is, for $i \geq 1$, the following conditions hold.*

(1) *$f_i^{-1}(Y) \subset \cup\mathcal{S}_{i+1}$ for each $Y \in \mathcal{S}_i$.*
(2) *If $G(f_i^{-1}) \cap (Y \times Z)$ is non-empty for $(Y, Z) \in (\mathcal{S}_i \times \mathcal{S}_{i+1})$, then it is a union of continuum-valued functions.*
(3) *If (a, x) and (b, x) are two points of $G(f_i^{-1})$, then there exists a continuum $L_i \subset G(f_i^{-1})$ containing (a, x) and (b, x) such that either $c_2(L_i) = \{x\}$ or $c_2(L_i) \subset Y_{i+1}$ for some $Y_{i+1} \in \mathcal{S}_{i+1}$.*

Then G'_n is connected for each $n \geq 1$.

Proof. Given $n \geq 1$, we consider the partial graph $G'\{f_n^{-1}, \ldots, f_1^{-1}\}$ of the reverse sequence of inverse functions as defined in Section 5 of [4]. We apply the methods illustrated and used in Section 5 of [4] to get that G'_n is connected. □

Corollary 4 generalizes Corollary 1.8 in [5] and Corollary 1 in [7].

Corollary 4. *Let* $X = \varprojlim\{X_i, f_i\}$*, where for each* $i \geq 1$*,* X_i *is a continuum, and* $f_i \colon X_{i+1} \to X_i$ *is a surjective, upper semi-continuous, set-valued function whose graph is connected. Suppose there exists a sequence* $\{\mathcal{S}_i\}_{i\geq 1}$ *of families of subcontinua of* X_i *such that either* f_i *satisfies the conditions in Theorem 1 for each* $i \geq 1$*, or* f_i^{-1} *satisfies the conditions in Corollary 4 for each* $i \geq 1$*. Then* X *is a continuum.*

4. Connectedness of partial graphs and inverse limits on $[0, 1]$

We note the following three immediate corollaries to Theorem 1, Corollary 3, and Corollary 4 respectively. Although, it is clear that these corollaries are special cases of the three general results, it is useful to have the corollaries so stated. See Examples 1, 2, and 5 in Section 5.

A *subinterval* of $[0, 1]$ is a subcontinuum of $[0, 1]$.

Corollary 5. *Let* $\{[0, 1], f_i\}$ *be an inverse sequence, where, for each* $i \geq 1$*,* f_i *is a surjective, upper semi-continuous, set-valued function whose graph is connected. Suppose there exists a sequence of subintervals* $\{I_i\}_{i\geq 1}$ *of* $[0, 1]$ *such that, for each* $i \geq 1$*, the following conditions are satisfied.*

(1) $f_i(I_{i+1}) \subset I_i$.
(2) $f_i|_{I_{i+1}} \colon I_{i+1} \to I_i$ *is a union of interval-valued functions.*
(3) *If* (a, t) *and* (b, t) *are two points of* $G(f_i)$*, then there exists a continuum* $L_i \subset G(f_i)$ *containing* (a, t) *and* (b, t) *such that either* $c_2(L_i) = \{t\}$*, or* $c_2(L_i) \subset I_i$.

Then G'_n *is connected for each* $n \geq 1$.

Corollary 6. *Let* $\{[0, 1], f_i\}$ *be an inverse sequence, where, for each* $i \geq 1$*,* f_i *is a surjective, upper semi-continuous, set-valued function whose graph is connected. Suppose there exists a sequence of subintervals* $\{I_i\}_{i\geq 1}$ *of* $[0, 1]$ *such that, for each* $i \geq 1$*,* f_i^{-1} *satisfies analogous conditions that* f_i *satisfied in Corollary 5. Then* G'_n *is connected for each* $n \geq 1$.

Many examples in the literature satisfy the conditions of Corollary 7. In fact, every connected inverse limit example in [2] satisfies Corollary 7.

Corollary 7. *Let* $X = \varprojlim\{[0, 1], f_i\}$*, where, for each* $i \geq 1$*,* f_i *is a surjective, upper semi-continuous, set-valued function whose graph is connected. Suppose there exists a sequence of subintervals* $\{I_i\}_{i\geq 1}$ *of* $[0, 1]$ *such that either* $f_i \colon X_i \to X_{i+1}$ *satisfies the conditions in Corollary 5 for each* $i \geq 1$*, or* $f_i^{-1} \colon X_i \to X_{i+1}$ *satisfies the conditions in Corollary 5 for each* $i \geq 1$*. Then* X *is connected.*

For I a subinterval of $[0, 1]$, we let $I^c = [0, 1] \setminus I$. Let I_0^c and I_1^c denote, respectively, the components of I^c that contain 0 and 1. We note that one or both of I_0^c and I_1^c may not exist.

Theorems 2 and 3 below are complementary results to Lemma 2 and Corollary 5, allowing for a bit more generality regarding the behavior of the bonding functions on a specified subinterval of $[0,1]$. Both Corollary 5 and Theorem 3 appear to have rather technical conditions, however, the conditions are quite easy to check, particularly if there is a single bonding function. We first establish a lemma that will be needed in the proof of Theorem 2.

Lemma 3. *Let $f\colon [0,1] \to [0,1]$ be a surjective, upper semi-continuous, set-valued function such that, on some subinterval I of $[0,1]$, $G(f) \cap (I \times I)$ is a union of interval-valued functions. If $g'\colon I \to [0,1]$ is interval-valued, $G(g') \subset G(f)$, and $y \in g'(x) \cap I_i^c$ for some $x \in I$ and $i \in \{0,1\}$, then there exists an interval-valued function $g\colon I \to I \cup I_i^c$ such that $(x,y) \in G(g) \subset G(f)$.*

Proof. Assume, without loss of generality, that $y \in g'(x) \cap I_0^c$. If $g'(I) \subset I \cup I_0^c$, then g' satisfies the conclusion, and the proof is complete. So, we assume $g'(I)$ meets both I_0^c and I_1^c.

Suppose $g'(x) \cap I \neq \emptyset$. Let $t \in g'(x) \cap I$. By hypothesis, we let $f'\colon I \to I$ be an interval-valued function with $(x,t) \in G(f')$. Note that $[y,t] \subset g'(x)$ since g' is interval-valued. Let K be the component of $f(x)$ that contains $[y,t]$. Let $g\colon I \to I \cup I_0^c$ be the interval-valued function where $g(x) = K \cap (I \cup I_0^c)$, and $g(z) = f'(z)$ for $z \neq x$. It is easy to see that g is the desired interval-valued function.

Suppose that $g'(x) \cap I = \emptyset$. Let $I = [u,v]$, and note that, by our assumption in the first paragraph of this proof, for some $z \in I \setminus \{x\}$, $v \in g'(z)$. Let r be the largest number in $[u,x)$ such that $v \in g'(r)$, and let s be the least number in $(x,v]$ such that $v \in g'(s)$. We observe that one and only one of r and s may not exist. By hypothesis, we may choose interval-valued functions $f_r\colon [u,r] \to [u,v]$ with $(r,v) \in G(f_r)$ and $f_s\colon [s,v] \to [u,v]$ with $(s,v) \in G(f_s)$. Let K_r and K_s be, respectively, the components of $f(r)$ and $f(s)$ that contain v. We define $g\colon I \to I \cup I_0^c$ as follows. Let $g(r) = K_r \cap (I \cup I_0^c)$, and $g(s) = K_s \cap (I \cup I_0^c)$. Otherwise, let

$$g(z) = \begin{cases} f_r(z) & \text{for } u \le z < r \\ g'(z) & \text{for } r < z < s \\ f_s(z) & \text{for } s < z \le v \end{cases}$$

It is straightforward to see that g is the desired interval-valued function. □

Example 4 in Section 5 illustrates the use of Theorem 2 and Theorem 3 below. It may be helpful to read Example 4 alongside reading Theorem 2 and its proof.

Theorem 2. *Let $\{[0,1], f_i\}$ be an inverse sequence, where, for each $i \geq 1$, f_i is a surjective, upper semi-continuous, set-valued function whose graph is connected. Suppose there exists an interval $I \subset [0,1]$ such that, for each $i \geq 1$, the following conditions are satisfied.*

(1) *$G(f_i) \cap (I \times I)$ is a union of interval-valued functions.*
(2) *$f_i|_I : I \to [0,1]$ is a union of interval-valued functions.*
(3) *For $(z,y) \in G(f_i) \cap (I_j^c \times J)$ where $j \in \{0,1\}$ and $J \in \{I_0^c, I, I_1^c\}$, there exists an interval-valued function $g_i : I_j^c \cup I \to I \cup J$ such that $(z,y) \in G(g_i) \subset G(f_i)$.*

Then, for each $n \geq 1$,

(a) *$F_n|_I : I \to G'_{n-1}$ is a union of continuum-valued functions, and*
(b) *if $p = (p_{n+1}, p_1, \ldots, p_n) \in G(F_n)$ with $p_{n+1} \in I_j^c$ for some $j \in \{0,1\}$, then there exists a continuum-valued function $\hat{F}_n : I_j^c \cup I \to G'_{n-1}$ with $p \in G(\hat{F}_n) \subset G(F_n)$.*

Proof. We use induction on n. Let $n = 1$. Recall, from item 4 of Section 2, that $F_1 = f_1$. By (2) of the hypothesis, $f_1|_I$ is a union of interval-valued functions. So, (a) is satisfied for F_1. Let $(p_2, p_1) \in G(f_1)$ with $p_2 \in I_j^c$ for some $j \in \{0,1\}$. Now, $p_1 \in J$ for some $J \in \{I_0^c, I, I_1^c\}$. By (3) of the hypothesis, there exists an interval-valued function $g_1 : I_j^c \cup I \to I \cup J$ such that $(p_2, p_1) \in G(g_1) \subset G(f_1)$. So, (b) is satisfied for F_1.

Assume, for some $n \geq 1$, properties (a) and (b) hold for the set-valued function $F_n : [0,1] \to G'_{n-1}$. We show that properties (a) and (b) hold for $F_{n+1} : [0,1] \to G'_n$.

(a) Let $p = (p_{n+2}, p_1, \ldots, p_{n+1}) \in G(F_{n+1})$ with $p_{n+2} \in I$. We consider the two cases, whether $p_{n+1} \in I$ or $p_{n+1} \in I_j^c$ for some $j \in \{0,1\}$.

Suppose $p_{n+1} \in I$. By (1) of the hypothesis, there exists an interval-valued function $g_{n+1} : I \to I$ such that $(p_{n+2}, p_{n+1}) \in G(g_{n+1}) \subset G(f_{n+1})$. By inductive assumption (a), there exists a continuum-valued function $\hat{F}_n : I \to G'_{n-1}$ such that $(p_{n+1}, p_1, \ldots, p_n) \in G(\hat{F}_n) \subset G(F_n|_I)$. We define $\hat{F}_{n+1} : I \to G'_n$ by $\hat{F}_{n+1}(z) = G(\hat{F}_n|^{-1}_{g_{n+1}(z)})$. By item 4 in Section 2, we see that $\hat{F}_{n+1}$ is continuum-valued. Also, $p \in G(\hat{F}_{n+1}) \subset G(F_{n+1}|_I)$. So, $F_{n+1}|_I$ is a union of continuum-valued functions.

Suppose $p_{n+1} \in I_j^c$ for some $j \in \{0,1\}$. We assume, without loss of generality, that $j = 0$. So, $(p_{n+2}, p_{n+1}) \in I \times I_0^c$. By (2) of the hypothesis, there exists an interval-valued function $g'_{n+1} : I \to [0,1]$ such that $(p_{n+2}, p_{n+1}) \in G(g'_{n+1}) \subset G(f_{n+1}|_I)$. By Lemma 3, there exists an interval-valued function $g_{n+1} : I \to I_0^c \cup I$ such that $(p_{n+2}, p_{n+1}) \in G(g_{n+1}) \subset G(f_{n+1}|_I)$. Since $p_{n+1} \in I_0^c$, by inductive assumption (b), there exists a continuum-valued function $\hat{F}_n : I_0^c \cup I \to G'_{n-1}$ with $p \in$

$G(\hat{F}_n) \subset G(F_n)$. We define $\hat{F}_{n+1} \colon I \to G'_n$ analogously as in the previous paragraph, again getting that $F_{n+1}|_I$ is a union of continuum-valued functions.

(b) Let $p = (p_{n+2}, p_1, \ldots, p_{n+1}) \in G(F_{n+1})$ with $p_{n+2} \in I_j^c$ for some $j \in \{0,1\}$. Now, $p_{n+1} \in J$ for some $J \in \{I_0^c, I, I_1^c\}$. By (3) of the hypothesis, there exists an interval-valued function $g_{n+1} \colon I_j^c \cup I \to I \cup J$ such that $(p_{n+2}, p_{n+1}) \in G(g_{n+1}) \subset G(f_{n+1})$.

If $J = I$, then $g_{n+1} \colon I_j^c \cup I \to I$ is interval-valued, and we apply inductive assumption (a) to get a continuum-valued function $\hat{F}_n \colon I \to G'_{n-1}$ such that $(p_{n+1}, p_1, \ldots, p_n) \in G(\hat{F}_n) \subset G(F_n|_I)$. If $J = I_k^c$ for some $k \in \{0,1\}$, then $g_{n+1} \colon I_j^c \cup I \to I_k^c \cup I$ is interval-valued, and we apply inductive assumption (b) to get a continuum-valued function $\hat{F}_n \colon I_k^c \cup I \to G'_{n-1}$ with $(p_{n+1}, p_1, \ldots, p_n) \in G(\hat{F}_n) \subset G(F_n)$. In either case, we define the continuum-valued function $\hat{F}_{n+1} \colon I_j^c \cup I \to G'_n$ analogously as in the proof of property (a), getting that property (b) holds for F_{n+1}. □

Theorem 3. *Let $\{[0,1], f_i\}$ be an inverse sequence, where, for each $i \geq 1$, f_i is a surjective, upper semi-continuous, set-valued function whose graph is connected. Suppose there exists an interval $I \subset [0,1]$ such that, for each $i \geq 1$, the following conditions are satisfied.*

1. *$G(f_i) \cap (I \times I)$ is a union of interval-valued functions.*
2. *$f_i|_I \colon I \to [0,1]$ is a union of interval-valued functions.*
3. *For $(z,y) \in G(f_i) \cap (I_j^c \times J)$ where $j \in \{0,1\}$ and $J \in \{I_0^c, I, I_1^c\}$, there exists an interval-valued function $g_i \colon I_j^c \cup I \to I \cup J$ such that $(z,y) \in G(g_i) \subset G(f_i)$.*
4. *If (a,x) and (b,x) are two points of $G(f_i)$, then there exists a continuum $L_i \subset G(f_i)$ containing (a,x) and (b,x) such that either $c_2(L_i) = \{x\}$, or $c_2(L_i) \subset I$.*

Then G'_n is connected for each $n \geq 1$; hence, $\varprojlim\{[0,1], f_i\}$ is a continuum.

Proof. Using Theorem 2 and Lemma 1, the proof is analogous to the proof of Corollary 2. □

Corollary 8. *Let $\{[0,1], f_i\}$ be an inverse sequence, where, for each $i \geq 1$, f_i is a surjective, upper semi-continuous, set-valued function whose graph is connected. Suppose, for each $i \geq 1$, f_i^{-1} satisfies analogous conditions that f_i satisfied in Theorem 3. Then G'_n is connected for each $n \geq 1$; hence, $\varprojlim\{[0,1], f_i\}$ is a continuum.*

5. Examples

In this section, we provide examples to illustrate the use of our results to easily determine connectedness of inverse limits when the set-valued bonding functions satisfy certain properties. We revisit some well-known examples, and look at some new ones. There are no other results presently in the literature that immediately show connectedness of the examples.

Example 1. The functions g and f, whose graphs are pictured in Figure 1, are defined and discussed in Examples 2.8 and 2.9, respectively, in [2]. Ingram proves that each of g and f produces a connected inverse limit when used as the single bonding function in an inverse sequence on $[0, 1]$. We observe that g^{-1} and f, respectively, satisfy the three conditions in Corollaries 6 and 5. It then follows from Corollary 7 that the two inverse limits, using one of g or f as a single bonding function, are continua.

To see that g^{-1} satisfies Corollary 6, let $\{I_i\}_{i\geq 1}$ be the constant sequence where each $I_i = [\frac{3}{4}, 1]$. Clearly, $g^{-1}([\frac{3}{4}, 1]) \subset [\frac{3}{4}, 1]$, and $g^{-1}|_{[\frac{3}{4},1]}$ is a union of two mappings. So, conditions (1) and (2) are satisfied. For condtion (3), it is easy to see that for two points (a, t) and (b, t) in the graph of g^{-1}, there is a continuum in the graph of g^{-1} containing them whose projection into the second coordinate is a subset of $[\frac{3}{4}, 1]$. Hence, by Corollary 7, $\varprojlim\{[0, 1], g\}$ is a continuum.

To see that f satisfies Corollary 5, let $\{I_i\}_{i\geq 1}$ be the constant sequence where each $I_i = [0, \frac{1}{4}]$. As in the previous paragraph, it is easy to check that f satisfies the three conditions in Corollary 5, making $\varprojlim\{[0, 1], f\}$ a continuum.

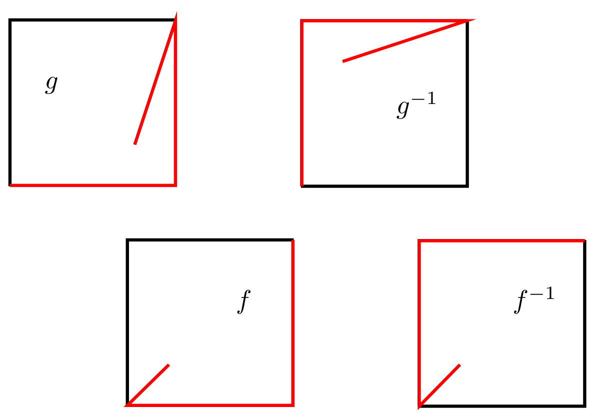

Figure 1. Graphs of g, f, and their inverses in Example 1.

Example 2. We consider the collection of functions f_a, for $\frac{1}{2} \leq a \leq 1$, pictured in Figure 2. In Example 2.24 in [2], Ingram discusses the function $f_{\frac{1}{2}}$ and its associated non-connected inverse limit. He attributes the example to V. Nall.

For $\frac{1}{2} < a \leq 1$, we consider the constant sequence $\{[0, \frac{1}{2}]\}_{i\geq 1}$, and we note that f_a satisfies the three condtions in Corollary 5. Hence, by Corollalry 7, $\varprojlim\{[0,1], f_a\}$ is a continuum for $\frac{1}{2} < a \leq 1$.

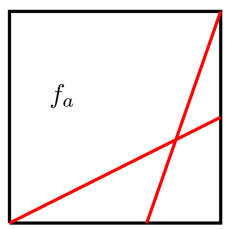

Figure 2. Graph of f_a for $a > \frac{1}{2}$ in Example 2.

Example 3. In this example, the function g_1 whose graph is pictured in Figure 3 does not satisfy the conditions of Corollary 5, but it does satisfy the conditions of Theorem 1, and hence produces a connected inverse limit. The graph of g_1 is the union of four straight line segments joining the five points, $(\frac{1}{4}, \frac{1}{4})$ to $(0, 0)$ to $(1, 0)$ to $(0, 1)$ to $(\frac{1}{2}, 1)$.

To satisfy the conditions of Theorem 1, we let $\mathcal{S}_i = \{[0, \frac{1}{4}], [\frac{3}{4}, 1]\}$ for each $i \geq 1$. For condition (1), we see that $g_1([0, \frac{1}{4}]) \subset [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and $g_1([\frac{3}{4}, 1]) \subset [0, \frac{1}{4}]$. For condition (2), each of $G(g_1) \cap ([0, \frac{1}{4}] \times [0, \frac{1}{4}])$, $G(g_1) \cap ([0, \frac{1}{4}] \times [\frac{3}{4}, 1])$, and $G(g_1) \cap ([\frac{3}{4}, 1] \times [0, \frac{1}{4}])$ is a union of two mappings. For condition (3), if (a, x) and (b, x) are two points of $G(g_1)$, we see that there exists a continuum L such that either $c_2(L) = \{1\}$, or $c_2(L) \subset [0, \frac{1}{4}]$. Hence, by Corollary 5, $\varprojlim\{[0,1], g_1\}$ is a continuum.

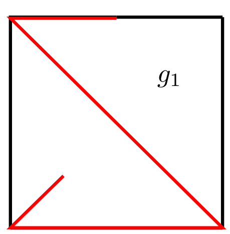

Figure 3. Graph of g_1 in Example 3.

Example 4. The function g_2 whose graph is pictured in Figure 4 does not satisfy either of Theorem 1 or Corollary 5. It does, however, satisfy Theorem 3, and hence produces a connected inverse limit. The graph of g_2 is the union of four straight line segments. One from $(0,0)$ to $(\frac{1}{4},0)$, a second from $(\frac{1}{4},0)$ to $(1,\frac{3}{4})$, a third from $(\frac{1}{2},\frac{1}{2})$ to $(\frac{3}{4},\frac{1}{2})$, and a fourth from $(\frac{1}{2},\frac{1}{2})$ to $(1,1)$. We let $I=[\frac{1}{2},1]$, and $I_0^c=[0,\frac{1}{2})$. We note that the conditions of Theorem 3 are satisfied.

Clearly, (1) and (2) are satified by g_2 on the interval I. For condition (3), let $(z,y)\in G(g_2)\cap(I_0^c\times J)$ for some $J\in\{I_0^c,I\}$. It follows that $z\in[0,\frac{1}{2})$ and $y\in J=[0,\frac{1}{2})$. We see that there is a mapping $g\colon[0,1]\to[0,1]$ such that $(z,y)\in G(g)\subset G(g_2)$. For condition (4), if (a,x) and (b,x) are two points of $G(g_2)$, then there exists a continuum L containing them such that either $c_2(L)=\{0\}$ or $c_2(L)\subset[\frac{1}{2},1]$. Hence, by Theorem 3, $\varprojlim\{[0,1],g_2\}$ is a continuum.

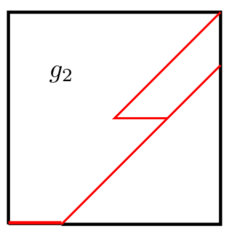

Figure 4. Graph of g_2 in Example 4.

Example 5. We consider an inverse sequence with alternating functions p and q whose graphs are pictured in Figure 5. The graph of p is the union of the graph of f from Example 1 and the line segment from $(\frac{1}{4},\frac{1}{4})$ to $(\frac{1}{4},\frac{1}{2})$. The graph of q is the union of three line segments. One from $(\frac{1}{2},0)$ to $(0,\frac{1}{4})$, a second from $(0,\frac{1}{4})$ to $(\frac{1}{2},\frac{1}{4})$, and a third from $(\frac{1}{2},\frac{1}{4})$ to $(1,1)$. Let X and Y be, respectively, the inverse limits on $[0,1]$ of the alternating inverse sequences of functions $p,q,p,q,\ldots$ and $q,p,q,p,,\ldots$. We observe that Corollary 5 is satisfied in both cases, and hence, by Corollary 7, X and Y are continua.

For the sequene $p,q,p,q,\ldots$, let $I_{2i-1}=[0,\frac{1}{2}]$ and $I_{2i}=[0,\frac{1}{4}]$ for $i\geq 1$. We have that $p([0,\frac{1}{4}])=[0,\frac{1}{2}]$ and $p|_{[0,\frac{1}{4}]}$ is a union of continuum-valued functions. Also, if (a,t) and (b,t) are two points of $G(p)$, then there exists a continuum $L\subset G(p)$ such that $c_2(L)\subset[0,\frac{1}{2}]$. Furthermore, we have that $q([0,\frac{1}{2}])=[0,\frac{1}{4}]$ and $q|_{[0,\frac{1}{2}]}$ is a union of two mappings. Also, if (a,t) and (b,t) are two points of $G(q)$, then $t=\frac{1}{4}$ and there exists a continuum

$L \subset G(q)$ such that $c_2(L) = \{\frac{1}{4}\}$. So, the conditions of Corollary 5 are satisfied, and hence, X is a continuum.

A similar argument shows that Corollary 5 holds for the sequence $q, p, q, p, \ldots$. Hence, Y is a continuum.

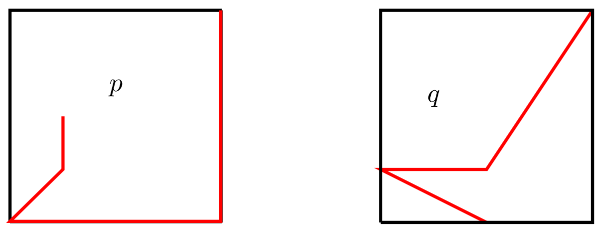

Figure 5. Graphs of p and q in Example 5.

In our last example, we show that the limit of an inverse sequence, where each bonding function is either f or f^{-1} from Example 1, is a continuum. Such an inverse limit may, in general, not be connected, even if f is a mapping.

Example 6. Given the functon f from Example 1, let $X = \varprojlim\{[0,1], f_i\}$, where, for each $i \geq 1$, $f_i = f$ or $f_i = f^{-1}$. Then X is a continuum. Let $I = [0, \frac{1}{4}]$. We use induction on n, and Corollary 2 to show that for each $n \geq 1$, the function $F_n|_I \colon I \to G'_{n-1}$ is a union of continuum-valued functions, and $G(F_n^{-1}) = G'_n$ is connected.

Let $n = 1$. Then $F_1 = f_1$. Since f_1 is either f or f^{-1}, we see, in Figure 1, that, in either case, $F_1|_I$ is a union of interval-valued functions. Also, $G(F_1)$ is connected; so, G'_1 is connected.

Assume for some $n \geq 1$, $F_n|_I$ is a union of continuum-valued functions, and G'_n is connected. By item 3 of Section 2, G'_j is connected for each $1 \leq j \leq n$. We note that, for $f_{n+1} \in \{f, f^{-1}\}$, if (a, x) and (b, x) are two points of $G(f_{n+1})$, then there exists a continuum L containing (a, x) and (b, x) such that either $c_2(L) = \{1\}$ or $c_2(L) \subset I$. In either case, by item 2(c) of Section 2 or by inductive assumption, $F_n|_{c_2(L)}$ is a union of continuum-valued functions. So, by Corollary 2, G'_{n+1} is connected.

We now show that $F_{n+1}|_I$ is a union of continuum-valued functions. Let $p = (p_{n+2}, p_1, \ldots, p_{n+1}) \in G(F_{n+1})$ with $p_{n+2} \in I$. There are two cases to consider.

Case 1. Suppose $f_{n+1} = f$. Then, $f_{n+1}(I) = I$ and $f_{n+1}|_I$ is a union of two mappings, namely $\mathrm{id}|_I$ and the constant 0 function on I. Since, by inductive assumption, $F_n|_I$ is a union of continuum-valued functions, we let $\hat{F}_n\colon I \to G'_{n-1}$ be a continuum-valued function whose graph contains the point $(p_{n+1}, p_1, \ldots, p_n)$.

If $p_{n+1} = 0$, define $\hat{F}_{n+1}\colon I \to G'_n$ by $\hat{F}_{n+1}(t) = \hat{F}_n(0) \times \{0\}$. We note that $G(\hat{F}_{n+1}) \subset G(F_{n+1}|_I)$, and $\hat{F}_{n+1}$ is continuum-valued since $\hat{F}_n(0)$ is a continuum. Also, $p = (p_{n+2}, p_1, \ldots, p_n, 0) \in G(\hat{F}_{n+1})$.

If $p_{n+1} \neq 0$, then $p_{n+1} = p_{n+2}$. Define $\hat{F}_{n+1}\colon I \to G'_n$ by $\hat{F}_{n+1}(t) = \hat{F}_n(t) \times \{t\}$. Analogously, as in the previous paragraph, $\hat{F}_{n+1}$ is continuum-valued, and $p \in G(\hat{F}_{n+1}) \subset G(F_{n+1}|_I)$. So, $F_{n+1}|_I$ is a union of continuum-valued functions as desired.

Case 2. Suppose $f_{n+1} = f^{-1}$. If $p_{n+2} = 0$, let $\hat{F}_n\colon I \to G'_{n-1}$ be a continuum-valued function whose graph contains the point $(0, 0, \ldots, 0)$, and define $\hat{F}_{n+1}(0) = G'_n$, and $\hat{F}_{n+1}(t) = \hat{F}_n(t) \times \{t\}$ for $0 < t \leq \frac{1}{4}$. By inductive assumption, G'_n is a continuum, and, by definition, $\hat{F}_n(t)$ is a continuum for each $0 < t \leq \frac{1}{4}$. So, $\hat{F}_{n+1}$ is continuum-valued. Also, it is clear that $p \in G(\hat{F}_{n+1}) \subset G(F_{n+1}|_I)$.

If $p_{n+2} \neq 0$, then either $p_{n+1} = p_{n+2} \in I$ or $p_{n+1} = 1$. If $p_{n+1} = p_{n+2}$, let $\hat{F}_n\colon I \to G'_{n-1}$ be a continuum-valued function whose graph contains the point $(p_{n+1}, p_1, \ldots, p_n)$, and define $\hat{F}_{n+1}(t) = \hat{F}_n(t) \times \{t\}$. It follows that $\hat{F}_{n+1}$ is the desired continuum-valued function.

If $p_{n+1} = 1$, then either $p_i = 1$ for all $1 \leq i \leq n+1$, or there exist a largest $1 \leq j < n+1$ such that $p_j \neq 1$. If the former is the case, define $\hat{F}_{n+1}(t) = (1, 1, \ldots, 1, t)$ for $t \in I$. In this case, $\hat{F}_{n+1}$ is a mapping, and $p \in G(\hat{F}_{n+1}) \subset G(F_{n+1}|_I)$. If the latter is the case, then $f_j = f$, and $f_j(1) = [0,1]$. So, we define $\hat{F}_{n+1}(t) = G'_{j-1} \times (1, \ldots, 1, t)$ for $t \in I$. It is clear, in this case, that $\hat{F}_{n+1}$ is continuum-valued, and $p \in G(\hat{F}_{n+1}) \subset G(F_{n+1}|_I)$.

The inductive proof is complete, and it follows that X is a continuum.

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