To appear in
Topology Proceedings

FOLDERS OF CONTINUA

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Abstract. This article is motivated by the following unsolved fixed point problem of G. R. Gordh, Jr. If a continuum X admits a map onto an arc such that the preimage of each point is either a point or an arc, then must X have the fixed point property? We call such a continuum an arc folder. This terminology generalizes naturally to the concept of a continuum folder.

We give several partial solutions to Gordh's problem. The answer is yes if X is either planar, one dimensional, or an approximate absolute neighborhood retract. We establish basic properties of both continuum folders and arc folders. We provide several specific examples of arc folders, and give general methods for constructing continuum folders. Numerous related questions are raised for further research.

1. Introduction

The main focus of this paper is arc folders; that is, continua admitting maps onto an arc with point preimages being an arc or a point. In conversation (circa 1980) with a number of topologists, G. R. Gordh, Jr. asked the following question, which is still open.

Question 1. *Do all arc folders have the fixed point property?*

We find this question challenging and intriguing, and the class of arc folders interesting and more diverse than its rather restrictive definition may suggest. In this paper, we give some partial answers to Gordh's

2010 *Mathematics Subject Classification.* Primary 54F15, 54B15, 54D80; Secondary 54C10, 54B17.

Key words and phrases. continuum folder, arc folder.

question. We establish some general properties and unexpected examples of arc folders and their generalizations.

A *continuum* is a non-empty, compact, connected metric space. A *map* or *mapping* is a continuous function. A mapping $f\colon X \to Y$ is *monotone* if, for each $y \in Y$, $f^{-1}(y)$ is connected. We study the class $\mathcal{M}$ of continua that admit a monotone mapping onto $[0,1]$. If $X \in \mathcal{M}$ and $\eta\colon X \to [0,1]$ is surjective and monotone, we refer to $[0,1]$ as the *base space* and to each $\eta^{-1}(t)$, for $t \in [0,1]$, as a *fiber* of X. This structure gives rise to an upper semi-continuous decomposition of X, where the fibers are elements of the decomposition. The class $\mathcal{M}$ is inverse-invariant with respect to monotone maps; that is, if X is in $\mathcal{M}$ and Y admits a monotone map onto X, then Y is in $\mathcal{M}$. Metric brush spaces with core an arc are in $\mathcal{M}$ [34]; so, dendrites are in $\mathcal{M}$. The arc of pseudo-arcs [3], [14], [23], the pseudo-hairy arc [46], and the hairy arc [1] are intricate members of $\mathcal{M}$ that have received recent attention.

For eighty years, there has been considerable interest in the special case of irreducible continua X in $\mathcal{M}$, see for example [19], [21], [22], [25], [27, Section 48], [28], [30], [31], [37], [39], [49], and [50]. E.S. Thomas's article [49] provides an extensive investigation of this case.

Definition 1. We refer to a member X of $\mathcal{M}$ together with a monotone surjective map $\eta\colon X \to [0,1]$ as a *continuum folder*. So, a continuum folder is a pair (X,η), but we will typically refer to X as a continuum folder, with an assumed monotone map $\eta\colon X \to [0,1]$. Let $\mathcal{G}$ be a class of continua. If each fiber of X is either a point or belongs to $\mathcal{G}$, we call X a $\mathcal{G}$ *folder* or a *folder of continua from* $\mathcal{G}$.

Definition 2. For M a continuum, an $\{M\}$ folder, which we denote simply by M *folder*, is a continuum folder where each fiber is either a point or is homeomorphic to M.

Definition 3. If the decomposition of a continuum folder X into its fibers is a continuous decomposition, we call X a *continuous continuum folder*.

A familiar, simple example of a continuous M folder is $[0,1]\times M$, which is sometimes called an M *cylinder*.

There are arc folders, tree folders, disk folders, folders of absolute retracts, folders of chainable continua, etc. Our particular interest is arc folders.

In Section 2, we establish general properties of continuum folders. Subsections 3.1 and 3.2 provide general methods for constructing examples. Section 4 contains results specific to arc folders.

2. Continuum folders

For X a compact metric space, and $B \subset X$, we let $\mathrm{cl}(B)$ and $\dim(B)$ denote, respectively, the closure of B, and the inductive dimension of B as defined in [24] and [42]. If continua X and Y are homeomorphic, we write $X \overset{T}{\approx} Y$.

Lemma 1. *If X is a folder of continua and $t \in (0,1)$, then $\mathrm{cl}(\eta^{-1}([0,t))) \cap \eta^{-1}(t)$ and $\mathrm{cl}(\eta^{-1}((t,1])) \cap \eta^{-1}(t)$ are continua. For $t = 0$, the latter set is a continuum, and for $t = 1$, the former set is a continuum.*

Proof. Let $0 < t \leq 1$. The set $\mathrm{cl}(\eta^{-1}([0,t))) \cap \eta^{-1}(t)$ is closed, so we will show that it is also connected. Assume the contrary. Let $\mathrm{cl}(\eta^{-1}([0,t))) \cap \eta^{-1}(t)$ be the union of two disjoint, non-empty, closed sets A and B. Let C and D be disjoint open sets in X such that $A \subset C$ and $B \subset D$. Since each fiber of X is a continuum, there exists $r \in [0,t)$ such that for each $s \in [r,t)$, either $\eta^{-1}(s) \subset C$ or $\eta^{-1}(s) \subset D$. Let $s \in [r,t)$ such that $\eta^{-1}(s) \subset C$. Since $\eta^{-1}(s)$ separates X, it follows that $\eta^{-1}([0,s]) \cup \mathrm{cl}(\eta^{-1}([s,t)) \cap C)$ and $\mathrm{cl}(\eta^{-1}([s,t)) \cap D)$ are non-empty, disjoint, closed sets whose union is $\mathrm{cl}(\eta^{-1}([0,t)))$. However, since η is monotone, $\eta^{-1}([0,t))$ is connected and $\mathrm{cl}(\eta^{-1}([0,t)))$ is a continuum, which is a contradiction.

A similar argument holds when $0 \leq t < 1$. □

A continuum X is *acyclic* if for each $n \geq 1$, the n^{th} Čech cohomology group of X, $H^n(X;\mathbb{Z})$, is trivial. A continuum X is *unicoherent* if whenever X is the union of two subcontinua H and K, we have that $H \cap K$ is connected.

Proposition 1. *If X is a folder of acyclic continua, then X is acyclic. Also, X is unicoherent.*

Proof. That X is acyclic follows from the Vietoris-Begle theorem, see [48, Theorem 15, page 344]. Since $H^1(X;\mathbb{Z})$ is trivial, by Bruschlinsky's theorem [13, Theorem 8.1], each map from X to S^1 is inessential (homotopic to a constant map). So, X is unicoherent. □

Proposition 2. *If X is a folder of continua with trivial shape and $\dim(X)$ is finite, then X has trivial shape.*

Proof. This follows from R.B Sher [47, Theorem 11]. Or, see Theorem 9.3 on page 352 in [6]. □

Proposition 3. *If X is a folder of continua, each with dimension less than or equal n, then $\dim(X) \leq n+1$.*

Proof. This follows from [42, Theorem 22.1, page 139]. □

Proposition 4. *If X is a hereditarily unicoherent folder of continua, then each indecomposable subcontinuum of X is contained in some fiber of X.*

Proof. Suppose X contains an indecomposable subcontinuum Y that is not contained in a fiber of X. Then Y must meet two fibers $\eta^{-1}(r)$ and $\eta^{-1}(s)$, where $0 \leq r < s \leq 1$. We can choose a, b with $r \leq a < b \leq s$ so that two distinct composants H and L of Y meet both $\eta^{-1}(a)$ and $\eta^{-1}(b)$. Let E and F be continua in H and L, respectively, such that $\eta^{-1}(a) \cap E \neq \emptyset \neq \eta^{-1}(b) \cap E$, and $\eta^{-1}(a) \cap F \neq \emptyset \neq \eta^{-1}(b) \cap F$. Note that $\eta^{-1}(a) \cup E \cup \eta^{-1}(b)$ and $\eta^{-1}(a) \cup F \cup \eta^{-1}(b)$ are two subcontinua of X whose intersection $\eta^{-1}(a) \cup \eta^{-1}(b)$ is not connected. So, X contains a non-unicoherent subcontinuum, which is a contradiction. □

A continuum X is *irreducible* if there exist points p and q in X such that no proper subcontinuum of X contains both p and q. In this case, we say that X is *irreducible between the points p and q*. We note that if a continuum folder X is irreducible, then it is irreducible between points $p \in \eta^{-1}(0)$ and $q \in \eta^{-1}(1)$.

Proposition 5. *Each irreducible folder of* 1*-dimensional continua is* 1*-dimensional.*

Proof. This proposition follows almost immediately from Theorem 1 in [28]. One only needs to check that the "fibres" in Theorem 1 are subsets of the 1-dimensional fibers in our folder decomposition. The A. Lelek and D. Zaremba fibres are "layers" of the minimal decomposition discussed by K. Kuratowski in §48(IV) of [27], called "tranches" in an earlier edition of his book. See specifically Theorem 3 on page 200 in [27], or see Theorem 3 on page 8 in [49].

To complete the proof, let X be an irreducible folder of 1-dimensional continua, and let G be the minimal decomposition of X. Since each element of G is contained in a fiber of X, and each fiber of X is 1-dimensional, it follows from [28, Theorem 1] that X is 1-dimensional. □

Proposition 6. *A folder of chainable continua is chainable if and only if it is atriodic.*

Proof. Let X be a folder of chainable continua. If X is chainable, it is known that X is atriodic. So, we prove the converse.

Suppose X is atriodic. Then, in particular, X is not a triod. By Proposition 1, X is unicoherent. By Sorgenfrey's theorem (see [43, 11.34, page 216]), it follows that X is irreducible.

By Proposition 5, the dimension of X is one. By Proposition 2, X has trivial shape. Thus, we have that X is tree-like. So, X is hereditarily unicoherent. By Proposition 4, each indecomposable subcontinuum of X is in some fiber of X. Hence, by [14, Theorem 2], X is chainable. □

Question 2. *Is each irreducible arc folder embeddable in the plane?*

A continuum X is a λ-*dendroid* if it is hereditarily unicoherent and hereditarily decomposable. H. Cook [11] has shown that all λ-dendroids are tree-like.

Proposition 7. *If X is a 1-dimensional folder of λ-dendroids, then X is a λ-dendroid.*

Proof. Since tree-likeness is equivalent to trivial shape in the case of 1-dimensional continua, it follows from Proposition 2 that X is tree-like. Hence, X is hereditarily unicoherent. Since λ-dendroids are hereditarily decomposable, it follows from Proposition 4 that X contains no indecomposable subcontinuum; that is, X is hereditarily decomposable. We have that X is a λ-dendroid. □

A continuum X has the *fixed point property* (fpp) if each mapping $f\colon X \to X$ has a fixed point; that is, a point $x \in X$ such that $f(x) = x$.

Corollary 1. *If X is a 1-dimensional folder of λ-dendroids, then X has the fpp.*

Proof. This follows from Proposition 7 and R. Mańka's result in [32] that λ-dendroids have the fpp. □

A 1-dimensional continuum folder may fail to have the fpp, even when it is irreducible and all of its fibers have the fpp (see [19], [21], and [37]).

Let X and A be continua with $A \subset X$. If $r\colon X \to A$ is a mapping such that $r|_A$ is the identity mapping on A, then r is called a *retraction* and A is called a *retract* of X. The continuum A is an *absolute retract (AR)* if whenever A is embedded as a subset of the Hilbert cube Q, there is a retraction r of Q onto the embedded copy of A.

Below we give an example of an AR folder without the fpp, where all fibers are arcs, except $\eta^{-1}(1)$ which is a disk.

Example 1. *Let X be the cone over a simple spiral to the unit circle in the plane. Then X is an AR folder that does not have the fpp. Furthermore, the fiber $\eta^{-1}(1)$ is a disk and all other fibers of X are arcs.*

Proof. R.J. Knill showed in [26] that X admits a fixed-point-free mapping. We show that X is an AR folder.

First, we embed X in $\mathbb{R}^3$. Let C be the unit circle in the xy-plane, and let S be a simple spiral in the xy-plane such that $q = (2, 0, 0)$ is the endpoint of S, $S \cap C = \emptyset$, and $S \cup C$ is a compactification of S. Let $v = (0, 0, 1)$.

For each point p in the xy-plane, let $\ell_p = \{tv+(1-t)p \mid 0 \le t \le 1\}$. For convenience, we let $t \cdot p$ denote the point $tv + (1 - t)p$ for each $0 \le t \le 1$.

Note that the third coordinate of $t \cdot p$ is t for all points p in the xy-plane. So, for all such p, $0 \cdot p = p$ and $1 \cdot p = v$. For each subset B of the xy-plane, let $\operatorname{cone}(B) = \bigcup_{p \in B} \ell_p$.

Let $X = \operatorname{cone}(S \cup C)$. To see that X is an AR folder, we decompose $\operatorname{cone}(S)$ into a union of arcs; and let $\operatorname{cone}(C)$, which is topologically a disk, be a single member of the decomposition. Let $A_q = \{q\}$. For $p \in S \setminus \{q\}$, let A_p be the arc in S with endpoints q and p. For $0 < t < 1$, let $t \cdot A_p = \{t \cdot z \mid z \in A_p\}$. Let $g \colon S \to [0, 1)$ be a homeomorphism. Finally, for $p \in S$, let $L_p = \{t \cdot p \mid 0 \leq t \leq g(p)\} \cup g(p) \cdot A_p$. Note that $L_q = \{q\}$, and otherwise, L_p is a nondegenerate arc. Also, it is easy to see that the collection $\{L_p \mid p \in S\} \cup \{\operatorname{cone}(C)\}$ is an upper semi-continuous decomposition of X. Letting $\eta \colon X \to [0, 1]$ be the mapping such that $\eta(L_p) = g(p)$ and $\eta(\operatorname{cone}(C)) = \{1\}$, we see that X is an AR folder. Although we have one degenerate fiber, namely L_q, it is clear that the decomposition of $\operatorname{cone}(S)$ could be modified to have no degenerate fibers. □

A compactum X is *weakly aposyndetic* if, for each point $x \in X$, there exist a continuum K in $X \setminus \{x\}$ such that K has nonempty interior. A point x in a compactum X is a *cofilament point* if $X \setminus \{x\}$ contains only continua with empty interior. Dendroids that do not contain a cofilament point are weakly aposyndetic.

Proposition 8. *If X is a 1-dimensional continuous folder of dendroids that do not contain cofilament points, then X is an arc.*

Proof. By Proposition 7, X is a λ-dendroid. Since the decomposition of X into fibers is continuous, the monotone mapping $\eta \colon X \to [0, 1]$ is open. Each fiber that is nondegenerate is weakly aposydetic. So, it follows from Theorem 2.3 in [45], that η is a homeomorphism. Hence, X is an arc. □

Corollary 2. *Each 1-dimensional continuous folder of dendrites is an arc.*

Question 3. *Can Proposition 8 be generalized to include all dendroids?*

The next section provides methods for constructing continuum folders.

3. Examples of threaded continuum folders

Definition 4. A *thread* in a continuum folder X is a continuous selector of $\eta \colon X \to [0, 1]$; that is, an arc A in X such that $A \cap \eta^{-1}(t)$ is degenerate for each $t \in [0, 1]$. The folder X is called *threaded* if each point $x \in X$ belongs to a thread in X.

Proposition 9. *Each threaded continuum folder is continuous.*

Proof. Let X be a threaded continuum folder. It suffices to show that $\eta\colon X \to [0,1]$ is an open mapping. Let U be an open set in X, and let $x \in U$. Since X is threaded, there exists an arc A in X such that $x \in A$ and $A \cap \eta^{-1}(t)$ is degenerate for each $t \in [0,1]$. We see that $\eta(x)$ is in the interior of $\eta(U \cap A)$, which is a subset of $\eta(U)$. It follows that $\eta(U)$ is open. Hence, η is an open mapping, and the decomposition of X into fibers $\eta^{-1}(t)$, for $t \in [0,1]$, is continuous. □

Proposition 10. *An arc is the only one-dimensional threaded arc folder.*

Proof. Let X be a one-dimensional threaded arc folder. By Proposition 9, X is a continuous arc folder. By Corollary 2, X is an arc. □

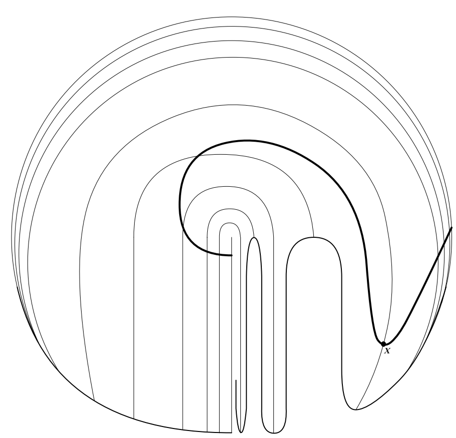

FIGURE 1. A non-locally connected threaded arc folder

Example 2. *Not all threaded arc folders are locally connected.*

Proof. Let W be the Warsaw circle together with its bounded complementary domain as in Figure 1. The darker arc is a thread that contains the given point x and whose intersection with every fiber is degenerate. Clearly, W is not locally connected.

There is a second embedding of the Warsaw circle in the plane that defines a different continuum when its bounded complementary domain is added. Although this continuum admits an arc folder decomposition, it does not admit one that is threaded. □

Question 4. *Do all locally connected threaded arc folders have the fpp?*

Subsection 3.1 introduces a class of continuum folders constructed from mappings between continua. If the mappings are surjective, then these examples are threaded continuum folders. Subsection 3.2 introduces a class of threaded continuum folders that are constructed from inverse limits of continua.

3.1. **Mapping cylinders.**

Definition 5. Let $f\colon M \to N$ be a mapping between continua. The *mapping cylinder* $C(f)$ of f is the adjunction space $([0,1]\times M)\cup_f N$ with quotient map $p\colon ([0,1]\times M)\cup N \to C(f)$ that identifies each point $(0,x)$ in $[0,1]\times M$ with $f(x)$ in N. Otherwise, p embeds $(0,1]\times M$ as an open set in $C(f)$.

Discussion of adjunction spaces and mapping cylinders can be found, respectively, in Chapter VI (Section 6) and Chapter XVIII (Section 4) in [12].

Definition 6. If $M \overset{T}{\approx} N$, we refer to $C(f)$ as an M *mapping cylinder.* In particular, if $M \overset{T}{\approx} N \overset{T}{\approx} [0,1]$, we refer to $C(f)$ as an *arc mapping cylinder.*

Proposition 11. *Let $f\colon M \to N$ be a mapping between continua. The mapping cylinder of f is a continuum folder. Furthermore, if $M \overset{T}{\approx} N$, then $C(f)$ is an M folder. If f is surjective, $C(f)$ is a threaded continuum folder.*

Proof. Define the map $\eta\colon C(f) \to [0,1]$ as follows. Let $\eta(p(t,x)) = t$ for $0 < t \le 1$, and let $\eta(p(y)) = 0$ for $y \in N$. It is clear that η is a monotone map with fibers that are homeomorphic to M for $t > 0$, and with $\eta^{-1}(0)$ homeomorphic to N. So, $C(f)$ is a continuum folder, and clearly, if $M \overset{T}{\approx} N$, then $C(f)$ is an M folder.

For surjective mappings $f\colon M \to N$, the collection of threads $\{p([0,1]\times \{x\}) \mid x \in M\}$ makes $C(f)$ a threaded continuum folder. □

Definition 7. Given $f\colon M \to N$, and quotient map $\eta\colon C(f) \to [0,1]$, we call $\eta^{-1}(0)$ and $\eta^{-1}(1)$, which are, respectively, homeomorphic to N and M, the *left side of C(f)* and the *right side of C(f).*

Let X and A be continua with $A \subset X$, and let $j\colon A \hookrightarrow X$ be the inclusion mapping. A *deformation retraction* of X into A is a homotopy $H\colon X \times [0,1] \to X$ from the identity mapping on X to $j \circ r\colon X \to X$, where r is a retraction of X onto A. In this case, A is called a *deformation retract* of X. If $H(a,t) = a$ for all $a \in A$ and all $t \in [0,1]$, then H is a *strong deformation retraction*, and A is called a *strong deformation retract* of X.

Proposition 12. *Let $f\colon M \to N$ be a mapping between continua. Then N and $C(f)$ have the same fundamental group.*

Proof. It is easy to see that N is a strong deformation retract of $C(f)$ (see [12, 4.2, page 369]). Hence, $C(f)$ has the same fundamental group as N, see [41, Theorem 58.3, page 361]. □

Proposition 13. *Each arc mapping cylinder is 2-dimensional and embeddable in $\mathbb{R}^3$.*

Proof. Let $X = C(f)$ be an arc mapping cylinder. Since X contains a copy of $(0,1] \times [0,1]$, it is clear that X is 2-dimensional. We embed a copy of X in $\mathbb{R}^3$.

Let $I_0 = \{0\} \times \{0\} \times [0,1]$ in $\mathbb{R}^3$, and let $I_1 = \{(1, s, f(s)) \mid s \in [0,1]\}$. For each $t \in [0,1]$, let A_t be the union of the line intervals in $\mathbb{R}^3$ from the point $(0,0,t)$ in I_0 to all points in the set $\{(1,s,t) \mid s \in f^{-1}(t)\} \subset I_1$. It is easy to see that the defining intervals in A_t all lie in the plane $\mathbb{R}^2 \times \{t\}$, all have the point $(0,0,t)$ in common, and otherwise do not intersect. Let $Y = \cup_{t\in[0,1]} A_t$. Clearly, $Y \overset{T}{\approx} X$. □

Proposition 14. *Let $f\colon M \to N$ be a mapping between finite dimensional ARs. Then $C(f)$ is an AR, and hence, has the fpp.*

Proof. By [4, Corollary 10.5, page 122], a finite dimensional compactum is an AR if and only if it is locally contractible and contractible in itself. We first show that $C(f)$ is locally contractible.

If $(t,x) \in C(f)$ for $t > 0$, then (t,x) lies in $[s,1] \times M$ for some $0 < s < t$. Since $[s,1] \times M$ is an AR and ARs are locally contractible, it follows that $C(f)$ is locally contractible at (t,x).

Let $(0,x) = f(x) \in N$ be a point of the base of $C(f)$. Let V be a neighborhood of $(0,x)$ in $C(f)$. Let U be a neighborhood of $(0,x)$ in $V \cap N$ and let U_0 be a connected open set in N such that $(0,x) \in U_0 \subset U$, U_0 is contractible to $(0,x)$ in U, and $L = C(f|_{f^{-1}(U_0)}) \cap \eta^{-1}([0,s)) \subset V$ for some $s > 0$. Now, L is open in $C(f)$ and contains $(0,x)$. Let $H\colon C(f) \times [0,1] \to N$ be the natural deformation retraction. Applying H to L and thereafter contracting U_0 to $(0,x)$ in U gives us that L is contractible to $(0,x)$ in V.

Since N is an AR, it is contractible in itself. So, we use the deformation retraction H from $C(f)$ into N, and follow it with a contraction of N to a point. Hence, $C(f)$ is contractible in itself and, thus, is an AR. $\square$

Although Proposition 14 gives us that every AR mapping cylinder has the fpp, it is not the case that all AR folders have the fpp (recall Example 1).

3.2. **Mapping cylinders of inverse sequences.** We use inverse sequences and inverse limits throughout this section. Definitions and general properties of these notions can be found in [29, Subsections 2.1 – 2.3].

For an inverse sequence $\{X_i, g_i^{i+1}\}$ with surjective bonding maps, we combine the mapping cylinders $C(g_i^{i+1})$ as follows. We assume that all of the $C(g_i^{i+1})$'s are embedded in a single space, and $C(g_i^{i+1}) \cap C(g_j^{j+1}) = \emptyset$ if $j > i+1$. If $j = i+1$, we assume the intersection $C(g_i^{i+1}) \cap C(g_j^{j+1})$ is the right side of $C(g_i^{i+1})$ identified with the left side of $C(g_j^{j+1})$.

Definition 8. Let $X = \varprojlim\{X_i, g_i^{i+1}\}$ be an inverse limit of continua with surjective bonding maps. For each $n \geq 1$, let $A_{n+1} = \cup_{i=1}^{n} C(g_i^{i+1})$. Note that the sequence $\{A_n\}$ is nested. Also, for each $n \geq 2$, since X_n is a strong deformation retract of $C(g_n^{n+1})$, there is a natural retraction $r_n^{n+1} \colon A_{n+1} \to A_n$. Let $C(\{X_i, g_i^{i+1}\}) = \varprojlim\{A_i, r_i^{i+1}, i \geq 2\}$. We call $C(\{X_i, g_i^{i+1}\})$ *the mapping cylinder of the inverse sequence* $\{X_i, g_i^{i+1}\}$.

Remarks. In the terminology introduced by Marsh and Prajs [35, 36], by its definition, $C(\{X_i, g_i^{i+1}\})$ has an internal inverse limit structure with retractions for bonding maps (see specifically the definitions on pages 1211 and 1212, and Theorem 2.1). Furthermore, X is topologically a subset of $C(\{X_i, g_i^{i+1}\})$. To see this, let $x = (x_1, x_2, \ldots)$ be a point of X. Identify $x_i \in X_i$ with $\hat{x}_i = (g_1^i(x_i), g_2^i(x_i), \ldots, x_i, x_i, \ldots)$ in $C(\{X_i, g_i^{i+1}\})$. Then we have that $x = \lim_{i\geq 1} \hat{x}_i$. So, $x \in C(\{X_i, g_i^{i+1}\})$. Under this identification $C(\{X_i, g_i^{i+1}\}) = \bigcup_{i\geq 1} C(g_i^{i+1}) \cup X$, and each X_i is contained in $C(\{X_i, g_i^{i+1}\})$ as the left side of $C(g_i^{i+1})$.

By compressing the images of the quotient maps $\eta_i \colon C(g_i^{i+1}) \to [0,1]$ to a null sequence of amalgamated intervals, and mapping X to 1, we see that $C(\{X_i, g_i^{i+1}\})$ is a continuum folder. Specifically, let $\eta \colon C(\{X_i, g_i^{i+1}\}) \setminus X \to [0,1)$ be defined by

$$\eta(x) = \frac{1}{i(i+1)}\eta_i(x) + \frac{i-1}{i}, \text{ for } x \in C(g_i^{i+1}). \text{ Let } \eta(X) = 1.$$

By definition, each point $x \in X = \varprojlim\{X_i, g_i^{i+1}\}$ is a thread $(x_1, x_2, ...)$ in $X_1 \times X_2 \times ...$ of the inverse sequence $\{X_i, g_i^{i+1}\}$. Under our identification, X, each X_i, and each $C(g_i^{i+1})$ are contained in $C(\{X_i, g_i^{i+1}\})$ and $\lim_{i\geq 1} x_i = x$ in $C(\{X_i, g_i^{i+1}\})$. For each $i \geq 1$, we have $x_i, x_{i+1} \in C(g_i^{i+1})$, and there is an arc $p_i([0,1] \times \{x_{i+1}\})$ in $C(g_i^{i+1})$ from x_i to x_{i+1} (see Definition 5), which we denote by $[x_i, x_{i+1}]$. For $n \geq 2$, let $[x_1, x_n] = \cup_{i=1}^{n}[x_i, x_{i+1}]$, and let $[x_1, x_n) = [x_1, x_n] \setminus \{x_n\}$. The union $\alpha(x) = \{x\} \cup \bigcup_{i=1}^{\infty}[x_i, x_{i+1}]$ becomes a thread in the continuum folder $C(\{X_i, g_i^{i+1}\})$ from x_1 to x. The collection $\{\alpha(x) \mid x \in X\}$ of threads makes $C(\{X_i, g_i^{i+1}\})$ a threaded continuum folder. We note that for $n \geq 2$, $\eta(\alpha(x) \setminus [x_1, x_n)) = [\frac{n-1}{n}, 1]$. So, the lengths of the right end segments of the threads $\alpha(x)$ limit to zero as n approaches infinity.

Proposition 15. *Let $C(\{X_i, g_i^{i+1}\})$ be a mapping cylinder of an inverse sequence, where each A_i has the fpp, then $C(\{X_i, g_i^{i+1}\})$ has the fpp. So, in particular, if the factor spaces X_i are ARs, then $C(\{X_i, g_i^{i+1}\})$ has the fpp.*

Proof. This result follows from the first paragraph of the Remarks above, and from Theorems 1 and 3 in [35]. Also, it is straightforward to prove this theorem, since the projection mappings from $C(\{X_i, g_i^{i+1}\})$, as an internal inverse limit, to the A_i's are $\frac{1}{2^i}$-retractions. □

Definition 9. Let $X = \varprojlim\{X_i, g_i^{i+1}\}$, and suppose X admits a mapping $f\colon X \to Z$. We let $\hat{X}$ denote the copy of X in $C(\{X_i, g_i^{i+1}\})$, and we let $\hat{f}\colon \hat{X} \to Z$ denote the mapping that is conjugate to $f\colon X \to Z$. We modify the mapping cylinder of $\{X_i, g_i^{i+1}\}$ by identifying the point inverses $\hat{f}^{-1}(z)$, for $z \in Z$, to points, and taking the quotient topology on $C(\{X_i, g_i^{i+1}\})/_{\hat{f}}$. This has the effect of giving a compactification of $\cup_{i\geq 1} C(g_i^{i+1})$ that has remainder Z rather than X. We will call such a space a *modified mapping cylinder of* $\{X_i, g_i^{i+1}\}$ *(with right-end fiber Z)*, denoted by $C(\{X_i, g_i^{i+1}\})/_{\hat{f}}$. We let $\gamma\colon C(\{X_i, g_i^{i+1}\}) \to C(\{X_i, g_i^{i+1}\})/_{\hat{f}}$ be the quotient mapping.

By defining $\overline{\eta}\colon C(\{X_i, g_i^{i+1}\})/_{\hat{f}} \to [0,1]$ to be the natural mapping making the diagram below commute, we see that $C(\{X_i, g_i^{i+1}\})/_{\hat{f}}$ is also a continuum folder.

$$\begin{array}{ccc} C(\{X_i, g_i^{i+1}\}) & \xrightarrow{\gamma} & C(\{X_i, g_i^{i+1}\})/_{\hat{f}} \\ & \eta \searrow \quad \swarrow \overline{\eta} & \\ & [0,1] & \end{array}$$

If Z and all X_i are in the same class $\mathcal{G}$ of continua, then $C(\{X_i, g_i^{i+1}\})/_{\hat{f}}$ is a $\mathcal{G}$ folder. The collection $\{\gamma(\alpha(\hat{x})) \mid \hat{x} \in \hat{X}\}$ makes $C(\{X_i, g_i^{i+1}\})/_{\hat{f}}$ a threaded continuum folder. We note that, for $\hat{x} \in \hat{X}$, the sequence of arcs $\{\gamma(\alpha(\hat{x})) \setminus [x_1, x_n)\}_{n \geq 2}$ forms a null sequence since $\overline{\eta}(\gamma(\alpha(\hat{x})) \setminus [x_1, x_n)) = [\frac{n-1}{n}, 1]$ for $n \geq 2$. This follows from the diagram above and the last few sentences at the end of the Remarks after Definition 8.

Proposition 16. *Each modified mapping cylinder of an inverse sequence on ARs with right-end fiber an AR is a locally connected, threaded, continuous AR folder. In particular, each modified mapping cylinder of an inverse sequence on* $[0, 1]$ *with right-end fiber an arc is a locally connected, threaded, continuous arc folder.*

Proof. Let $Y = C(\{X_i, g_i^{i+1}\})/_{\hat{f}}$ be a modified mapping cylinder of an inverse sequence of ARs. We noted above that Y is threaded. Proposition 9 gives us that Y is a continuous AR folder. To see that Y is locally connected, let $y \in Y$. If $y \in A_n$ for some $n \geq 2$, then Y is locally connected at y since A_i is an AR for all $i \geq 2$. Suppose $y = \gamma(\hat{x}) \in Z$ for some $\hat{x} \in \hat{X}$. We note that if $\{y_n\}$ is a sequence of points in Y converging to y, there exists a null sequence of arcs $\{\beta_n\}$ such that for each $n \geq 1$, both y_n and y are in β_n. This follows from the fact that Z is an AR, and from the remarks previous to this proposition about the right end arcs in threads of Y. It follows that Y is connected im kleinen at $y = \gamma(\hat{x})$. Hence, Y is locally connected since Y is connected im kleinen at every point (see pages 47-49 in [29]). □

In the remainder of this section, the reader may find it helpful to have reference [36] at hand, paying particular attention to Theorem 2.1 and its proof, and to Corollary 3.1.

We now consider the special case of mapping cylinders on inverse sequences where the inverse limit $X = \varprojlim\{X_i, g_i^{i+1}\}$ is retractably AR-like. This case arises when X can be expressed as an inverse limit on ARs with bonding maps that are r-maps (see page 1211 in [36]), and it includes all Knaster continua (inverse limits on $[0,1]$ with open bonding maps) as examples.

Let $X = \varprojlim\{X_i, g_i^{i+1}\}$ with surjective r-maps for bonding maps. By Theorem 2.1 and its proof in [36], there is a nested increasing sequence $\{\hat{X}_i\}$ in X, where for each $i \geq 1$, $\hat{X}_i \overset{T}{\approx} X_i$, and $h_i \circ g_i \colon X \to \hat{X}_i$ is a $\frac{1}{2^i}$-retraction, where $h_i \colon X_i \to \hat{X}_i$ is the embedding defined in Theorem 2.1 and g_i is the projection mapping of X onto X_i. Furthermore, for each $i \geq 1$, we have that $g_i \circ h_i$ is the identity mapping on X_i. For convenience, we let $\hat{g}_i = h_i \circ g_i$. It follows from this construction, or from Corollary 3.1 in [36], that X is retractable AR-like.

Whenever X is an inverse limit on ARs, it follows from Definition 8 and Corollary 3.1 in [36] that $C(\{X_i, g_i^{i+1}\})$ is retractably AR-like, specifically, onto the nested sequence $\{A_i\}$. For $k \geq 1$, recall that g_k denotes the projection mapping of X onto X_k. We show that, if additionally the bonding maps g_i^{i+1} are r-maps, then we can construct a different internal structure on $C(\{X_i, g_i^{i+1}\})$ so that, for all $k \geq 1$, the modified inverse sequence mapping cylinders $C(\{X_i, g_i^{i+1}\})/_{\hat{g}_k}$ will also be retractably AR-like. It follows that all of these types of modified inverse sequence mapping cylinders will have the fpp. We verify these claims in Propositions 17 and 18, and Corollary 3 below.

Proposition 17. *Let $X = \varprojlim\{X_i, g_i^{i+1}\}$, where for each $i \geq 1$, X_i is an AR, and g_i^{i+1} is an r-mapping. Then $C(\{X_i, g_i^{i+1}\})$ is retractably AR-like onto a nested sequence of ARs different from the sequence $\{A_i\}$.*

Proof. By definition of an r-mapping, for each $i \geq 1$, there exists an embedding $j_i \colon X_i \to X_{i+1}$ such that $g_i^{i+1} \circ j_i$ is the identity mapping on X_i. We will show that $C(\{X_i, g_i^{i+1}\})$ is retractably AR-like onto a sequence determined by the internal inverse limit structure of X. From the discussion preceding this proposition, we assume that $\hat{X}$ is a copy of X in $C(\{X_i, g_i^{i+1}\})$, each $\hat{X}_i = h_i(X_i)$ is a copy of X_i in $\hat{X}$, and we denote points $h_i(z)$ in $\hat{X}_i$ by $\hat{z}$.

Let $C(\{X_i, g_i^{i+1}\})$ and the sequence $\{A_n\}$ be given as in Definition 8. For each $n \geq 2$, let $C_n = \cup\{\alpha(\hat{x}) \mid \hat{x} \in \hat{X}_n\}$. We note that for each $n \geq 2$, $\mathrm{cl}(C_n \setminus A_n) \overset{T}{\approx} X_n \times [\frac{n-1}{n}, 1]$. So, each C_n is an AR since both A_n and $\mathrm{cl}(C_n \setminus A_n)$ are ARs and their intersection X_n is an AR. Also, $\{C_n\}$ is a nested increasing sequence lying in $C(\{X_i, g_i^{i+1}\})$.

For each $n \geq 1$, we define the retraction $\rho_n \colon C(\{X_i, g_i^{i+1}\}) \to C_n$ as follows. For each point $\hat{x} \in \hat{X}$, and each point $z \in \alpha(\hat{x})$, we let $\rho_n(z) = (\eta|_{\alpha(\hat{g}_n(\hat{x}))})^{-1} \circ \eta(z)$. We see that ρ_n is well-defined since η is one-to-one on threads. Since, for each $n \geq 2$, $\hat{g}_n$ is a $\frac{1}{2^n}$-retraction, it follows that ρ_n is a $\frac{1}{2^n}$-retraction. Hence, $C(\{X_i, g_i^{i+1}\})$ is retractably AR-like onto the nested sequence $\{C_i\}$. □

Our main purpose, for the internal constructions on $C(\{X_i, g_i^{i+1}\})$ in the proof of Proposition 17 above, is to see that we can maintain the retractably AR-likeness for any modified $C(\{X_i, g_i^{i+1}\})/_{\hat{f}}$, where f is one of the projection maps $g_k\colon X \to X_k$. We first introduce notation that will be helpful for transitioning between $C(\{X_i, g_i^{i+1}\})$ and $C(\{X_i, g_i^{i+1}\})/_{\hat{g}_k}$.

Fix $k \geq 1$. Let $Y = C(\{X_i, g_i^{i+1}\})$, $Y_k = C(\{X_i, g_i^{i+1}\})/_{\hat{g}_k}$, and let $\gamma\colon Y \to Y_k$ be the quotient mapping. Since $Y \setminus \hat{X}$ and $Y_k \setminus \overline{\eta}^{-1}(1)$ are homeomorphic open subsets of, respectively, Y and Y_k, we will not distinguish notationally between points of these two sets. So, for sets $D \subset Y \setminus \hat{X}$, D will also denote its copy in $Y_k \setminus \overline{\eta}^{-1}(1)$. If a set $D \subset Y$ meets $\hat{X}$, then we let $\overline{D} = \gamma(D)$. Thus, $\overline{\eta}^{-1}(1)$ is denoted by $\overline{X}_k$, distinguishing it in Y_k from the copy of X_k that is the left side of $C(g_k^{k+1})$.

Note that for $\hat{x} \in \hat{X}$, the thread $\alpha(\hat{x})$ in Y becomes $\overline{\alpha(\hat{x})}$ in Y_k, which is an arc in Y_k with endpoints $g_1(x) \in X_1$ and $\overline{g_k(x)} \in \overline{X}_k$. Furthermore, we have that $\overline{\alpha(\hat{x})} = \mathrm{cl}(\cup_{n\geq 1}[g_n(x), g_{n+1}(x)])$ in Y_k. We observed earlier that $\overline{\alpha(\hat{x})}$ is a *thread* in Y_k.

Proposition 18. *Let $X = \varprojlim\{X_i, g_i^{i+1}\}$, where for each $i \geq 1$, X_i is an AR, and g_i^{i+1} is an r-mapping. Then, for each $k \geq 1$, $C(\{X_i, g_i^{i+1}\})/_{\hat{g}_k}$ is retractably AR-like.*

Proof. We adopt the notation and terminology in the paragraphs immediately above. In Y_k, for $n \geq k$, note that $\mathrm{cl}(\overline{C}_n \setminus A_n) \overset{T}{\approx} C(g_k|_{X_n})$. So, for $n \geq k$, $\overline{C}_n$ is an AR in Y_k. For each point $\hat{x} \in \hat{X}$, and each point $z \in \alpha(\hat{x}) \setminus \{\hat{x}\}$, we let $\overline{\rho}_n(z) = \rho_n(z)$, and let $\overline{\rho}_n(\gamma(\hat{x})) = \hat{g}_k(\hat{x})$. Since $g_k = g_k^n \circ g_n$, we have that, for $\hat{x} \in \hat{X}$, $\hat{g}_k(\hat{g}_n(\hat{x})) = \hat{g}_k(\hat{x})$, establishing the continuity of $\overline{\rho}_n$. It is clear that each $\overline{\rho}_n$ for $n \geq k$ is a $\frac{1}{2^n}$-retraction. Hence, Y_k is retractably AR-like onto the nested sequence $\{\overline{C}_n\}$. □

If X is a compactum and there exists an embedding of X into a compactum Y, we will denote the embedded copy of X by X'. A compactum X is an *approximate absolute retract (AAR)* if whenever X is embedded in a compactum Y, for each $\epsilon > 0$, there is a mapping $f\colon Y \to X'$ such that $d(x, f(x)) < \epsilon$ for all $x \in X'$. A compactum X is an *approximate absolute neighborhood retract (AANR)* if whenever X is embedded in a compactum Y, for each $\epsilon > 0$, there exists a neighborhood U of X' and a mapping $f\colon U \to X'$ such that $d(x, f(x)) < \epsilon$ for all $x \in X'$. AARs and AANRs have been studied extensively, see for example [5], [7], [9], [15], [16], [17], and [44].

Corollary 3. *Let* $X = \varprojlim\{X_i, g_i^{i+1}\}$*, where for each* $i \geq 1$*,* X_i *is an AR, and* g_i^{i+1} *is an* r*-map. Then, for each* $k \geq 1$*,* $C(\{X_i, g_i^{i+1}\})/_{\hat{g}_k}$ *is an approximate absolute retract and has the fpp.*

Proof. That $C(\{X_i, g_i^{i+1}\})/_{\hat{g}_k}$ is an approximate absolute retract follows from Proposition 18 and Theorem 2.3 in [10]. That $C(\{X_i, g_i^{i+1}\})/_{\hat{g}_k}$ has the fpp follows from Theorems 1 and 3 in [35]. □

Corollary 4. *Let* $X = \varprojlim\{[0,1], g_i^{i+1}\}$ *be a Knaster continuum. That is, for each* $i \geq 1$*,* g_i^{i+1} *is an open mapping. Then, for each* $k \geq 1$*,* $C(\{[0,1], g_i^{i+1}\})/_{\hat{g}_k}$ *has the fpp.*

Proof. This follows from the fact that the open bonding maps g_i^{i+1} are r-maps. □

The following questions are special cases of Question 4.

Question 5. *Do all modified mapping cylinders of inverse sequences on* $[0,1]$ *with right-end fiber an arc have the fpp?*

The authors have found the following more specific open question particularly interesting. A proof in this special case may provide insight for a general approach to answering Gordh's Question 1.

Question 6. *If the pseudo arc* P *is expressed as an inverse limit on* $[0,1]$*, does each of its modified mapping cylinders with right-end fiber an arc have the fpp?*

4. Arc folders

The first two propositions of this section list, respectively, properties of arc folders and arc mapping cylinders that follow immediately from propositions established in Sections 2 and 3. For the reader's convenience, we list the appropriate propositions in parenthesis at the beginning of each statement.

Proposition 19. *Let* X *be an arc folder.*

1. (*Prop.*1) X *is acyclic and unicoherent.*
2. (*Prop.*3) X *has either dimension one or dimension two.*
3. (*Prop.*2) X *has trivial shape.*
4. (*Prop.*5) *If* X *is irreducible, then* $\dim(X) = 1$*.*
5. (*Prop.*6) X *is atriodic if and only if* X *is chainable.*
6. (*Prop.*7) *If* $\dim(X) = 1$*, then* X *is a* λ*-dendroid.*
7. (*Prop.*8) *If* X *is continuous and* $\dim(X) = 1$*, then* X *is an arc.*

Proposition 20. *Let* X *be an arc mapping cylinder. Then* X *is an arc folder that has the following additional properties.*

(1) (*Prop.*12) *X has trivial fundamental group.*
(2) (*Prop.*13) *X is 2-dimensional and embeddable in* $\mathbb{R}^3$.

Proposition 21. *For an arc folder X, the following conditions are equivalent.*

(a) *X is one-dimensional.*
(b) *X is a λ-dendroid.*
(c) *X is tree-like.*
(d) *X is hereditarily unicoherent.*
(e) *X is hereditarily decomposable.*

Proof. That (a), (b), and (c) are equivalent follows from Proposition 19(6), and well-known facts about tree-like continua and λ-dendroids.

S. Mazurkiewicz [38] has shown that 2-dimensional continua contain nondegenerate indecomposable subcontinua (also see [27, Remark 2]). So, if X is hereditarily decomposable, then $\dim(X) = 1$. That is, (e) implies (a). Since (a) is equivalent to (b), we have that (a) implies (d) and (e). So, (a) and (e) are equivalent. If X is hereditarily unicoherent, then by Proposition 4, X is hereditarily decomposable. So, (d) implies (e), and the proof is complete. □

Example 3. *There exist arc folders that are neither planar nor disk-like.*

Proof. Consider the open map $f\colon [0,1] \to [0,1]$ that linearly maps $[0,\frac{1}{3}]$ to $[0,1]$, $[\frac{1}{3},\frac{2}{3}]$ to $[1,0]$, and $[\frac{2}{3},1]$ to $[0,1]$. Note that $C(f)$ contains a "thumbtack". That is, the wedge of a disk and an arc, where an endpoint of the arc is an interior point of the disk. It follows that $C(f)$ is neither disk-like nor embeddable in the plane. By Proposition 11, $C(f)$ is an arc folder. □

Example 3 is 2-dimensional. In the example below, we define a tree-like arc folder in $\mathbb{R}^3$ that cannot be embedded in the plane.

Example 4. *A non-planar tree-like arc folder.*

Proof. For each pair of points p and q in $\mathbb{R}^3$, let $[p,q]$ denote the straight-line interval from p to q. Let C be the Cantor ternary set in $[0,1]$.

Let $H = [(-1/2,0,1),(-1/2,0,0)]$, $I = [(-1,0,0),(0,0,0)]$, and $J = [(0,0,0),(1,0,0)]$.

For each $x \in C \setminus \{0\}$, let

$$K_x = [(x,0,0),(-1,x,0)] \ \text{ and } \ L_x = [(x,0,0),(-1,-x,0)].$$

Let $M = H \cup I \cup J \cup \bigcup\{K_x \cup L_x |\ x \in C \setminus \{0\}\}$.

Since M is arcwise connected and hereditarily unicoherent, M is a tree-like continuum [11]. In fact, M is a dendroid. To see that M is an arc-folder, define a map $\eta : M \to [0,1]$ such that

(1) η sends $H \setminus \{(-1/2, 0, 0)\}$ homeomorphically onto $[0, 1/2)$,
(2) $\eta(I) = 1/2$,
(3) if $(x, 0, 0) \in J$ and $x \notin C$, then $\eta((x, 0, 0)) = 1/2 + x/2$, and
(4) if $(x, 0, 0) \in J$ and $x \in C \setminus \{0\}$, then $\eta(K_x \cup L_x) = 1/2 + x/2$.

To see that M cannot be embedded in $\mathbb{R}^2$, assume the contrary. Let h be a homeomorphism of M into $\mathbb{R}^2$. Let G be an open disk in $\mathbb{R}^2$ containing $h(0, -1/2, 0)$ whose closure misses $h(J) \cup \{h(-1/2, 0, 1), h(-1, 0, 0)\}$.

Let A be the closure of the $h(-1/2, 0, 0)$-component of $G \cap h(H)$. Note that K_x and L_x limit on I as x in C approaches 0. Hence there is a positive number r such that for each $s \in C \cap (0, r]$

(5) both $h(K_s)$ and $h(L_s)$ intersect G, and

(6) neither $h(K_s)$ nor $h(L_s)$ abut $h(I \cup J)$ from the same side as A (see page 180 of [40]).

Let s be a point of $C \cap (0, r)$ that is inaccessible from $J \setminus C$ such that both $h(K_s)$ and $h(L_s)$ intersect G.

Let U be the arc in $h(I \cup J)$ that is irreducible between $h(s)$ and $h(\mathrm{cl}(G))$.

Let V be the arc in $h(K_s)$ that is irreducible between $h(s)$ and $h(\mathrm{cl}(G))$.

Let W be the arc in $h(L_s)$ that is irreducible between $h(s)$ and $h(\mathrm{cl}(G))$.

Let u, v, and w be the endpoints of U, V, and W, respectively, that are opposite $h(s)$.

Let B be the arc in the boundary of G that misses A and is irreducible about $\{u, v, w\}$.

Assume without loss of generality that $\{h(s), v\}$ separates u from w in the simple closed curve $\Delta = U \cup B \cup W$. Let Ω denote the bounded complementary domain of Δ. Let t be a point of $C \cap (0, s)$ such that the arc X in $h(L_t) \cap \mathrm{cl}(\Omega)$ irreducible between $h(t)$ and B has an endpoint in B between v and w. Since X and V are disjoint, this contradicts Theorem 28 on page 156 of [40]. Hence M is not embeddable in the plane. □

Question 7. *Can every arc folder be embedded in $\mathbb{R}^3$?*

We finish this section with results and questions that relate to the fpp for arc folders. Every tree-like arc folder has the fpp (by Proposition 21 and Mańka [32]). We show in Proposition 22 below that every planar arc folder has the fpp, but first we prove a reduction theorem for fixed-point-free maps on arc folders.

A mapping $f\colon X \to Y$ is *universal* if for each mapping $g\colon X \to Y$, there exists a point $x \in X$ such that $f(x) = g(x)$.

Theorem 1. *Suppose (X, η) is an arc folder that admits a fixed-point-free mapping f. Then there exists a subfolder $X_1 = \eta^{-1}([a, b])$ of X and a fixed-point-free mapping $f_1\colon X_1 \to X_1$, where*
(1) *X_1 is arcwise connected.*

Moreover, if X is continuous, the subfolder X_1 and the fixed-point-free mapping f_1 can be chosen so that we have the following additional property.
(2) *For each $t \in [a,b]$, t is in the interior of $\eta(f_1(\eta^{-1}(t)))$ relative to $[a,b]$.*

Proof. (1) Let $f\colon X \to X$ be a fixed-point-free mapping. Since η is universal, there exists a point z in X where $\eta(z) = \eta f(z)$. Let A be the arc component of X that contains $B = \eta^{-1}(\eta(z))$. We note that B is nondegenerate since $\{z, f(z)\} \subset B$. Also, $B \cup f(B) \subset A$, $f(A) \subset A$, and $A \neq B$. Furthermore, if $A \cap \eta^{-1}(t) \neq \emptyset$ for some $0 \leq t \leq 1$, then $\eta^{-1}(t) \subset A$. That is, $A = \eta^{-1}\eta(A)$.

Since A is connected, $\eta(A)$ has one of the following forms $[r,s)$, $[r,s]$, (r,s), or $(r,s]$, for some $0 \leq r < s \leq 1$. We consider three cases.

Case 1. Suppose $\eta(A) = [r,s]$. Then $A = \eta^{-1}([r,s])$, and since $f(A) \subset A$, A is the desired subfolder and $f|_A$ is the desired fixed-point-free mapping.

Case 2. Suppose $\eta(A) = [r,s)$. Let $K = \mathrm{cl}(A) \cap \eta^{-1}(s)$. By Lemma 1, K is connected.

Since $f(A) \subset A$, it follows from the continuity of f that $f(K) \subset \mathrm{cl}(A)$. So, if some point x in K maps into $\eta^{-1}(s)$, we have that $f(K) \subset K$. But then, since K is an arc, f has a fixed point in K, a contradiction. So, $f(K) \cap \eta^{-1}(s) = \emptyset$. It follows in this case that $f(K) \subset A$. In fact, $f(\eta^{-1}(s)) \subset A$. Hence, by continuity and compactness, there exists some b with $r < b < s$ such that $f(\eta^{-1}(b)) \subset \eta^{-1}([r,b))$. Let $\ell\colon \eta^{-1}([r,s]) \to \eta^{-1}([r,b])$ be a retraction, where $\ell(\eta^{-1}([b,s])) = \eta^{-1}(b)$, and consider the map $f_1 = \ell \circ f|_{\eta^{-1}([r,b])}$. We note that $f_1(\eta^{-1}([r,b])) \subset \eta^{-1}([r,b])$, and f_1 is fixed point free by choice of b. So, we have the desired closed subfolder of X.

Case 3. Suppose $\eta(A) = (r,s)$. Let $J = \mathrm{cl}(A) \cap \eta^{-1}(r)$, and, as in Case 2, $K = \mathrm{cl}(A) \cap \eta^{-1}(s)$. As in Case 2, we can deduce that $f(J) \cup f(K) \subset \mathrm{cl}(A) \subset \eta^{-1}([r,s])$and $f(J) \cap \eta^{-1}(r) = \emptyset = f(K) \cap \eta^{-1}(s)$. It follows that we can pick a and b, with $r < a < b < s$, so that $f(\eta^{-1}(a)) \subset \eta^{-1}((a,s])$ and $f(\eta^{-1}(b)) \subset \eta^{-1}([r,b))$. Let $g\colon \eta^{-1}([r,s]) \to \eta^{-1}([a,b])$ be a retraction, where $g(\eta^{-1}([r,a])) = \eta^{-1}(a)$ and $g(\eta^{-1}([b,s])) = \eta^{-1}(b)$. Define $f_2 = g \circ f|_{\eta^{-1}([a,b])}$. As in Case 2, we have the desired result for the closed subfolder $\eta^{-1}([a,b])$ and the map f_2.

(2) We now assume that (X,η) is a continuous arc folder, and $f\colon X \to X$ is a fixed-point-free mapping. By (1), we may also assume that X is arcwise connected. By the continuity of (X,η) there exists the largest number $a \in [0,1]$ such that $\eta(f(\eta^{-1}(a)))) \subset [a,1]$. Similarly, there exists the smallest number $b \in [a,1]$ such that $\eta(f(\eta^{-1}(b))) \subset [0,b]$. Let $X_1 = \eta^{-1}([a,b])$, and $r : X \to X_1$ be a retraction such that $r(\eta^{-1}([0,a)) = \eta^{-1}(a)$ and

$r(\eta^{-1}((b,1]) = \eta^{-1}(b)$. The map $f_1 : X_1 \to X_1$ defined by $f_1(x) = r(f(x))$ is fixed-point-free. By the definition of a and b, t is an interior point of $\eta(f_1(\eta^{-1}(t)))$ relative to $[a,b]$ for each $t \in [a,b]$. Since X is arcwise connected and each fiber of X separates X, it is easy to see that the subfolder X_1 is arcwise connected.

□

Proposition 22. *If X is a planar arc folder, then X has the fpp.*

Proof. By Theorem 1, we may assume that X is an arcwise connected, planar arc folder. Suppose there exists a simple closed curve S in X that does not bound a disk in X. Then the bounded complementary domain of S contains a point that is not in X. Via the Schoenflies theorem, we can retract X to S. It follows that X admits an essential map to $S \overset{T}{\approx} S^1$, contradicting, via Bruschlinsky's theorem [13, Theorem 8.1], that X is acyclic. Hence, each simple closed curve S in X bounds a disk, and therefore, the fundamental group of X is trivial. We have that X is simply connected. Hagopian [18, Theorem 9.1] has shown that simply connected planar continua have the fpp. □

In [8], K. Borsuk introduced the notions of nearly extendable maps and NE-sets, which are related to the notions of AARs and AANRs. He showed that for a continuum X with trivial shape, X is an NE-set if and only if X is an AANR. He also showed that NE-sets with trivial shape have the fpp. So, we have the following fixed point result.

Proposition 23. *If an arc folder X is an AANR, then X has the fpp.*

Question 8. *Is each modified mapping cylinder of an inverse sequence on $[0,1]$ with right-end fiber an arc an AANR?*

If Question 8 has an affirmative answer, then so does Question 5.

We end the paper with five questions that are typical in continuum fixed point theory. Variants of these questions have received considerable interest for over sixty years. For specific results and related questions, see the survey articles [2], [20], and [33].

Question 9. *Does every disk-like arc folder have the fpp?*

Question 10. *Suppose X is an arc folder where each of $\eta^{-1}([0,\frac{1}{2}])$ and $\eta^{-1}([\frac{1}{2},1])$ has the fpp. Does X have the fpp?*

Question 11. *Suppose X is an arc folder where $\eta^{-1}([0,\frac{1}{2}])$ is topologically a disk and $\eta^{-1}([\frac{1}{2},1])$ has the fpp. Does X have the fpp?*

Question 12. *Suppose X is an arc folder where each proper subfolder $\eta^{-1}([a,b])$ has the fpp. Does X have the fpp?*

Question 13. *Do all arc folders have the fpp for homeomorphisms?*

Acknowledgement. The authors wish to thank Michael Heacock for providing Figure 1.

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