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TREE-LIKE CONTINUA WITH INVARIANT COMPOSANTS UNDER FIXED-POINT-FREE HOMEOMORPHISMS

C. L. HAGOPIAN, M. M. MARSH, AND J. R. PRAJS

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Abstract. Using an example of D. P. Bellamy, we define a 3-composant tree-like continuum admitting a fixed-point-free homeomorphism that sends each composant onto itself. This continuum is used to define an indecomposable tree-like continuum that admits a composant-preserving fixed-point-free homeomorphism. A map extension theorem is proved and applied in this construction.

Suppose X is a plane continuum and f is a map of X that sends each arc-component of X into itself. In 1988, Hagopian [3] proved that f has a fixed point if X is tree-like or indecomposable. Solenoids show this is not true for nonplanar indecomposable continua. However, in 1998 Hagopian [4] proved every tree-like continuum has the fixed-point property for arc-component-preserving maps. Recently, Hagopian [5] proved that every composant-preserving map of an indecomposable k-junctioned tree-like continuum has a fixed point. He used tree-chain covers to get his results. Question 1 of [5] asks if this theorem can be generalized to every indecomposable tree-like continuum. We use an example of Bellamy [1] to answer this question in the negative. In fact, we define an indecomposable tree-like continuum that admits a composant-preserving fixed-point-free homeomorphism. It is not known if there exists a tree-like plane continuum or an indecomposable plane continuum that admits a composant-preserving fixed-point-free map.

A *continuum* is a nonempty compact connected metric space. A continuum is *indecomposable* if it is not the union of two proper subcontinua. Let x be a point of a nondegenerate continuum X. The *x-composant* of X is the union of all proper subcontinua of X that contain x. If X is indecomposable, then X is the union of uncountably many dense disjoint composants. Given $\epsilon > 0$, a mapping $f\colon X \to Y$ is an *ϵ-mapping* if $\operatorname{diam}(f^{-1}(y)) < \epsilon$ for each $y \in Y$. A continuum X is *tree-like* if for each $\epsilon > 0$, there exist a tree Y and an ϵ-mapping of X onto Y. A continuum X is tree-like if and only if for each $\epsilon > 0$ there is an ϵ-tree-chain covering X. A tree-like continuum X is *k-junctioned* if k is the least integer such that for every positive number ϵ there is an ϵ-tree-chain covering X with k junction links.

We refer to a locally compact noncompact metric space P as a *parameter space* and denote by $P \cup \{\infty\}$ the one-point compactification of P. A map $\alpha : P \to P$ of a parameter space to itself is called *infinity preserving* provided that whenever

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a sequence $\{p_n\} \subset P$ converges to ∞ in $P \cup \{\infty\}$ (denoted by $p_n \to \infty$), it follows that $\alpha(p_n) \to \infty$ in $P \cup \{\infty\}$.

Our first result is a general map extension theorem. We need a special case of this theorem, namely Corollary 1, to construct our example. It may be helpful for the reader to note the statement of Corollary 1 before proceeding with Theorem 1 and its proof.

Theorem 1. *Let P be a parameter space and X a compact metric space with metric d. Suppose that $g : P \to X$, $f : X \to X$, and $\alpha : P \to P$ are maps such that α is infinity preserving and $\lim d(fg(p_n), g\alpha(p_n)) = 0$ whenever $p_n \to \infty$. Then there exist a compact metric space $\hat{X}$, with metric μ, containing X, an embedding $h : P \to \hat{X}$ with $h(P) = \hat{X} - X$, and a map $\hat{f} : \hat{X} \to \hat{X}$ such that*

(a) $\lim \mu(h(p_n), g(p_n)) = 0$ *whenever* $p_n \to \infty$,
(b) $\hat{f}|X = f$ *and* $\hat{f}(\hat{X} - X) \subset \hat{X} - X$, *and*
(c) $\hat{f}h(p) = h\alpha(p)$ *for each* $p \in P$.

Moreover, if f and α are homeomorphisms, then $\hat{f}$ is also. If P and X are connected, then so is $\hat{X}$.

Proof. Let ρ be a metric on $P \cup \{\infty\}$ with $\operatorname{diam}(P \cup \{\infty\}) < 1$. In the product $X \times (P \cup \{\infty\}) \times [0, 1]$, let μ denote the product metric and define two disjoint sets, topological copies $X_1 = X \times \{\infty\} \times \{0\}$ and $P_1 = \{(g(p), p, \rho(p, \infty)) : p \in P\}$ of X and P, respectively. We show that $\hat{X} = X_1 \cup P_1$ is the desired space, where, in the final conclusion, we identify X_1 with its projection X. For the sake of the proof we use distinct symbols, X_1 and X. Let $f_1 : X_1 \to X_1$ and $g_1 : P \to X_1$ be the maps corresponding to $f : X \to X$ and $g : P \to X$, respectively. That is, $f_1(x, \infty, 0) = (f(x), \infty, 0)$ for $x \in X$, and $g_1(p) = (g(p), \infty, 0)$ for $p \in P$. The map $h(p) = (g(p), p, \rho(p, \infty))$ defines an embedding $h : P \to \hat{X}$ with $h(P) = P_1$. If $p_n \to \infty$ for $\{p_n\} \subset P$, we have $h(p_n) = (g(p_n), p_n, \rho(p_n, \infty))$ and $g_1(p_n) = (g(p_n), \infty, 0))$, and thus (a) clearly holds.

If a sequence p_n in P has no convergent subsequence, that is, $p_n \to \infty$, then $g_1(p_n)$ has a subsequence converging in X_1, and thus $h(p_n)$ has a convergent subsequence in $\hat{X}$ by (a). This shows that $\hat{X}$ is compact. Since P_1 has a limit point in X_1, in the case when X and P are connected, it follows that $\hat{X}$ is connected.

Define $\hat{f} : \hat{X} \to \hat{X}$ by $\hat{f}(x) = f_1(x)$ for $x \in X_1$, and $\hat{f}(x) = h\alpha h^{-1}(x)$ for $x \in P_1$. By definition $\hat{f}|X_1 = f_1$, $\hat{f}(\hat{X} - X_1) = \hat{f}(P_1) \subset P_1 = \hat{X} - X_1$, and $\hat{f}h(p) = h\alpha(p)$ for $p \in P$.

To verify the continuity of $\hat{f}$ we note that X_1 is closed in $\hat{X}$ and show that $\lim \hat{f}(x_n) = \hat{f}(x_0)$ for every sequence $\{x_n\} \subset P_1$ converging to some $x_0 \in X_1$. For such a sequence x_n, let $p_n = h^{-1}(x_n)$ and note that $x_n = (g(p_n), p_n, \rho(p_n, \infty))$ and $x_0 = (y_0, \infty, 0)$, where $y_0 = \lim g(p_n)$. We have $p_n \to \infty$. The map α is infinity preserving, and thus $\alpha(p_n) \to \infty$. By the continuity of f, $\lim fg(p_n) = f(y_0)$. Since $\lim d(fg(p_n), g\alpha(p_n)) = 0$, it follows that

$$\lim \hat{f}(x_n) = \lim h\alpha h^{-1}(x_n) = \lim h\alpha(p_n) = \lim(g\alpha(p_n), \alpha(p_n), \rho(\alpha(p_n), \infty))$$

$$= \lim(fg(p_n), \infty, 0) = (f(y_0), \infty, 0) = \hat{f}(x_0)\ .$$

We note in the definition of $\hat{f}$ that if f and α are homeomorphisms, then $\hat{f}$ is also. □

A set S in a topological space X is said to be *uniquely arcwise connected* provided that for every two distinct points $p, q \in S$ there is a unique arc in S having p and q as its endpoints. Let X be a continuum with metric d. A *topological ray* (or a *ray*) in X is a uniquely arcwise connected subset R of X for which there exists a one-to-one surjective mapping $\alpha : [0, \infty) \to R$. A *topological line* (or a *line*) in X is a uniquely arcwise connected subset L of X for which there exists a one-to-one surjective mapping $\beta : \mathbb{R} \to L$. The maps α and β in these definitions are called *parametrizations* of R and L, respectively. The assumption of the unique arcwise connectivity on a ray R (or a line L) ensures a unique arc order structure on R (or L). Thus, for example, we avoid calling simple closed curves (and some other continua) rays. This order agrees with the order structure of $[0, \infty)$ as the domain of any parametrization of R.

Rays R_1 and R_2 in a space X are called *asymptotic* if there exist parametrizations $\alpha_1 : [0, \infty) \to R_1$ and $\alpha_2 : [0, \infty) \to R_2$ such that $d(\alpha_1(t), \alpha_2(t)) \to 0$ as $t \to \infty$.

We now state a special case of Theorem 1, which reflects the form of this theorem that will be used in our construction.

Corollary 1. *Let R_1 be a ray in a continuum X and let $f : X \to X$ be a homeomorphism such that $f(R_1)$ is a ray asymptotic to R_1. Then there exist a continuum $\hat{X} = X \cup R$, where R is a ray in $\hat{X}$ asymptotic to R_1 and satisfying $R \cap X = \emptyset$, and a homeomorphism $\hat{f} : \hat{X} \to \hat{X}$ such that $\hat{f}|X = f$ and $\hat{f}(R) = R$.*

Proof. Let $\alpha_1 : [0, \infty) \to R_1$ and $\alpha_2 : [0, \infty) \to f(R_1)$ be parametrizations such that $d(\alpha_1(t), \alpha_2(t)) \to 0$ as $t \to \infty$. We apply Theorem 1 to the parameter space $P = [0, \infty)$. Let $g = \alpha_1 : P \to X$. Note that $\alpha = \alpha_2^{-1} f \alpha_1 : P \to P$ is a homeomorphism and α is infinity preserving. We have that $d(fg(t), g\alpha(t)) = d(f\alpha_1(t), \alpha_1\alpha_2^{-1} f\alpha_1(t)) = d(\alpha_2\alpha_2^{-1} f\alpha_1(t), \alpha_1\alpha_2^{-1} f\alpha_1(t)) \to 0$ as $t \to \infty$. The existence of $\hat{X}$ and $\hat{f} : \hat{X} \to \hat{X}$ follows from Theorem 1. □

If X and Z are continua with $X \subset Z$, then X is *terminal* in Z provided that for each continuum $K \subset Z$ intersecting X either $K \subset X$ or $X \subset K$. Notice in Corollary 1 that if R_1 limits on X, then X is terminal in $\hat{X}$. This structure will be common throughout the paper.

Proposition 1. *Let $f : X \to X$ be a fixed-point-free homeomorphism on a continuum X and $Y = X \cup R$, where R is a ray with endpoint e limiting on X but disjoint with X; that is, $R \cap X = \emptyset$ and $\overline{R} \cap X = X$. Let $\hat{f} : Y \to Y$ be a homeomorphism such that $\hat{f}|X = f$. Then the wedge sum Z (at the point e) of two disjoint copies of Y admits a fixed-point-free homeomorphism h. Furthermore, Z has exactly three composants, each of which is invariant under h.*

Proof. We note that $\hat{f}(X) = X$ and $\hat{f}(R) = R$. Since R has unique endpoint e, we have $\hat{f}(e) = e$. Let $Y_1 = X_1 \cup R_1$ and $Y_2 = X_2 \cup R_2$ be disjoint copies of Y. Let $Z = Y_1 \vee Y_2$ be the wedge sum of Y_1 and Y_2 such that the endpoints of R_1 and R_2 are identified to the single point e_0. Let $f_i : X_i \to X_i$ and $\hat{f}_i : Y_i \to Y_i$ for $i \in \{1, 2\}$ be the corresponding homeomorphisms as given in the hypothesis. Define $g\colon Z \to Z$ so that $g(x) = \hat{f}_1^{-1}(x)$ for $x \in Y_1$ and $g(x) = \hat{f}_2(x)$ for $x \in Y_2$. Thus $g : Z \to Z$ is a homeomorphism, $g(e_0) = e_0$, $g(X_i) = X_i$, and $g(R_i) = R_i$ for $i \in \{1, 2\}$.

For notational convenience, since $R_1 \vee R_2$ is a topological line, we identify $R_1 \vee R_2$ with $\mathbb{R}$ in such a way that $e_0 = 0$, $R_1 = (-\infty, 0]$ and $R_2 = [0, \infty)$. Since $g(\mathbb{R}) = \mathbb{R}$, $g(X_2) = X_2$, and g is fixed-point-free on X_2, it follows that either $g(x) > x$ for all sufficiently large x or $g(x) < x$ for all sufficiently large x. Assume $g(x) > x$ for all $x \in [N, \infty)$, where N is some positive number. (The other case is similar.)

Thus $\hat{f}_2$, which coincides with g on R_2, moves sufficiently large numbers toward X_2. The maps $\hat{f}_1$, $\hat{f}_2$ are both congruent to $\hat{f} : Y \to Y$, and $\hat{f}_1^{-1}$ coincides with g on R_1. Thus g moves all numbers less than or equal to some $M \in (-\infty, 0]$ outward away from X_1. In other words, $g(x) > x$ for $x \in (-\infty, M]$. We have $x < g(x) < 0 < y < g(y)$ for all $x \in (-\infty, M]$ and $y \in [N, \infty)$. Define $h(x) = g(x)$ for $x \in X_1 \cup (-\infty, M] \cup [N, \infty) \cup X_2$ and extend h linearly on $[M, N]$. Then h is strictly increasing on $\mathbb{R}$, moving points away from X_1 and toward X_2. Note that Z has exactly three composants, $X_1 \cup (R_1 \vee R_2)$, $X_2 \cup (R_1 \vee R_2)$, and Z itself. Also, $h : Z \to Z$ is a fixed-point-free homeomorphism such that $h(X_1) = X_1$, $h(X_2) = X_2$, and $h(\mathbb{R}) = \mathbb{R}$. □

We now construct an example of a 3-composant tree-like continuum that admits a fixed-point-free homeomorphism that leaves composants invariant. To begin the construction, we need

1. an indecomposable tree-like continuum Y in which one composant S is topologically the union of a fan F and a topological ray R that meet only at the point v, which is the branchpoint of F and the endpoint of R, and
2. a fixed-point-free map g on Y such that
   a. $g(E) = E$, where E is the endpoint set of F, and
   b. if $c \in E$, then g restricted to $[c, v] \cup R$ is a homeomorphism onto $[g(c), v] \cup R$.

Examples satisfying these conditions have been described in [1, 8, 9]. We will use Bellamy's tree-like continuum [1].

Example 1. There is a tree-like continuum with exactly three composants that admits a fixed-point-free homeomorphism h leaving each composant invariant.

Proof. Let B be Bellamy's continuum and let X be Bellamy's second indecomposable tree-like continuum after applying the Fugate-Mohler technique [2]. That is, $X = \varprojlim\{B, g\}$, where g is the fixed-point-free map on B. We recall that each composant of X is either a ray or a line. Let S be the composant of B that contains a fan F with endpoint set E, and let v be the branchpoint of F. We have the properties listed above. Additionally, since g is fixed-point-free, $g(v)$ is in $R \setminus \{v\}$.

For $p, q \in B$, let $[p, q]$ be the unique arc in B from p to q, if it exists. Let α be a homeomorphism from $[1, 2]$ to $[v, g(v)]$ with $\alpha(1) = v$. For $1 \le t \le 2$ and $n \in \mathbb{N} \cup \{0\}$, extend α to $[1, \infty)$ letting $\alpha(t + n) = g^n(x)$ whenever $x \in [v, g(v)]$ and $\alpha(t) = x$. Note that $\alpha(t+n) = g^n(\alpha(t))$ for all $t \ge 1$ and $n \in \mathbb{N} \cup \{0\}$. If $c \in E \subset B$, let $R_c = [c, v] \cup R$. Take a homeomorphism f_c from $[0, 1]$ to $[c, v]$ with $f_c(0) = c$, and extend it to $[0, \infty)$ letting $f_c(t + 1) = \alpha(t + 1)$. Observe that $f_c : [0, \infty) \to R_c$ parametrizes R_c, $f_c(1) = v$, $f_c(t + 2) = g(f_c(t + 1))$, and $f_c(t + 1) = f_{c'}(t + 1)$ for all $t \in [0, \infty)$ and $c, c' \in E$.

Define $\sigma : X \to X$ to be the shift homeomorphism $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$, which is fixed-point-free [2]. Let $\{a_n\} \subset E$ be such that $g(a_{n+1}) = a_n$. Letting

$\mathbf{a} = (a_1, a_2, \dots)$, $b_n = a_{n+1}$ for $n \geq 1$, and $\mathbf{b} = (b_1, b_2, \dots)$, we notice that $\mathbf{a}, \mathbf{b} \in X$ and $\sigma(\mathbf{a}) = \mathbf{b}$. Moreover, $g(R_{a_{n+1}}) = R_{a_n}$ and $g|R_{a_{n+1}} : R_{a_{n+1}} \to R_{a_n}$ is a bijection by Property 2b above.

Let $\mathbf{R_a} = \varprojlim\{R_{a_n}, g|R_{a_n}\}$ and $\mathbf{R_b} = \varprojlim\{R_{b_n}, g|R_{b_n}\}$. Notice that both $\mathbf{R_a}$ and $\mathbf{R_b}$ are subspaces of X and $\sigma(\mathbf{R_a}) = \mathbf{R_b}$. In fact they are rays, and we define their corresponding parametrizations $\mathbf{f_a} : [0, \infty) \to \mathbf{R_a}$ and $\mathbf{f_b} : [0, \infty) \to \mathbf{R_b}$ as follows. Let $\mathbf{f_a}(t)$ and $\mathbf{f_b}(t)$ be the unique points in $\mathbf{R_a}$ and $\mathbf{R_b}$ having their first coordinates equal to $f_{a_1}(t)$ and $f_{b_1}(t)$, respectively. Given $\epsilon > 0$ we choose a natural number n such that each two points in X having their first n coordinates equal are within a distance less than ϵ. For $t > n$ the first n coordinates of both $\mathbf{f_a}(t)$ and $\mathbf{f_b}(t)$ are $f_{a_1}(t) = f_{b_1}(t)$, $f_{a_2}(t-1) = f_{b_2}(t-1)$, $\dots$, $f_{a_n}(t-n+1) = f_{b_n}(t-n+1)$, and thus $\mathbf{R_a}$ and $\mathbf{R_b}$ are asymptotic. Since the projections R_{a_n} and R_{b_n} of $\mathbf{R_a}$ and $\mathbf{R_b}$, respectively, are dense in B, the rays $\mathbf{R_a}$ and $\mathbf{R_b}$ are dense in X.

Hence, by Corollary 1, we let R be a ray limiting on $\mathbf{R_a}$ and let $\hat{\sigma}\colon X \cup R \to X \cup R$ be a homeomorphic extension of $\sigma\colon X \to X$ such that $\hat{\sigma}(R) = R$. Applying Proposition 1, the result follows. □

A map $f : X \to Y$ is *atomic* if $f^{-1}(y)$ is a terminal continuum in X for each $y \in Y$. Note that an atomic map must be surjective. The following known proposition is easy to prove.

Proposition 2. *Let $f : X \to Y$ be a nonconstant atomic map between continua X and Y. Then X is indecomposable if and only if Y is indecomposable.*

Example 2. There is an indecomposable tree-like continuum that admits a fixed-point-free homeomorphism leaving each composant invariant.

Proof. Let Z be the continuum guaranteed by Proposition 1 with X being the second Bellamy continuum, as defined in the proof of Example 1. Thus Z is a compactification of the real line with two copies of X, X_1 and X_2, as the remainder of this compactification, and with the real line limiting on X_1 when $x \to -\infty$, and on X_2 when $x \to \infty$. Let $h : Z \to Z$ be the fixed-point-free homeomorphism as in Proposition 1 with $h(x) > x$ for each number x. Consider the product $C \times Z$, where C is the Cantor set defined in $[0, 1]$ as usual. In this product we identify each pair of points (c_1, y_1) and (c_2, y_2) whenever either

(1) $y_1 = y_2$, $y_1 \in X_2$ and $c_1 + c_2 = 1$ or
(2) $y_1 = y_2$, $y_1 \in X_1$ and $3^n c_1 + 3^n c_2 = 5$ for some $n \in \{1, 2, \dots\}$.

Let $q : C \times Z \to T$ be the quotient map of this identification with the quotient space T. Figure 1 shows a schematic picture of the fourth approximation of this construction. The homeomorphism $h : Z \to Z$ induces a homeomorphism $\hat{h} : T \to T$ that leaves invariant each copy of X in T, that is, the continua $q(c \times X_1)$ and $q(c \times X_2)$. These copies are terminal in T because X_1 and X_2 are terminal in Y. If we apply another quotient map and identify these copies to points, the quotient space is the "simplest indecomposable continuum", the bucket handle B_H, which is indecomposable. This last quotient map is atomic, and thus T is indecomposable by Proposition 2. Since $\dim T = 1$, the bucket handle continuum B_H is tree-like, and the fibers of this atomic map are tree-like, it follows that T is tree-like [7, (6.14), p. 18]. Clearly, $\hat{h}$ preserves the composants of T. □

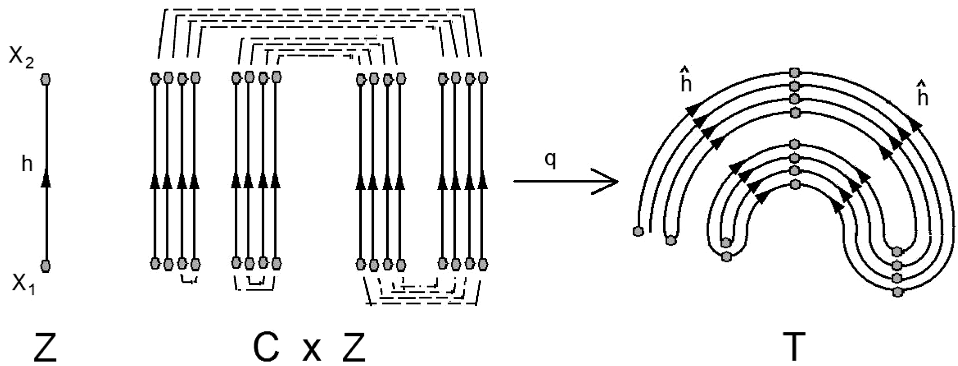

Figure 1

Comments. In Example 2 we note that T admits an atomic map onto Knaster's simplest indecomposable chainable continuum B_H [6, page 107] with the preimage of each point of B_H being either a point of T or a copy of Bellamy's continuum X. In our example, each copy of Bellamy's continuum is left invariant by the homeomorphism $\hat{h}$.

Question 1. Must every fixed-point-free composant-preserving map (homeomorphism) of an indecomposable tree-like continuum leave some proper subcontinuum invariant?

It is not known if Bellamy's continuum X is embeddable in the plane. The open question below, with the tree-like assumption included, was raised in [5]. It was answered in the case when composants are arcwise connected in [3].

Question 2. Must every composant-preserving map of an indecomposable (tree-like) plane continuum have a fixed point?

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Department of Mathematics and Statistics, California State University, Sacramento, 6000 J Street, Sacramento, California 95819
E-mail address: hagopian@csus.edu

Department of Mathematics and Statistics, California State University, Sacramento, 6000 J Street, Sacramento, California 95819
E-mail address: mmarsh@csus.edu

Department of Mathematics and Statistics, California State University, Sacramento, 6000 J Street, Sacramento, California 95819
E-mail address: prajs@csus.edu