COMPACTA ADMITTING RETRACTIONS CLOSE TO THE IDENTITY MAP

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ABSTRACT. We develop techniques for determining certain structural properties of inverse limits on compacta. In particular, we show if easily observable properties of the bonding mappings are present, then one can identify (1) nested sequences of subcompacta of the inverse limit space whose members are copies of subcompacta of the factor spaces, and (2) a sequence of retractions of the inverse limit space onto members of this nested sequence that converges uniformly to the identity mapping. Applications to inverse limits on continua are discussed. In the special case of a continuum X that admits such a sequence of retractions onto arcs, we establish properties that describe the nature of proper subcontinua of X.

1. Introduction and Definitions. A *compactum* is a compact metric space. A *continuum* is a connected compactum. If each proper subcontinuum of a continuum X is an arc, we call X an *arc continuum.* A *mapping* or *map* is a continuous function. Let $\{X_i\}_{i\geq 1}$ be a sequence of compacta, and $\{g_i^{i+1}\colon X_{i+1} \to X_i\}_{i\geq 1}$ be a sequence of mappings. We refer to $\{X_i, g_i^{i+1}\}_{i\geq 1}$ as an *inverse sequence*, and its *inverse limit* X, denoted $\varprojlim\{X_i, g_i^{i+1}\}$, is the subset of $\prod_{i\geq 1} X_i$ given by $X = \{(x_1, x_2, \ldots)|\ x_i = g_i^{i+1}(x_{i+1})$ for all $i \geq 1\}$. We use inverse sequences and inverse limits throughout the paper. General properties of these notions can be found in [**13,** p.7-14], [**19,** Sections 2.1-2.3], or [**30,** Part One, Sec.II].

The study of continua expressed as inverse limits has been a central theme in continuum theory for around 60 years. Since inverse limits on intervals can be used to represent complicated continua in a simple way, and since they can appear as attractors in dynamical systems [**29**], one dimensional inverse limits have been of particular interest. In

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fact, R. F. Williams **[32]** showed that all hyperbolic one-dimensional attractors are inverse limits of maps on branched one-manifolds. Many remarkable examples have been constructed using inverse limits. A few are the R. D. Anderson and G. Choquet **[1]** tree-like continuum that contains no chainable subcontinua, H. Cook's **[7]** continuum that, other than the identity mapping, only admits constant self-mappings, W. T. Ingram's **[12]** simple-triod-like arc continuum with positive span, R.M. Schori's **[31]** arclike continuum that contains a copy of every arclike continuum, and J.H. Case and R.E. Chamberlin's **[6]** figure-eight-like continuum that admits no essential map to the unit circle, but does admit an essential map to the figure eight. Ingram discusses the first four of these examples, as well as some others, in Sections 3 and 5 in **[13]**.

Also, understanding the structure of continua, that are expressed as inverse limits on relatively simple continua, such as intervals, trees, graphs, disks, and other locally connected continua, by investigating properties of the bonding mappings has been a standard practice. Particularly, for inverse limits on intervals, there are some very detailed and thorough investigations that involve certain families of unimodal bonding mappings, see **[3]**, **[4]**, **[5]**, **[10]**, **[13]**, **[14]**, and **[15]**. The results in these references demonstrate that there is a remarkable abundance of diversity among inverse limit spaces, even when bonding maps are chosen from a rather restrictive family of maps. The proof of Ingram's Conjecture **[2]** that no two inverse limits on $[0, 1]$ using bonding maps from the "tent map" family that have unequal slopes between 1 and 2, are homeomorphic highlights this fact.

Many techniques, particularly in the one-dimensional setting, have been developed to help determine the structure of inverse limits. These include identifying certain subcontinua of the inverse limit space, determining folding patterns and asymptotic behavior of arc components, and determining if the inverse limit space can be approximated from within by simple locally connected subcontinua. The first two methods can be found among the references listed in the preceding paragraph. The third method is considered in **[8]**, **[9]**, **[16]**, **[20]**, **[25]**, **[26]**, **[27]**, and **[28]**.

We develop general theorems for inverse sequences on compacta that facilitate investigations of the structure of the associated inverse limits. In some cases, the techniques implied by our theorems may be reminiscent of tools used in some of the references cited for the one-dimensional investigations, but we emphasize that our results apply to the much larger class of inverse limits on compacta. Given an inverse sequence on compacta, easily observable properties of the bonding maps restricted to certain subsequences of subcompacta of the factor spaces, can immediately produce nested sequences of copies of these subcompacta in the inverse limit space. The unions of these nested sequences are typically dense in the inverse limit space or in some identifiable subcompactum of the inverse limit space. Furthermore, in some cases, the nested sequences can approximate the inverse limit X in a strong manner, namely by having retractions, arbitrarily close to the identity mapping, from X onto members of the nested sequences.

Let X and Y be compacta. A map $f\colon X \to Y$ is an *r-map* if there exists a map $g\colon Y \to X$ such that $f \circ g$ is the identity map on Y. Given $\epsilon > 0$, a mapping $f\colon X \to Y$ is an *ϵ-mapping* if for each point $y \in Y$, $\mathrm{diam}\,(f^{-1}(y)) < \epsilon$. If K is a closed subset of X, a mapping $f\colon X \to K$ is a *retraction* if $f(x) = x$ for each $x \in K$. An *ϵ-retraction* $r\colon X \to K$ is a retraction that is also an ϵ-mapping.

For a class of compacta $\mathcal{G}$, a compactum X is *$\mathcal{G}$-like* if for each $\epsilon > 0$, X admits an ϵ-mapping onto a member of $\mathcal{G}$. The teminology suggests that X is, in some manner, like the compacta in $\mathcal{G}$. It is not unreasonable to think of a compactum X as being even more $\mathcal{G}$-like if the ϵ-maps are retractions onto copies of members of $\mathcal{G}$ that lie in X. Perhaps even most $\mathcal{G}$-like if for each x in X there are members of $\mathcal{G}$ containing x onto which X admits ϵ-retractions. Interestingly, we show in Theorem 6 that if X is everywhere retractably arclike (defined below), then X is either an arc or an arc continuum. If X is an arclike, arc continuum that is not an arc, then X is indecomposable. So, the continua that are most arclike are at positions in the spectrum of arclike continua that we sometimes think of as being far apart.

Perhaps some of the tools developed herein can be used to distinguish between inverse limits on compacta or continua outside of the well-understood families of inverse limits on intervals with unimodal bonding maps, and also outside of the one-dimensional setting. We provide examples in Section 6 that illustrate this.

Let $\mathcal{G}$ be a class of compacta. Let X be a compactum with metric d. We say that X is *internally $\mathcal{G}$-like* if for each $\epsilon > 0$, there is a subset M

of X with $M \in \mathcal{G}$, and a map $f\colon X \to M$ such that $d(f(x), x) < \epsilon$ for all $x \in X$. If X is internally $\mathcal{G}$-like and, additionally, the mappings f are retractions, we say that X is *retractably $\mathcal{G}$-like.* The compactum X is *retractable $\mathcal{G}$-like onto a nested sequence $\{K_i\}_{i\geq 1}$ of compacta* if for each $i \geq 1$, $K_i \in \mathcal{G}$, $K_i \subset K_{i+1}$, and there is a retraction $f_i\colon X \to K_i$ where the sequence $\{f_i\}_{i\geq 1}$ converges uniformly to the identity mapping on X. Discussion and results related to these notions can be found in [**25**] and [**26**].

Hereafter, we denote sequences $M_1, M_2, M_3, \ldots$ by $\{M_i\}$ or $\{M_n\}$. For sequences $M_k, M_{k+1}, M_{k+2}, \ldots$, we use the notation $\{M_i\}_k$ or $\{M_n\}_k$. Next, we offer some variations of the definitions above. We say that X is *internally $\mathcal{G}$-like at the point* $x \in X$ if for each $\epsilon > 0$, there exists a subcompactum M of X that is in $\mathcal{G}$ and contains x, and there exists a mapping $f\colon X \to M$ with $d(f(y), y) < \epsilon$ for all $y \in X$. We say that X is *everywhere internally $\mathcal{G}$-like* if X is internally $\mathcal{G}$-like at each point of X. We say that X is *retractably $\mathcal{G}$-like at the point* $x \in X$ if for each $\epsilon > 0$, there exists a subcompactum M of X that is in $\mathcal{G}$ and contains x, and there exists a retraction $r\colon X \to M$ with $d(r(y), y) < \epsilon$ for all $y \in X$. We say that X is *everywhere retractably $\mathcal{G}$-like* if X is retractably $\mathcal{G}$-like at each point of X. Note that if X is retractable $\mathcal{G}$-like onto a nested sequence $\{K_i\}$ of compacta, then X is retractably $\mathcal{G}$-like at each point of $\cup_{i\geq 1} K_i$.

We note the following obvious implications.

$$X \text{ is everywhere retractably } \mathcal{G}\text{−like } \Rightarrow$$

$$X \text{ is retractably } \mathcal{G}\text{−like at a point } x \in X \;\Rightarrow$$

$$X \text{ is retractably } \mathcal{G}\text{−like } \Rightarrow X \text{ is internally } \mathcal{G}\text{−like } \Rightarrow\; X \text{ is } \mathcal{G}\text{−like}$$

A continuum X has the *fixed point property* if every self map f of X has a fixed point; that is, a point $x \in X$ such that $f(x) = x$. Throughout, we let $\mathcal{G}$ be an arbitrary class of compacta, and we let $\mathcal{F}$ be the class of continua with the fixed point property. The notation $X \overset{T}{\approx} Y$ will indicate that X is homeomorphic to Y.

Each of the various forms of internally $\mathcal{G}$-like and retractably $\mathcal{G}$-like is considerably stronger than simply being $\mathcal{G}$-like. For example, each arclike continuum X with no dense arc component is not retractably arclike at a point $x \in X$, see Lemma 4 and Observation 7 in §5. Even the standard topologist's $\sin(1/x)$-curve can be modified so that the modified version is not internally arclike. Also, in reference to Theorem 3 below, there are many continua that are $\mathcal{F}$-like and admit fixed-point-free mappings.

If $\epsilon > 0$, $f\colon X \to X$ is a mapping, and $d(f(x), x) < \epsilon$ for all $x \in X$, we write $d(f, \mathrm{id}) < \epsilon$, where id is the identity mapping on X. Note that having a subcompactum M of X and a retraction $r\colon X \to M$ with $d(r, \mathrm{id}) < \epsilon$ for each $\epsilon > 0$ is equivalent to having a subcompactum M of X and an ϵ-retraction $r\colon X \to M$ for each $\epsilon > 0$.

We begin by noting three theorems from **[25]** and **[26]**. See these two references for definitions of internally and retractaby $\mathcal{G}$-representable, which involve expressing X as an inverse limit on subcompacta. The first two theorems characterize certain inverse limit representations of compacta with being internally or retractably $\mathcal{G}$-like. The third theorem shows that continua that are internally $\mathcal{F}$-like must have the fixed point property.

Theorem 1. **[25,** Theorem 3] A compactum X is internally $\mathcal{G}$-like if and only if it is internally $\mathcal{G}$-representable.

Theorem 2. **[26,** Corollary 3.1] A compactum X is retractably $\mathcal{G}$-like onto a nested sequence of compacta if and only if it is retractably $\mathcal{G}$-representable.

Theorem 3. **[25,** Theorem 1] If a continuum X is internally $\mathcal{F}$-like, then X has the fixed point property.

The definitions and some of the results below were established in the more general setting of inverse limits with set-valued bonding functions in **[24]**. It is helpful and convenient to have the terminology and notation in a more specialized form for ordinary inverse limits. In particular, in **[24]**, we defined a k-tail sequence as a certain sequence formed from an inverse sequence on compacta with upper semi-continuous set-valued bonding functions. The partial graph (Mahavier product) through the first k factors, and certain properties of the bonding functions for $i \geq k$, were shown to produce subcompacta of the inverse limit space that are homeomorphic to subcompacta of the partial graphs. For ordinary inverse sequences, the underlying idea is to use these special k-tail sequences to produce a nested sequence of subcompacta of the inverse limit space, each member of which is homeomorphic to a subcompactum of the k^{th} factor space.

Let $\{X_i, g_i^{i+1}\}$ denote an inverse sequence on compacta with surjective bonding mappings $g_i^{i+1}\colon X_{i+1} \to X_i$ for $i \in \mathbb{N}$. For $i < j$, we let $g_i^j\colon X_j \to X_i$ denote the composition mapping $g_i^{i+1} \circ \ldots \circ g_{j-1}^j$. It is customary to let $g_i^i\colon X_i \to X_i$ denote the identity mapping $\mathrm{id}\colon X_i \to X_i$. For $X = \varprojlim\{X_i, g_i^{i+1}\}$, we assume that $\mathrm{diam}\,(X_i) = 1$ for each $i \geq 1$, and let d denote the usual metric on $\prod_{i\geq 1} X_i$. So, each projection map $g_i\colon X \to X_i$ is a $\frac{1}{2^i}$-map.

Definition 1. Let $\{X_i, g_i^{i+1}\}$ be an inverse sequence on compacta with surjective bonding mappings. Fix $k \geq 1$. Suppose $\{Y_i\}_k$ is a sequence of non-degenerate compacta such that for each $i \geq k$,

(i) $Y_i \subset g_i^{i+1}(Y_{i+1})$, and
(ii) $g_i^{i+1}|_{Y_{i+1}}$ is a homeomorphism onto its image.

We call $\{Y_i\}_k$ a *k-tail sequence of homeomorphically-covered compacta in the inverse sequence* $\{X_i, g_i^{i+1}\}$. For simplicity of terminology, we say that $\{Y_i\}_k$ is an *hc-sequence*, and the notation makes it clear in which factor the hc-sequence begins. If $k = 1$, we say that $\{Y_i\}$ is a *complete* hc-sequence.

As mentioned, we use hc-sequences to generate nested sequences of subcompacta of the inverse limit space, as defined in Definition 2 below.

Definition 2. Let $X = \varprojlim\{X_i, g_i^{i+1}\}$, and let $\{Y_i\}_k$ be an hc-sequence in the inverse sequence $\{X_i, g_i^{i+1}\}$. Fix $n \geq k$, and let $A_n^n = g_n^{n+1}(Y_{n+1})$. For $1 \leq i < n$, let $A_i^n = g_i^n(A_n^n)$, and for $i > n$, let $A_i^n = \left(g_{i-1}^i|_{Y_i}\right)^{-1}(A_{i-1}^n)$. Note that $A_{n+1}^n = Y_{n+1}$, and by hypothesis, A_{n+1}^n is homeomophic to A_n^n. In fact, for $i > n$, A_i^n is homeomophic to A_{i-1}^n and, through finitely many compositions, to A_n^n. Let $L_n = \varprojlim\{A_i^n, g_i^{i+1}|_{A_{i+1}^n}\}$. It is useful to note that $g_n(L_n) = A_n^n$, and $g_{n+1}(L_n) = Y_{n+1}$. We call $\{L_n\}_k$ *the nested sequence of subcompacta of* X *generated by the hc-sequence* $\{Y_i\}_k$.

Definition 3. Let $\{Y_i\}_k$ be an hc-sequence in the inverse sequence $\{X_i, g_i^{i+1}\}$. If for each $i \geq k$, $g_i^{i+1}(Y_{i+1}) = X_i$, then we call $\{Y_i\}_k$ a *surjective hc-sequence.*

Let $X = \varprojlim\{X_i, g_i^{i+1}\}$. Suppose $\{Y_i\}_k$ is an hc-sequence in $\{X_i, g_i^{i+1}\}$, and $\{L_n\}_k$ is the nested sequence of subcompacta of X generated by $\{Y_i\}_k$. We make the following observations. Observation 1 follows easily from Definitions 1 and 2. We provide short proofs for Observations 2 and 3.

Observation 1. The sequence $\{L_n\}_k$ is nested increasing in X. That is, $L_n \subset L_{n+1}$ for $n \geq k$. Also, by the definition of L_n, for a given $n \geq k$, if $x \in L_n$, we note that all coordinates of x are uniquely determined by the n^{th} coordinate of x.

Observation 2. Concerning the relationship between $\{Y_i\}_k$ and $\{L_n\}_k$, we observe the following.

(a) If $Y_i = g_i^{i+1}(Y_{i+1})$ for each $i \geq k$, then $\{L_n\}_k$ is a constant sequence with $L_n \overset{T}{\approx} Y_k$ for each $n \geq k$.
(b) If there exists $n \geq k$ such that $g_{n+1}^{n+2}(Y_{n+2}) \neq Y_{n+1}$, then L_n is a proper subset of L_{n+1}. So, if $g_{i+1}^{i+2}(Y_{i+2}) \neq Y_{i+1}$ for all $i \geq k$, it follows that $\{L_n\}_k$ is strictly increasing.

Proof. (a) Under our assumption, we note from Definition 2 that, for $n \geq k$, L_n is the limit of the inverse sequence

$$g_1^k(Y_k) \longleftarrow \ldots \longleftarrow g_{k-1}^k(Y_k) \longleftarrow Y_k \longleftarrow Y_{k+1} \longleftarrow \ldots,$$

where the bonding maps are g restricted to the factor spaces indicated. So, clearly $\{L_n\}_k$ is a constant sequence. Since for each $i \geq k$, $g_i^{i+1}|_{Y_{i+1}}$ is a homeomorphism, we have that $L_n \overset{T}{\approx} Y_k$ for each $n \geq k$.

(b) Suppose there exists $n \geq k$ such that $A_{n+1}^{n+1} \neq Y_{n+1}$. Let $x \in L_{n+1}$ with $g_{n+1}(x) \in A_{n+1}^{n+1} \setminus Y_{n+1}$. Since $g_{n+1}(L_n) = Y_{n+1}$ as noted in Definition 2, $x \notin L_n$. So, by Observation 1, the proof of (b) is complete. □

Observation 3. The inverse sequence $\{X_i, g_i^{i+1}\}$ contains a surjective hc-sequence if and only if for some $k \geq 1$, g_i^{i+1} is an r-map for each $i \geq k$.

Proof. Suppose $\{Y_i\}_k$ is a surjective hc-sequence in $\{X_i, g_i^{i+1}\}$. Then we simply note that, for $i \geq k$, $g_i^{i+1} \circ (g_i^{i+1}|_{Y_{i+1}})^{-1} \colon X_i \to X_i$ is the identity map. That is, g_i^{i+1} is an r-map for each $i \geq k$. The opposite implication is clear. $\square$

We provide the familiar example below to illustrate the definitions in this section.

Example 1. Let $X = \varprojlim\{[0,1], g\}$, where g is the "tent map". That is, $g\colon [0,1] \to [0,1]$ is the open mapping whose graph consists of the two line segments with endpoints $(0,0)$ and $(\frac{1}{2}, 1)$, and $(\frac{1}{2}, 1)$ and $(1,0)$. The arclike continuum X is commonly referred to as the *Buckethandle* or *Horseshoe continuum.* A picture of X can be found in [**17,** page 205]. Below are three complete surjective hc-sequences and their respective generated nested sequences of subcontinua (arcs in this case) in X. In each inverse sequence, the right-most $[0,1]$ is the n^{th} factor in the sequence, and is also A_n^n for the associated inverse sequence. In the notation for this example, if we write $[c,d] \xleftarrow{g} [a,b]$, we are indicating that $[a,b]$ and $[c,d]$ are subintervals of $[0,1]$, and that the restriction of g to $[a,b]$ has image $[c,d]$.

The constant hc-sequence $\{[0, \frac{1}{2}]\}$ generates, for each $n \geq 1$, an arc $L_n \subset X$. The arc L_n is the inverse limit of the sequence

$$[0,1] \xleftarrow{g} \dots \xleftarrow{g} [0,1] \xleftarrow{g} [0, \frac{1}{2}] \xleftarrow{g} [0, \frac{1}{4}] \xleftarrow{g} [0, \frac{1}{8}] \longleftarrow \dots.$$

The constant hc-sequence $\{[\frac{1}{2}, 1]\}$ generates, for each $n \geq 1$, an arc $K_n \subset X$, which is the inverse limit of the sequence

$$[0,1] \xleftarrow{g} \dots \xleftarrow{g} [0,1] \xleftarrow{g} [\frac{1}{2}, 1] \xleftarrow{g} [\frac{1}{2}, \frac{3}{4}] \xleftarrow{g} [\frac{5}{8}, \frac{3}{4}] \longleftarrow \dots.$$

The alternating hc-sequence $[0, \frac{1}{2}], [\frac{1}{2}, 1], [0, \frac{1}{2}], \dots$ generates, for each even $n \geq 1$, an arc $J_n \subset X$, which is the inverse limit of the sequence

$$[0,1] \xleftarrow{g} \dots \xleftarrow{g} [0,1] \xleftarrow{g} [0, \frac{1}{2}] \xleftarrow{g} [\frac{3}{4}, 1] \xleftarrow{g} [\frac{3}{8}, \frac{1}{2}] \longleftarrow \dots.$$

The reader should note the inverse sequence for J_n when n is odd.

2. Homeomorphically-covered sequences. We begin this section with an addition to Theorem 2 that establishes an equivalence between a compactum being retractably $\mathcal{G}$-like and being representable as the inverse limit of an inverse sequence on compacta in $\mathcal{G}$ that contains a surjective hc-sequence. The theorem follows immediately from Observation 3 and Corollary 3.1 in **[26]**.

Theorem 4. Let X be a compactum and $\mathcal{G}$ a class of compacta. The following statements are equivalent.

(i) X is retractably $\mathcal{G}$-like onto a nested sequence of subcompacta.
(ii) X is retractably $\mathcal{G}$-representable.
(iii) $X \overset{T}{\approx} \varprojlim\{X_i, g_i^{i+1}\}$, where $X_i \in \mathcal{G}$ for each $i \geq 1$, and $\{X_i, g_i^{i+1}\}$ contains a surjective hc-sequence.

The focus of this section is to determine how to "spot" subcompacta of a compactum X that are retractably $\mathcal{G}$-like, when X has a given inverse limit representation, say $X = \varprojlim\{X_i, g_i^{i+1}\}$ with factor spaces that may or may not be members of $\mathcal{G}$. We show that finding hc-sequences in $\{X_i, g_i^{i+1}\}$ is important to this endeavor. For example, we may have a continuum X expressed as an inverse limit on trees, but if we find an hc-sequence of arcs in the inverse sequence with appropriate properties, we can identify subcontinua of X that are retractably arclike, and perhaps even discover that X itself is retractably arclike. The second example in Example 5 of Section 6 illustrates this.

Lemma 1. Let $X = \varprojlim\{X_i, g_i^{i+1}\}$. Suppose $\{Y_i\}_k$ is an hc-sequence, and $\{L_n\}_k$ is the nested sequence of subcompacta of X generated by $\{Y_i\}_k$. Then, for each $n \geq k$, the projection map $g_n|_{L_n} : L_n \to A_n^n = g_n^{n+1}(Y_{n+1})$ is a homeomorphism.

Proof. Fix $n \geq k$. Suppose $x, y \in L_n$ and $g_n(x) = g_n(y)$. By definition of L_n, $g_n(x) \in A_n^n$. Also, $g_i(x) = g_i^n g_n(x) = g_i^n g_n(y) = g_i(y)$ for $1 \leq i \leq n$. By construction of L_n, each $g_i(x)$ for $i > n$ is uniquely determined by $g_n(x)$, and similarly for $g_i(y)$ for $i > n$. Thus, since $g_n(x) = g_n(y)$, it follows that $g_i(x) = g_i(y)$ for $i > n$. Hence, we have that $x = y$. Therefore, $g_n|_{L_n}$ is one-to-one into A_n^n. By construction of L_n, it is clear that for $a \in A_n^n$, we can pick $x \in L_n$ where $g_n(x) = a$. □

(∗) Hereafter, when $\{L_n\}_k$ is the nested sequence of subcompacta of an inverse limit X generated by an hc-sequence $\{Y_i\}_k$, we let $\gamma_n \colon A_n^n \to L_n$ denote the inverse homeomorphism of $g_n|_{L_n}$ for each $n \geq k$.

Lemma 2 is a partial generalization of R. Bennett's theorem, see [**13,** Theorem 2.16] or [**15,** Theorem 3.4]. The "monotone (increasing)" assumption in Bennett's theorem has been generalized to the existence of an hc-sequence, and the inverse sequence on intervals has been generalized to an arbitrary inverse sequence on compacta. We give a complete generalization of Bennett's theorem for inverse limits with a single bonding mapping in Theorem 6, parts (1) and (2), in §3.

Lemma 2. Let $X = \varprojlim\{X_i, g_i^{i+1}\}$. Suppose $\{Y_i\}_k$ is an hc-sequence, and $\{L_n\}_k$ is the nested sequence of subcompacta of X generated by $\{Y_i\}_k$. Suppose Z is a subcompactum of X, and there exist an integer $m \geq 0$ and an increasing sequence $\{n_i\}$ of integers such that, for each $i \geq 1$, $g_{n_i-m}^{n_i}(A_{n_i}^{n_i}) = g_{n_i-m}(Z)$. Then $\overline{\bigcup_{n\geq k} L_n} = Z$. In particular, if for each $i \geq 1$, $g_{n_i-m}^{n_i}(A_{n_i}^{n_i}) = X_{n_i-m}$, then $\overline{\bigcup_{n\geq k} L_n} = X$.

Proof. Since A_n^n is only defined for $n \geq k$, it follows, by hypothesis, that $n_1 \geq k$.

$\subset$: Let $x \in \bigcup_{n\geq k} L_n$. We first show that $g_j(x) \in g_j(Z)$ for each $j \geq 1$. Fix $j \geq 1$. Since $x \in \bigcup_{n\geq k} L_n$ and $\{L_n\}_k$ is a nested increasing sequence, there exists $n \geq k$ such that for all $i \geq n$, $x \in L_i$. Pick n_i so that $n_i - m > \max\{j, n\}$. So, $x \in L_{n_i}$. Since $j < n_i - m$, using the definition of L_{n_i} and our hypothesis, we get that

$$\begin{aligned} g_j(L_{n_i}) = A_j^{n_i} &= g_j^{n_i}(A_{n_i}^{n_i}) \\ &= g_j^{n_i-m}(g_{n_i-m}^{n_i}(A_{n_i}^{n_i})) \\ &= g_j^{n_i-m}(g_{n_i-m}(Z)) = g_j(Z). \end{aligned}$$

Since $x \in L_{n_i}$, $g_j(x) \in g_j(L_{n_i}) = g_j(Z)$. Now, using a well-known inverse limit theorem (see [**13,** Theorem 1.9] or [**19,** Proposition 2.1.20]), we have that $\bigcup_{n\geq k} L_n \subset Z$. Since Z is a compactum, we have that $\overline{\bigcup_{n\geq k} L_n} \subset Z$.

$\supset$: Let $x \in Z$ and $\epsilon > 0$. Let n_i be large enough so that g_{n_i-m} is an ϵ-map. By hypothesis, there is a point $y_{n_i} \in A^{n_i}_{n_i}$ such that $g^{n_i}_{n_i-m}(y_{n_i}) = g_{n_i-m}(x)$. Let $y = \gamma_{n_i}(y_{n_i}) \in L_{n_i}$. So, $g_{n_i}(y) = y_{n_i}$ by definition of γ_{n_i}. Then $g_{n_i-m}(y) = g^{n_i}_{n_i-m} g_{n_i}(y) = g^{n_i}_{n_i-m}(y_{n_i}) = g_{n_i-m}(x)$. Since g_{n_i-m} is an ϵ-map, the distance from x to y is less than ϵ. Thus, x is a limit point of $\bigcup_{n\geq k} L_n$. That is, $x \in \overline{\bigcup_{n\geq k} L_n}$.

The proof is complete. □

Theorem 5. Let $X = \varprojlim\{X_i, g^{i+1}_i\}$. Suppose $\{Y_i\}_k$ is an hc-sequence, and $\{L_n\}_k$ is the nested sequence of subcompacta of X generated by $\{Y_i\}_k$. Then, for each $n \geq k$, $r_n = \gamma_n g_n|_{g_n^{-1}(A^n_n)} \colon g_n^{-1}(A^n_n) \to L_n$ is a retraction such that $d(r_n, \mathrm{id}|_{g_n^{-1}(A^n_n)}) < \frac{1}{2^n}$. In particular, if $A^n_n = X_n$ for some $n \geq k$, then $r_n = \gamma_n g_n \colon X \to L_n$ is a retraction such that $d(r_n, \mathrm{id}) < \frac{1}{2^n}$.

Proof. Fix $n \geq k$. If $x \in L_n$, then $r_n(x) = \gamma_n g_n(x) = x$ by definition of γ_n and L_n. So, r_n is a retraction onto L_n.

Let $x \in g_n^{-1}(A^n_n)$. Then, $g_n r_n(x) = g_n \gamma_n g_n(x) = g_n(x)$. So, $d(x, r_n(x)) < \frac{1}{2^n}$.

If $A^n_n = X_n$ for some $n \geq k$, then $g_n^{-1}(A^n_n) = X$ and the remaining statement in Theorem 5 is clear. □

Corollary 1. Let $X = \varprojlim\{X_i, g^{i+1}_i\}$, and let Z be a subcompactum of X. Suppose $\{Y_i\}_k$ is an hc-sequence with each $Y_i \in \mathcal{G}$, $\{L_n\}_k$ is the nested sequence of subcompacta generated by $\{Y_i\}_k$, and there exists an increasing sequence $\{n_i\}$ of positive integers such that, for each $i \geq 1$, $A^{n_i}_{n_i} = g_{n_i}(Z)$. Then Z is retractably $\mathcal{G}$-like onto the nested sequence of subcompacta $\{L_{n_i}\}$. Furthermore, Z is retractably $\mathcal{G}$-like at each point of $\cup_{n\geq k} L_n$. In particular, if $A^{n_i}_{n_i} = X_{n_i}$ for each $i \geq 1$, then X is retractably $\mathcal{G}$-like onto the nested sequence of subcompacta $\{L_{n_i}\}$, and X is retractably $\mathcal{G}$-like at each point of $\cup_{n\geq k} L_n$.

Proof. Since for each $n \geq k$, $Y_n \in \mathcal{G}$, and A^n_n is the homeomorphic image of Y_{n+1}, it follows that each $A^n_n \overset{T}{\approx} L_n \in \mathcal{G}$. By Lemma 2, we have that $\overline{\cup_{n\geq k} L_n} = Z$. For each $i \geq 1$, $Z \subset g_{n_i}^{-1}(A^{n_i}_{n_i})$ since $g_{n_i}(Z) = A^{n_i}_{n_i}$. So, by Theorem 5, we have that for each $i \geq 1$, there exists a retraction

$r_{n_i}\colon Z \to L_{n_i}$ with $d(r_{n_i}, \mathrm{id}|_Z) < \frac{1}{2^{n_i}}$. It follows that Z is retractably $\mathcal{G}$-like onto $\{L_{n_i}\}$. That Z is retractably $\mathcal{G}$-like at each point of $\cup_{n\geq k} L_n$ is clear. The last statement of Corollary 1 is also clear. □

Remark 1. We note an important difference between Lemma 2 and Corollary 1 when $m \neq 0$. Let the dyadic solenoid be given by $\Sigma = \varprojlim\{S^1, s\}$, where S^1 is the unit circle in $\mathbb{R}^2$, and s is the squaring map in the group structure on S^1. So, s is a 2-to-1 covering map of S^1. Let N be the shortest arc in S^1 with endpoints $(1,0)$ and $(0,1)$, and notice that the constant sequence $\{N\}$ is a complete hc-sequence, with $s(N) = A_n^n$ (the top half of S^1), and $s(A_n^n) = S^1$ for $n \geq 2$. So, by Lemma 2, the union of the nested sequence of arcs in Σ generated by $\{N\}$ is dense in Σ, but, of course, Σ is not retractably arclike.

Corollary 2. Let $X = \varprojlim\{X_i, g_i^{i+1}\}$ with each $X_i \in \mathcal{G}$. Suppose that for each $i \geq k$ and each $x \in X_{i+1}$, there exists a copy Y_{i+1} of X_i in X_{i+1} for which $x \in Y_{i+1}$ and $g_i^{i+1}\big|_{Y_{i+1}}\colon Y_{i+1} \to X_i$ is a homeomorphism. Then X is everywhere retractably $\mathcal{G}$-like onto a nested sequence of subcompacta.

Proof. Let $x \in X$, and for each $i \geq k$, let Y_{i+1} be a copy of X_i in X_{i+1} that contains $x_i = g_i(x)$, and so that $g_i^{i+1}\big|_{Y_{i+1}}\colon Y_{i+1} \to X_i$ is a homeomorphism. We note that $\{Y_i\}_k$ is a surjective hc-sequence. Also, by definition of the nested sequence $\{L_n\}_k$ of subcompacta generated by $\{Y_i\}_k$, we have that $x \in L_k$. To see this, recall that $L_k = \varprojlim\{A_i^k, g_i^{i+1}|_{A_{i+1}^k}\}$. We have that $x_k \in A_k^k = X_k$. So, for $i \leq k$, $x_i \in A_i^k = X_i$. Also, the point in L_k with k^{th} coordinate x_k is uniquely determined by x_k, as was observed in Observation 1. Since for each $i \geq k$, $x_i \in Y_i$ and $g_k^i(x_i) = x_k$, it follows that x is the unique point of L_k with k^{th} coordinate x_k. So, X is retractably $\mathcal{G}$-like at x onto the nested sequence $\{L_n\}_k$. Hence, X is everywhere retractably $\mathcal{G}$-like onto a nested sequence of subcompacta. □

Each inverse limit on $[0,1]$ with open bonding mappings is a simple example to which Corollary 2 applies. Such continua are called *Knaster continua*. We show later, in Theorem 11, that each member of a certain family of unimodal maps on $[0,1]$ is everywhere retractably arc-like. This family includes the well-known 3-endpoint indecomposable chainable continuum, see [**30,** Example 1.10, page 8]. Theorem 11 also demonstrates that the converse of Corollary 2 does not hold.

Corollary 3. Let $X = \varprojlim\{X_i, g_i^{i+1}\}$. Suppose $\{Y_i\}_k$ is an hc-sequence with each $Y_i \in \mathcal{G}$, and $\{L_n\}_k$ is the nested sequence of subcompacta generated by $\{Y_i\}_k$. Suppose there exists $m \in \mathbb{N}$ such that for each $\delta > 0$ and each $n \geq m+1$, there is a retraction $\rho_n\colon X_n \to A_n^n$ such that $d(g_{n-m}^n \rho_n(x), g_{n-m}^n(x)) < \delta$ for each $x \in X_n$. Then for each $\epsilon > 0$, there exists $n \in \mathbb{N}$ and a retraction $r_n\colon X \to L_n$ such that $d(r_n, \mathrm{id}) < \epsilon$. Thus, X is retractably $\mathcal{G}$-like, at each point of $\cup_{n\geq k} L_n$, onto a nested subsequence of $\{L_n\}_k$.

Proof. Let m be given as in the hypothesis and let $\epsilon > 0$. Let n be large enough so that $n > m$ and g_{n-m} is an ϵ-map. Pick $\delta > 0$ so that $d(x,y) \geq \epsilon$ in X implies that $d(g_{n-m}(x), g_{n-m}(y)) \geq \delta$ in X_{n-m}. Let $\rho_n\colon X_n \to A_n^n$ be a retraction such that $d(g_{n-m}^n \rho_n(x), g_{n-m}^n(x)) < \delta$ for each $x \in X_n$. Let $r_n\colon X \to L_n$ be given by $r_n = \gamma_n \rho_n g_n$. For $x \in X$, $g_n r_n(x) = g_n \gamma_n \rho_n g_n(x) = \rho_n g_n(x)$. Now, by choice of ρ_n, $d(g_{n-m}^n \rho_n g_n(x), g_{n-m}^n(g_n(x))) = d(g_{n-m}^n \rho_n(g_n(x)), g_{n-m}(x)) < \delta$. So, we get that $g_{n-m}^n \rho_n g_n(x) = g_{n-m}^n g_n r_n(x) = g_{n-m} r_n(x)$, and it follows that $d(g_{n-m} r_n(x), g_{n-m}(x)) < \delta$. Hence, $d(x, r_n(x)) < \epsilon$.

It follows that X is retractably $\mathcal{G}$-like, at each point of $\cup_{n\geq k} L_n$, onto a nested subsequence of $\{L_n\}_k$. $\square$

Remark 2. If ρ_n is simply a mapping in the hypothesis of Corollary 3, then a completely analogous proof gives that for $\epsilon > 0$, r_n is a mapping ϵ-close to the identity. So, it follows, in this case, that X is internally $\mathcal{G}$-like at each point of $\cup_{n\geq k} L_n$.

3. Expanding pairs. We now look at a special case of the ideas associated with an hc-sequence in a given inverse sequence. Specifically, we will be interested in inverse sequences on a single compactum with a single bonding mapping. The terminology in the next paragraph was first introduced in [**23**].

Let M be a compactum, and let $g\colon M \to M$ be a surjective mapping. If N' and N are subcompacta of M where $N' \subset N$, and $g|_{N'}\colon N' \to N$

is a homeomorphism, we say that (N', N) is an *expanding pair* (with respect to g). If $N = M$, we say that (N', M) is a *surjective expanding pair*. Note that an expanding pair generates a complete sequence of homeomorphically-covered compacta in the inverse sequence $\{M, g\}$, namely the constant sequence $\{N'\}$. Recall that, by definition, for each $n \geq 1$, $A_n^n = N$. So, by Lemma 1 and the discussion that precedes it, the existence of an expanding pair of subcompacta (N', N) in M with respect to g, gives rise to a nested increasing sequence of homeomorphic copies of N, namely $\{L_n\}$, in the inverse limit space $X = \varprojlim\{M, g\}$. In Theorem 6 and Corollary 4, we restrict our attention to those complete hc-sequences that are generated by an expanding pair.

By Lemma 2, if either $N = M$ or the m-fold composition $g^m(N)$ is equal to M for some $m \geq 1$, then the closure of the union of the sequence $\{L_n\}$ is the inverse limit space.

If, in addition, either $N = M$ or there are retractions of M onto N that behave nicely with respect to g as in Corollary 3, then the inverse limit will be retractably N-like. We formalize these comments in Theorem 6 below. As mentioned earlier, Theorem 6, parts (1) and (2), generalize Bennett's theorem for an inverse sequence on a single compactum with a single bonding mapping.

Theorem 6. Let $X = \varprojlim\{M, g\}$, where M is a compactum and g is a surjective map. Suppose that (N', N) is an expanding pair of subcompacta with respect to g. Then we have the following.

(1) X contains a nested increasing sequence of subcompacta $\{L_n\}$, each member of which is homeomorphic to N under the restriction to L_n of the projection mapping g_n. For $n \geq 1$, we let $\gamma_n = (g_n|_{L_n})^{-1}$.

(2) Suppose there exists a subcompactum K of M and an integer $m \geq 1$ such that for each $n \geq m$, $g^n(N) = K$.
   (a) Then $\hat{K} = \varprojlim\{K, g|_K\} = \overline{\bigcup_{n\geq 1} L_n}$. Furthermore, $\overline{\bigcup_{n\geq 1} L_n} = X$ if and only if $K = M$.
   (b) Let $P = \overline{K \setminus N'}$, and suppose that $g(P) \subset P$. Then $\hat{P} = \varprojlim\{P, g|_P\} = \hat{K} \setminus \bigcup_{n\geq 1} L_n$.

(3) For each $n \geq 1$, $r_n = \gamma_n g_n|_{g_n^{-1}(N)} \colon g_n^{-1}(N) \to L_n$ is a retraction such that $d(r_n|_{g_n^{-1}(N)}, \mathrm{id}|_{g_n^{-1}(N)}) < \frac{1}{2^n}$. So, if $N = M$, then for

each $n \geq 1$, $r_n = \gamma_n g_n \colon X \to L_n$ is a retraction such that $d(r_n, \mathrm{id}) < \frac{1}{2^n}$.

(4) Suppose there exists $m \in \mathbb{N}$ such that for each $\epsilon > 0$, there is a retraction $\rho \colon M \to N$ such that $d(g^m \rho(x), g^m(x)) < \epsilon$ for each $x \in M$. Then for each $\epsilon > 0$, there exists $n \in \mathbb{N}$ and a retraction $r_n \colon X \to L_n$ such that $d(r_n, \mathrm{id}) < \epsilon$. Thus, X is retractably N-like, at each point of $\cup_{i \geq 1} L_i$, onto a nested sequence of copies of N.

Proof. (1) We note that the constant sequence $\{N'\}$ is an hc-sequence, and for each $n \geq 1$, $A_n^n = N$. So (1) follows from Lemma 1 and our definitions.

(2)(a) Since $K = g^{m+1}(N) = g(g^m(N)) = g(K)$, we have that $\hat{K} = \varprojlim\{K, g|_K\}$ is a subcompactum of X. Also, for each $n \geq m+1$, we have that $g_{n-m}^n(A_n^n) = g^m(N) = K$. Hence, the first statement of (2)(a) follows from Lemma 2. The second statement is clear.

(2)(b) $\subset$: Let $p = (p_1, p_2, \ldots) \in \hat{P}$. Since $\hat{P} \subset \hat{K}$, $p \in \hat{K}$. Suppose that $p \in L_n$ for some $n \geq 1$. Then by definition of L_n, $p_i \in N'$ for all $i > n$. So, by assumption, $p_i \notin P$ for all $i > n$, which contradicts that $p \in \hat{P}$.

$\supset$: Let $x = (x_1, x_2, \ldots) \in \hat{K} \setminus \bigcup_{n \geq 1} L_n$. So, x is not in L_n for all $n \geq 1$. Hence, $x_{n+1} \notin N' \setminus P$ for all $n \geq 1$. For if $x_{n+1} \in N' \setminus P$ for some $n \geq 1$, then $x_i \in N' \setminus P$ for all $i \geq n+1$, putting $x \in L_n$. Hence, $x_{n+1} \in P$ for all $n \geq 1$. So, $x \in \hat{P}$.

(3) Let $x \in L_n$. Since $g_n(x) \in N$, we get that $r_n(x) = \gamma_n g_n(x) = x$. So, r_n is a retraction onto L_n.

Suppose $x \in g_n^{-1}(N)$. Then $g_n r_n(x) = g_n \gamma_n g_n(x) = g_n(x)$. So, $d(x, r_n(x)) < \frac{1}{2^n}$.

(4) This follows immediately from Corollary 3. □

Corollary 4. Let $X = \varprojlim\{M, g\}$ and suppose (N', M) is a surjective expanding pair of subcompacta with respect to g. Then X is retractably M-like, at each point of $\cup_{n \geq 1} L_n$, onto a nested sequence of copies of M.

Proof. Since M is the second member of the expanding pair, it follows from Theorem 6(3) that for each retraction $r_n = \gamma_n g_n \colon X \to L_n$, we have $d(r_n, \mathrm{id}) < \frac{1}{2^n}$. So, the result follows. □

For inverse limits X of inverse sequences on a single compactum with a single surjective bonding mapping, we are often interested in how the shift homeomorphism behaves on X. While dynamics is not the focus of this paper, we make some observations about the behavior of the shift homeomorphism on nested sequences in X that are generated by expanding pairs or by hc-sequences.

Let M be a compactum, and let $g \colon M \to M$ be a surjective mapping. Let $X = \varprojlim\{M, g\}$, and let $\sigma \colon X \to X$ be the right shift homeomorphism. That is, for $x = (x_1, x_2, \ldots) \in X$, let $\sigma(x) = (g(x_1), x_1, x_2, \ldots)$. Let $n \geq 2$. We say that an hc-sequence $\{Y_i\}$ is *n-cyclic* if $Y_i = Y_j$ whenever $i \equiv j \pmod{n}$, and $\{Y_i\}$ is not m-cyclic for $1 \leq m < n$. If $\{Y_i\}$ is 2-cyclic, we say that $\{Y_i\}$ is an *alternating* hc-sequence.

Each of the observations below is readily apparent from the definition of σ and Definitions 1 and 2. Also, the two constant sequences generated by expanding pairs in Example 1 relate to Observation 4, and the alternating hc-sequence in Example 1 relates to Observation 6.

Observation 4. If $\{L_n\}$ is the nested sequence of compacta generated by the constant hc-sequence $\{N'\}$ that arises from an expanding pair (N', N), then for each $n \geq 1$, $\sigma(L_n) = L_{n+1}$. So, σ maps $\bigcup_{n\geq 1} L_n$ homeomorphically onto itself.

Observation 5. Let $Y_1, \ldots, Y_n, Y_1, \ldots, Y_n, \ldots$ be an n-cyclic hc-sequence in $\{M, g\}$. Referring back to Definition 1, we note that

$$g(Y_1), Y_1, Y_2, \ldots, Y_n, Y_1, Y_2, \ldots \quad \text{contains a 2−tail hc−sequence,}$$

since the n-cyclic hc-sequence starts in the second coordinate, and

$$g^2(Y_1), g(Y_1), Y_1, Y_2, \ldots, Y_n, Y_1, Y_2, \ldots \quad \text{contains a 3−tail hc−sequence,}$$

since the n-cyclic hc-sequence starts in the third coordinate, and so forth.

From the cyclic nature of the first hc-sequence, we have that the second sequence above is, in fact, a 1-tail sequence since $g|_{Y_1}$ is a homeomorphism. However, the third sequence may not be a 1-tail sequence since we do not know if $g|_{g(Y_1)}$ is a homeomorphism.

The first hc-sequence generates a nested sequence of compacta $\{L_n^1\}$. The second hc-sequence generates a nested sequence of compacta $\{L_n^2\}_2$. The third hc-sequence generates a nested sequence of compacta $\{L_n^3\}_3$, and so on. It follows that for each $i \geq 1$ and $n \geq 1$, $\sigma(L_n^i) = L_{n+1}^{(i+1)(\text{mod } n)}$. So, σ cycles through the sets $\bigcup_{n\geq 1} L_n^1$, $\bigcup_{n\geq 2} L_n^2$, ..., $\bigcup_{n\geq n} L_n^n$.

Observation 6. Let $Y_1, Y_2, Y_1, Y_2, \ldots$ be an alternating hc-sequence in $\{M, g\}$. Then if we define the generated nested sequences $\{L_n^1\}$ and $\{L_n^2\}_2$ as in Observation 5, we see that $\sigma(L_1^1) = L_2^2$, $\sigma(L_2^2) = L_3^1$, $\sigma(L_3^1) = L_4^2$, and so on. Hence, σ maps $\bigcup_{n\geq 1} L_n^1$ onto $\bigcup_{n\geq 2} L_n^2$, and vice versa.

4. Indecomposability of tree-like continua. A continuum X is *decomposable* if it is the union of two nonempty proper subcontinua. Otherwise, X is *indecomposable*. A continuum T is a *triod* if there exists subcontinua A_1, A_2, A_3, and K of T such that $T = A_1 \cup A_2 \cup A_3$, K is a proper subcontinuum of A_i for $i \in \{1, 2, 3\}$, and $K = A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3$. We refer to K as a *core* of T. A *simple triod* is a triod with each A_i an arc and K an endpoint of each A_i. A continuum T is a *tree* if it is a finite union of arcs, each pair of which is either disjoint or meets in a common endpoint, and it contains no simple closed curve.

If $\{T_i, g_i^{i+1}\}$ is an inverse sequence on trees, and we can find a subsequence of pairs of distinct arcs in the factor spaces that "expand" to a previous factor space under the bonding mappings, then the inverse limit space will be indecomposable. This subsequence of arcs, if it exists, should be easy to "spot" while looking for hc-sequences of arcs in $\{T_i, g_i^{i+1}\}$, since the hc-sequences of interest "expand" at each $i \geq k$.

Lemma 3 below is straightforward to verify. Theorem 7 is a direct consequence of D.P. Kuykendall's theorem for indecomposability of an inverse limit space, see **[18]** or **[13,** Theorem 6.4].

Lemma 3. Let A and B be arcs, whose interiors are disjoint, in a tree T . Then the following are equivalent.

(i) $A \cup B$ is contained in either an arc or a simple triod in T.
(ii) Three of the endpoints of A and B have the property that if a and b are two of them, then the arc $[a, b]$ in T contains one of A or B.

Theorem 7. Let $X = \varprojlim\{T_i, g_i^{i+1}\}$, where each T_i is a tree, and each bonding mapping g_i^{i+1} is surjective. Suppose there exist $m \geq 1$, and an increasing subsequence $\{n_i\}$ of $\mathbb{N}$ such that for each $i \geq 1$, Y_{n_i} and Z_{n_i} are arcs in T_{n_i}, where the interiors of Y_{n_i} and Z_{n_i} are disjoint, $Y_{n_i} \cup Z_{n_i}$ is contained in either an arc or a simple triod in T_{n_i}, and $g_{n_i-m}^{n_i}(Y_{n_i}) = T_{n_i-m} = g_{n_i-m}^{n_i}(Z_{n_i})$. Then X is indecomposable.

Proof. We show that Kuykendall's criteria for indecomposability of an inverse limit applies. Let $\epsilon > 0$ and $n \in \mathbb{N}$. Let n_i be large enough so that $n_i - m > n$ and $\frac{1}{2^{n_i-m}} < \epsilon$.

By Lemma 3, we can pick three endpoints p, v, and q of Y_{n_i} and Z_{n_i} having the property in (ii). Suppose that K is a subcontinuum of T_{n_i} containing two of the points p, v, and q. Then K must contain either Y_{n_i} or Z_{n_i}, and hence $g_{n_i-m}^{n_i}(K) = T_{n_i-m}$. So, $g_n^{n_i}(K) = g_n^{n_i-m} g_{n_i-m}^{n_i}(K) = T_n$. Clearly, $d(x, g_n^{n_i}(K)) = 0 < \epsilon$ for each $x \in T_n$. Hence, X is indecomposable. □

Corollary 5. Let $X = \varprojlim\{T, g\}$, where T is a tree, and g is surjective. Suppose that J', J and N', N are expanding pairs of arcs where J' and N' have disjoint interiors, and $J' \cup N'$ is contained in either an arc or a simple triod in T. If there exists $m \geq 1$ such that $g^m(J') = T = g^m(N')$, then X is indecomposable.

Remark 3. We point out that Ingram's triod-like arc continuum with positive span is an example that illustrates Corollary 5. On pages 29 and 30 in **[13]**, Ingram describes the example. In Ingram's notation, the arc from A/4 to A/2 and the $[OA]$-arm form an expanding pair, as do the arc from A/2 to 3A/4 and the $[OA]$-arm. These two expanding pairs satisfy the hypothesis of Corollary 5 with $m = 2$.

We will apply Corollary 5 several times in the examples in Section 6.

5. Retractably arclike continua and their proper subcontinua. In this section, we shift our focus to arclike continua. We investigate, in this setting, what may be considered the converse of our investigation thus far. That is, given a continuum X that is retractably arclike onto a nested sequence of arcs, we are interested in what structural properties of X, other than the equivalences given in Theorem 4, can be deduced.

Given a nested sequence of arcs $\{L_n\}$, we observe that for $m \geq 1$, $\bigcup_{n\geq 1} L_n = \bigcup_{n\geq m} L_n$. Hence, for convenience and simplicity of notation, throughout this section we let $\bigcup L_n$ denote $\bigcup_{n\geq 1} L_n$ for a nested sequence of arcs. For a continuum X and a point $p \in X$, the *composant of* p in X is the union of all proper subcontinua of X that contain p. A *topological ray* in X is a uniquely arcwise connected subset R of X for which there exists a one-to-one mapping from $[0, \infty)$ onto R. A *topological line* in X is a uniquely arcwise connected subset L of X for which there exists a one-to-one mapping from $\mathbb{R}$ onto L.

Suppose X is arclike, $x \in X$, and $\{M_i\}$ is a sequence of arcs in X, each member of which contains x. Since X is hereditarily unicoherent and atriodic, we note that $\bigcup M_i$ is either an arc, a topological ray, or a topological line. So, for $\bigcup M_i$, we choose an order $<$ on $\bigcup M_i$, induced by the order on $\mathbb{R}$, and we use standard interval and ray notation for subcontinua of $\bigcup M_i$.

Observation 7. If X is retractably arclike onto a sequence of arcs $\{L_n\}$, then $\overline{\bigcup L_n} = X$.

Proof. By definition, for each $n \geq 1$, there is a retraction $r_n \colon X \to L_n$, and the sequence $\{r_n\}$ converges uniformly to $\mathrm{id}\colon X \to X$. The observation follows. □

Lemma 4. If the continuum X is retractably arclike at a point x in X, then X is retractably arclike onto a nested sequence of arcs, each member of which contains x.

Proof. Let $\{L_n\}$ and $\{r_n \colon X \to L_n\}$ be given as in the definition. We assume that X is not an arc, since arcs trivially satisfy the conclusion.

Suppose there exists $m \geq 1$ such that $L_m \not\subset L_n$ for $n \geq m$. If there is no such m, then clearly we can pick a nested subsequence of

$\{L_n\}$ having the desired properties. We let $L_m = [a, b]$, with $a < b$ in the induced order on $\bigcup L_n$. For $n > m$, one of a or b is not in L_n. So, one of a or b, say a, is not in infinitely many of the L_n's. Since for any subsequence $\{L_{n_i}\}$, the sequence of retractions $\{r_{n_i}\}$ converges uniformly to id, we have, by Observation 7, that $\overline{\bigcup L_{n_i}} = X$, in which case it follows that $\bigcup L_{n_i}$ is a topological ray with endpoint a. Hence, we assume, without loss of generality, that $a \notin L_n$ for all $n > m$, and that $\bigcup_{n>m} L_n$ is a topological ray with endpoint a.

We pick $n_1 > m$ so that $L_{n_1} \cap (X \setminus [a, b]) \neq \emptyset$. By hypothesis, $x \in L_{n_1}$. Let $L_{n_1} = [a_1, b_1]$. We have that $a < a_1 < x < b < b_1$. We claim that there exists $n_2 > n_1$ such that L_{n_1} is a subset of the interior of L_{n_2}. Since $\bigcup_{n>m} L_n \neq [a, b_1]$, there exists $j > n_1$ such that L_j contains a point $y > b_1$. If for all such L_j, $a_1 \notin L_j$, then as we saw above $\overline{\bigcup_{n\geq j} L_n}$ would be a topological ray with endpoint a_1, and thus $a \notin X$, a contradiction. So, there must be an integer $n_2 > n_1$ as claimed. Now, it is clear that we can choose a nested sequence $\{L_{n_i}\}$ of $\{L_n\}$ so that X is retractably arclike onto $\{L_{n_i}\}$. □

Observation 8. If X is retractably arclike onto a sequence of arcs $\{L_n\}$, N is a subcontinuum of X, and $q \in \bigcup L_n \setminus N$, then there exists an integer m such that $q \notin r_n(N)$ for all $n \geq m$.

Proof. Suppose there is no such m. Note that $d(q, N) > 0$. Pick n large enough so that $d(r_n, \mathrm{id}) < d(q, N)$, and so that $q \in r_n(N)$. Let $u \in N$ such that $r_n(u) = q$. We get that $d(q, u) = d(r_n(u), u) < d(q, N)$, which is a contradiction. □

Theorem 8. Suppose X is retractably arclike onto a nested sequence of arcs $\{L_n\}$. If $y \in X \setminus \bigcup L_n$, and K is a proper subcontinuum containing y, then either $K \cap (\bigcup L_n)$ is empty, or $\overline{K \cap (\bigcup L_n)}$ is a topological ray not equal to $\bigcup L_n$. Furthermore, $K = \overline{K \cap (\bigcup L_n)}$.

Proof. By definition, for each $n \geq 1$, there is a retraction $r_n \colon X \to L_n$ with the sequence $\{r_n\}$ converging uniformly to $\mathrm{id} \colon X \to X$. Suppose $K \cap (\bigcup L_n) \neq \emptyset$. If $\bigcup L_n \subset K$, then by Observation 7, $X = \overline{\bigcup L_n} = \overline{K} = K$. So, K is not a proper subcontinuum, contradicting our assumption. Let $x \in (\bigcup L_n) \setminus K$.

Suppose that $[a,b]$ is an arc in $\bigcup L_n$ containing a point p with $a, b \in K$ and $p \notin K$. Then $K \cup [a,b]$ is not unicoherent, which is a contradiction. So, whenever two points of $\bigcup L_n$ belong to K, the arc in $\bigcup L_n$ between them is a subset of K. Hence, we may choose a linear order $<$ on $\bigcup L_n$ where $x < w = \text{glb}(K \cap (\bigcup L_n))$.

Suppose $z \geq w$ and the possibly degenerate arc $[w,z] = K \cap (\bigcup L_n)$. Then for some $j \geq 1$, $[x,z] \subset L_j$, and $K \cap L_j = [w,z]$. We note that z is an endpoint of L_n for all $n \geq j$, for otherwise, for some $n \geq j$, $K \cup L_n$ would be a triod in X. By Observation 8, there exists an integer m such that $x \notin r_n(K)$ for all $n \geq m$. So, for $n > \max\{j,m\}$, $r_n(K) \subset (x,z]$. Pick $n > \max\{j,m\}$ so that $d(r_n, \text{id}) < d(y,[x,z])$. But then we have that $d(y, r_n(y)) < d(y,[x,z])$ with $r_n(y) \in (x,z]$, which is a contradiction. It follows that for all $t > w$, the arc $[w,t] \subset K$. We write $K \cap (\bigcup L_n) = [w, \infty)$.

It remains to show that $K = \overline{[w,\infty)}$. The right-to-left inclusion is clear. By Observation 8, for large n, $r_n(K) \subset (x,\infty)$. Since $\{r_n\}$ converges uniformly to the identity mapping on X, it follows that each point of K not in $[w,\infty)$ is a limit point of $[w,\infty)$. So, the left-to-right inclusion holds, and the equality is established. □

Theorem 9. Suppose X is retractably arclike onto a nested sequence of arcs $\{L_n\}$, $y \in X \setminus \bigcup L_n$, and K is a proper subcontinuum containing y that meets $\bigcup L_n$. By Theorem 8, we may assume that $K \cap (\bigcup L_n) = [w,\infty)$ for some $w \in \bigcup L_n$, and that $K = \overline{[w,\infty)}$. Let $x \in \bigcup L_n \setminus [w,\infty)$, $[x,w]$ be the unique arc in $\bigcup L_n$ from x to w, and $K' = \bigcap_{t \geq w} \overline{[t,\infty)}$. Then we have the following.

(1) If N is a subcontinuum of $X \setminus [x,w]$ that contains y, then $N \subset K$. Hence, X is not retractably arclike at y.
(2) Either K is the compactification of $[w,\infty)$ with remainder K', or K' is indecomposable and $K = [w,v] \cup K'$ for some $v \geq w$ in $\bigcup L_n$.

Proof. We let $\{r_n \colon X \to L_n\}$ be the sequence of retractions given by the definition.

(1) Suppose $N \not\subset K$. Let $z \in N \setminus K$. Since $[w,\infty) \subset \bigcup L_n$, by Observation 8, we can pick an integer m such that for $n \geq m$, $x \notin r_n(K)$, and $L_n \cap [w,\infty) \neq \emptyset$. For $n \geq m$, $r_n|_{[w,\infty)\cap L_n} = \text{id}$, and

$[w,\infty) \cap L_n \subset K$; so, it follows that $r_n(K) \subset (x,\infty)$. Now choose $n \geq m$ such that $d(r_n, \mathrm{id}) < \min(d(x, K \cup N), d(z, K \cup [x,w])$. It follows that $r_n(z) \notin K \cup [x,w]$. So, $r_n(z) \in (-\infty, x)$. Since $y \in N \cap K$, $r_n(y) \in (x,\infty)$. We have that $x \in r_n(N)$. Let $u \in N$ such that $r_n(u) = x$. So, $u \in K \cup N$, and $d(u, r_n(u)) = d(u,x) < d(x, K \cup N)$, which is a contradiction.

Now suppose that N is an arc containing y. By Theorem 8, $N \cap [x,w] = \emptyset$. So, $N \subset K$. Hence, X is not retractably arclike at y.

(2) We note that K', defined in the hypothesis, is a nonempty subcontinuum of $K = \overline{[w,\infty)}$. We consider two cases.

Case 1. Suppose $K' \cap (\bigcup L_n) = \emptyset$. Then $K' \cap [w,\infty) = \emptyset$, and it follows that $K = K' \cup [w,\infty)$ is a compactification of $[w,\infty)$ with remainder K'.

Case 2. Suppose $K' \cap (\bigcup L_n) \neq \emptyset$. Then by Theorem 8, $K' \cap (\bigcup L_n) = [v,\infty)$ for some $v \geq w$, and $\overline{[v,\infty)} = K' = \bigcap_{t \geq v} \overline{[t,\infty)}$. It follows that $\overline{[t,\infty)} = K'$ for each $t \geq v$. Hence, by Theorem 8, if a subcontinuum N of K' meets $[v,\infty)$ and its complement in K', then $N = K'$. We have that each proper subcontinuum of K' is either a subset of $[v,\infty)$ or a subset of $K' \setminus [v,\infty)$.

Suppose that N is a proper subcontinuum of K' lying in $[v,\infty)$. Then N is an arc, say $N = [a,b]$. For $t > b$, $N \subset \overline{[t,\infty)}$. So, N has empty interior. Suppose that N is a proper subcontinuum of $K' \setminus [v,\infty)$. From the previous paragraph, we have that $K' = \overline{[v,\infty)}$. So, again N has empty interior. It follows that K' is indecomposable and $K = [w,v] \cup K'$. □

Theorem 10. Suppose X is retractably arclike onto a nested sequence of arcs $\{L_n\}$, and X is also retractably arclike at a point not in $\bigcup L_n$. Then X is indecomposable.

Proof. Let $\{r_n \colon X \to L_n\}$ be the sequence of retractions given by the definition. By assumption and Lemma 4, there is also a second nested sequence of arcs $\{J_n\}$ containing a point not in $\bigcup L_n$, and a sequence of retractions $\{\rho_n \colon X \to J_n\}$ with the sequence $\{\rho_n\}$ converging uniformly to $\mathrm{id} \colon X \to X$. By Observation 7, we have that $\overline{\bigcup L_n} = X = \overline{\bigcup J_n}$. It follows from Theorem 8 that $(\bigcup L_n) \cap (\bigcup J_n) = \emptyset$.

Suppose X is decomposable. Let $X = H \cup K$, where H and K are proper subcontinua of X. Since H is a proper subcontinuum of X, we have that $\bigcup L_n \not\subset H$. Similarly, $\bigcup L_n \not\subset K$. We have analogous statements for $\bigcup J_n$. It follows that $H \not\subset \bigcup L_n$, for otherwise, $\bigcup J_n \subset K$. Similarly, $H \not\subset \bigcup J_n$, and analogous statements hold for K. Hence, we have that each of H and K meet each of $\bigcup L_n$ and $\bigcup J_n$.

By Theorem 8, H intersects $\bigcup L_n$ in a ray $[w, \infty)$ for some $w \in \bigcup L_n$. Since $\bigcup L_n \subset H \cup K$, it follows that $(-\infty, w] \subset K$. So, there exists a point $v \geq w$ in $\bigcup L_n$ such that $K \cap \bigcup L_n = (-\infty, v]$, and $H \cap K \cap \bigcup L_n = [w, v]$. Similarly, and without loss of generality, we can choose an order on $\bigcup J_n$ so that $H \cap \bigcup J_n = [w', \infty)$, $K \cap \bigcup J_n = (-\infty, v']$, and $w' \leq v'$ in $\bigcup J_n$. So, $H \cap K \cap \bigcup J_n = [w', v']$. By unicoherence, $H \cap K$ is a proper subcontinuum of X that meets both $\bigcup L_n$ and its complement. This is a contradiction to Theorem 8 since $H \cap K$ meets $\bigcup L_n$ in an arc. □

We recall from the first paragraph of the paper that a continuum X is an arc continuum if every proper subcontinuum of X is an arc.

Corollary 6. If X is everywhere retractably arclike, then X is either an arc, or an indecomposable arc continuum.

Proof. Let K be a proper subcontinuum of X. Let $x \in K$. Since X is retractably arclike at x, X is retractably arclike onto a nested sequence of arcs $\{L_n\}$, each member of which contains x. So, K meets $\bigcup L_n$. Suppose K meets the complement of $\bigcup L_n$. Let $y \in K \setminus \bigcup L_n$. By Theorem 9(1), X is not retractably arclike at y, contradicting our hypothesis. So, $K \subset \bigcup L_n$. Hence, K is an arc.

So, X is an arc continuum, and it follows that either X is an arc or X is indecomposable. □

We note, from Remark 1, that Σ is an indecomposable arc continuum that is nowhere retractably arclike. So, the converse of Corollary 6 does not hold in general.

Question 1. Is every indecomposable arclike, arc continuum everywhere retractably arclike?

Example 2 illustrates that although a continuum X that is retractably arclike onto two distinct nested sequences of arcs is indecomposable, X must be everywhere retractably arclike to ensure that X is an arc continuum.

Example 2. We modify the Horseshoe continuum X of Example 1 so that it becomes retractably arclike everywhere except at points in the composant that contains an endpoint, and that composant will contain a topologist's $\sin(1/x)$-curve. So, the resulting continuum is not an arc continuum.

It should be intuitively clear that this construction can be done, but for the interested reader, we offer more detail. Let R be the composant of X that has an endpoint q. Let $x \in R \setminus \{q\}$. Let U be an open set contianing x such that $q \notin \overline{U}$, and U is homeomorphic to the product of a Cantor set C and the open segment $(0,1)$. So, $\overline{U} \overset{T}{\approx} C \times [0,1]$. Let p be the endpoint of $R \cap \overline{U}$ that separates q from x in R. Now, we replace the arc $\{p\} \times [0,1]$ with a continuum L that contains a $\sin(1/x)$-curve with limit interval, and has endpoints $(p,0)$ and $(p,1)$. All other arcs $\{t\} \times [0,1]$, for $t \in C \setminus \{p\}$, remain arcs, but "limit to" L in such a way that if $d(t,p) < \epsilon$, then there exists a retraction $f\colon L \cup (\{t\} \times [0,1]) \to \{t\} \times [0,1]$, where the endpoints of L are mapped to the endpoints of $\{t\} \times [0,1]$, and $d(x, f(x)) < \epsilon$ for $x \in L$. For exact details of how to accomplish this construction, see Section 3 in **[11]**. Specifically, we are replacing $\overline{U}$ with the S_p-mbox (or the $-S_p$-mbox) described in that reference. Let $\hat{X}$ be the continuum that results from this replacement for $\overline{U}$. Since X is everywhere retractably arclike (recall Corollary 2 and the remark after its proof), it follows that $\hat{X}$ is retractably arclike at each point $x \in \hat{X}$ that is not in the "new composant" $\hat{R}$ that contains q. However, by Theorems 8 and 9(1), $\hat{X}$ is not retractably arclike at points in $\hat{R}$. Also, $\hat{X}$ is not an arc continuum.

We end this section by showing that inverse limits on $[0,1]$ with a single bonding mapping chosen from a certain family of unimodal maps on $[0,1]$ are everywhere retractably arclike onto a nested sequence of arcs. The family of mappings in Theorem 11 are contained in the larger family of mappings on $[0,1]$ considered, in detail, by Ingram and Mahavier in **[15]**. Ingram summarizes properties of this family of mappings in **[13,** pages 45 & 46], although he does not discuss which members admit retractions to arcs that are close to the identity. The two pictured examples on page 37, and the examples discussed in the Remark at the bottom of page 37 in [**13**], are included in the family in Theorem 11 below. To the author's knowledge, heretofore it was not known that continua satisfying the conditions of Theorem 11 are everywhere retractably arclike, or even that the familiar three endpoint indecomposable arclike continuum is everywhere retractably arclike.

Theorem 11. Let $0 < v_1 < v_2 < \ldots < v_n < 1$. Let $g\colon [0,1] \to [0,1]$ be the piecewise linear, unimodal map whose graph contains the points $(0, v_1)$, $(v_n, 1)$, and $(1, 0)$ with $g^i(0) = v_i$, for $1 \leq i \leq n$. Then $X = \varprojlim\{[0,1], g\}$ is everywhere retractably arclike onto a nested sequence of arcs.

Proof. We prove the result for $n = 1$ and $n = 2$. Thereafter, the result will be clear.

For $n = 1$, we let $v_1 = v$ for simplicity of notation. We have that $g(0) = v$, $g(v) = 1$, and $g(1) = 0$. We note that

(a) $([v,1],[0,1])$ is a surjective expanding pair,
(b) $[0,v]$ is homeomorphically covered by $[v,1]$,
(c) $[v,1]$ is homeomorphically covered by both $[0,v]$ and $[v,1]$, and
(d) $g^{-1}([0,v)) \subset [v,1]$.

Let $x = (x_1, x_2, \ldots) \in X$. We define an hc-sequence containing the coordinates of x. Each term of the sequence will be either $[0,v]$ or $[v,1]$. Furthermore, $[v,1]$ will occur at least once in every two terms of the constructed sequence.

Suppose $x_1 \in [0,v)$. Let $Y_1 = [0,v]$, and note that by (d), Y_2 must be chosen to be $[v,1]$ in this case.

Suppose $x_1 \in [v,1]$. Let $Y_1 = [v,1]$. If $x_2 \in [0,v)$, let $Y_2 = [0,v]$, and if $x_2 \in [v,1]$, let $Y_2 = [v,1]$.

We may, for $n \geq 3$, continue the same procedure for choosing Y_n so that it is one of $[0,v]$ or $[v,1]$ and contains x_n. Clearly, by this procedure, $[v,1]$ will occur at least once in every two terms of the constructed sequence.

It follows that $\{Y_n\}$ is an hc-sequence with the properties claimed. It follows from properties (a) through (d), and Corollary 1 that X is retractably arclike onto a nested sequence of arcs containing x. The proof for $n = 1$ is complete.

For $n = 2$, we have that $g(0) = v_1$, $g(v_1) = v_2$, $g(v_2) = 1$, and $g(1) = 0$. We note that

(a) $([v_2, 1], [0, 1])$ is a surjective expanding pair,
(b) $[v_1, v_2]$ is homeomorphically covered by $[0, v_1]$,
(c) $[v_2, 1]$ is homeomorphically covered by $[v_1, v_2]$, and
(d) $g^{-1}([0, v_1)) \subset [v_2, 1]$.

Let $x = (x_1, x_2, \dots) \in X$. We define an hc-sequence containing the coordinates of x. Each term of the sequence will be either $[0, v_1]$, $[v_1, v_2]$ or $[v_2, 1]$. Furthermore, $[v_2, 1]$ will occur in every three terms of the constructed sequence.

Suppose $x_1 \in [0, v_1)$. Let $Y_1 = [0, v_1]$, and $Y_2 = [v_2, 1]$.

Suppose $x_1 \in [v_1, v_2)$. Let $Y_1 = [v_1, v_2]$. If $x_2 \in [0, v_1)$, let $Y_2 = [0, v_1]$, and if $x_2 \in [v_2, 1]$, let $Y_2 = [v_2, 1]$.

Suppose $x_1 \in [v_2, 1]$. Let $Y_1 = [v_2, 1]$. If $x_2 \in [v_1, v_2)$, let $Y_2 = [v_1, v_2]$, and if $x_2 \in [v_2, 1]$, let $Y_2 = [v_2, 1]$.

We may, for $n \geq 3$, continue the same procedure for choosing Y_n.

To see that $[v_2, 1]$ occurs in every three terms of the constructed sequence, suppose for some $n \geq 1$, $Y_n = [0, v_1]$ or $Y_n = [v_1, v_2]$. If $Y_n = [0, v_1]$, then by our chosing procedure, $x_n < v_1$, so $Y_{n+1} = [v_2, 1]$. So, we assume that $Y_n = [v_1, v_2]$. Again, by our procedure, it follows that $v_1 \leq x_n < v_2$, and either $Y_{n+1} = [v_2, 1]$, or $0 \leq x_{n+1} < v_1$, in which case $Y_{n+1} = [0, v_1]$, and Y_{n+2} must be $[v_2, 1]$.

We have that $\{Y_n\}$ is an hc-sequence with the properties claimed. It follows as in the first case that X is retractably arclike onto a nested sequence of arcs containing x. □

6. More examples. In this section, we provide examples, some of which are standard, well known continua, that illustrate the utilility of theorems and techniques developed herein. They show how certain structural properties of inverse limits and their subcontinua can be ascertained by readily observable properties of the bonding mappings in the inverse sequence.

In the first three examples, we revisit examples from **[23]**, **[21]** and **[22]** to see how theorems in this paper give alternative ways to quickly establish the fixed point property for the inverse limits considered, and to observe some other properties of these continua. It will aid the reader to have a copy of the three references, which give definitions, details, and figures.

Example 3. In **[23,** Theorem 4.1], it is shown that the inverse limit X of an inverse sequence on the 2-sphere S^2 with a single bonding map g is retractably disk-like onto a nested sequence of disks; see the proof of Theorem 4.1 beginning with the definition of r, or the first sentence after the proof. The bonding map g, which is defined in the first few lines of §4 in **[23]**, is a 2-to-1 covering map on the longitudinal circles of S^2 with the south pole as its fixed point. The proof uses the idea of Theorem 6(4), although the theorem is not explicitly stated there. It is observed in **[23]** that the bottom quarter of S^2 and the southern hemisphere form an expanding pair. From there, the nested sequence of disks in X is generated by this expanding pair, and for $\epsilon > 0$ it is shown that there is a retraction r of S^2 onto the southern hemisphere such that $d(g(x), gr(x)) < \epsilon$ for each $x \in S^2$. So, Theorem 6(4) applies. As a consequence of being retractably disk-like, X has the fixed point property (recall Theorem 3).

It is of interest to note that X can also be represented as an inverse limit on the projective plane with a single essential (not homotopic to a constant) bonding map.

Example 4. Let X be the inverse limit of the inverse sequence on a simple triod with alternating bonding mappings b_1 and b_2 as defined on page 141 in **[21]**. See Figure 1 for a diagram of b_1 and b_2; the smaller labeled edges are the images of the associated edges. We adopt the same notation used in **[21]**, and additionally, we let $[s,t]_i = \{(r,\theta) \in L_i \mid s \leq r \leq t\}$ for $i = 0, 2, 4$. Let $N' = [0, \frac{1}{3}]_0 \cup L_2 \cup L_4$. We note that (N', T) is a surjective expanding pair of simple triods for each of b_1 and b_2. So, the constant sequence $\{N'\}$ is a surjective hc-sequence in the inverse sequence. It follows from Corollary 1 that X is retractably simple-triod-like onto the nested sequence of simple triods in X generated by $\{N'\}$. So, by Theorem 3, X has the fixed point property.

We further note that $([0,\frac{1}{3}]_0, L_0)$ and $([\frac{1}{3},\frac{2}{3}]_0, L_0)$ are expanding pairs of arcs for each of b_1 and b_2, and that $b_1b_2b_1([0,\frac{1}{3}]_0) = T$ and $b_1b_2b_1([\frac{1}{3},\frac{2}{3}]_0) = T$. Also, $[0,\frac{1}{3}]_0 \cup [\frac{1}{3},\frac{2}{3}]_0$ is contained in an arc. So, for each even $n \geq 4$, three compositions of the bonding mappings applied to the arcs $[0,\frac{1}{3}]_0$ and $[\frac{1}{3},\frac{2}{3}]_0$ give T. Hence, by Theorem 7, X is indecomposable.

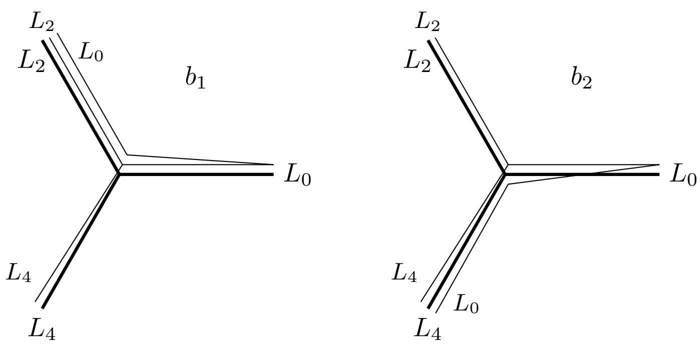

Figure 1. Schematic indications of the alternating bonding maps b_1 and b_2 in the inverse limit in Example 4.

Example 5. Let X and Y be, respectively, the inverse limits defined in Examples 1 and 2 in **[22]**. In both examples, the simple triod T is defined analogously as in **[21]**. So, we use notation as we did in Example 4. See Figure 2 for a diagram of the single bonding maps used in each of the two inverse limits X and Y; the smaller labeled edges are the images of the associated edges.

In the first example, let $N' = [0,\frac{1}{3}]_0 \cup [0,\frac{1}{3}]_2 \cup L_4$, and observe that (N', T) is a surjective expanding pair. Also, the constant sequences $\{[0,\frac{1}{3}]_2\}$ and $\{[\frac{1}{3},\frac{2}{3}]_2\}$ are hc-sequences with $b^3([0,\frac{1}{3}]_2) = T = b^3([\frac{1}{3},\frac{2}{3}]_2)$. So, a similar analysis as in Example 4, gives us that X is indecomposable, retractably simple-triod-like, and has the fixed point property.

In the second example of **[22]**, we have the same triod, but a different bonding mapping b. We note that letting $N' = [\frac{1}{2}, 1]_2$, $(N', L_0 \cup L_2)$ is an expanding pair of arcs with $b^2([\frac{1}{2}, 1]_2) = T$. So, by Theorem 6(2)(a), the union of the nested sequence of arcs in Y generated by the constant sequence $\{N'\}$ is dense in X. But, since $\{N'\}$ is not a surjective hc-sequence, which in fact it cannot be since N' is an arc and T is a triod, we do not know that X is retractably arclike. Nevertheless, we define the retraction $\rho\colon T \to L_0 \cup L_2$ by letting $\rho(r, \frac{4}{3}\pi) = (\frac{1}{4}r, 0)$ for points $(r, \frac{4}{3}\pi) \in L_4$, and observe that $b \circ \rho = b$. Hence, it follows from Theorem 6(4), that, in fact, Y is retractably arclike onto the nested sequence of arcs mentioned above. For the two constant sequences generated by the two expanding pairs $(N', L_0 \cup L_2)$ and $([\frac{1}{4}, \frac{1}{2}]_2, L_2)$, we have that b^3 applied to the first member of each expanding pair gives T. So, as we have seen, this implies that Y is indecomposable.

One can readily get a representation of Y as an inverse limit on arcs by determining the composition mapping $g\colon L_0 \cup L_2 \to L_0 \cup L_2$ defined by $g = \rho \circ b|_{L_0 \cup L_2}$. That the two inverse limit representations of Y give homeomorphic continua follows from Theorem 2.1.38 in **[19]**. Similar analyses can be considered for certain proper subcontinua of Y, e.g. the subcontinua $\varprojlim\{L_0 \cup L_4, g|_{L_0 \cup L_4}\}$ and $\varprojlim\{L_0 \cup [0, \dfrac{1}{3}]_2, g|_{L_0 \cup [0, \frac{1}{3}]_2}\}$.

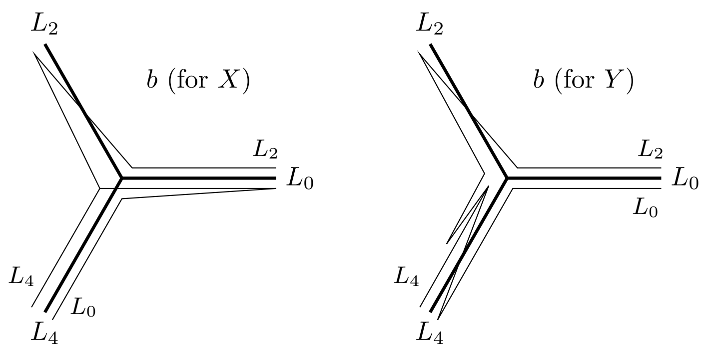

Figure 2. Schematic indications of the bonding map b in the two inverse limits in Example 5.

Example 6. In Remark 1, we commented on the dyadic solenoid $\Sigma = \varprojlim\{S^1, s\}$, where s is the 2-to-1 covering mapping on S^1. Let A be the arc in S^1 with endpoints $(0, -1)$ and $(0, 1)$ and containing $(1, 0)$. Let $p\colon S^1 \to S^1$ be reflection across the x-axis. We consider $X = \varprojlim\{S^1, g\}$, where g is the 2-to-1 mapping on S^1 such that $g = s$ on the arc A, and $g = p \circ s$ on $S^1 \setminus A$. Now, letting A' be the subarc of A with endpoints $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, we note that (A', A) is an expanding pair with $g^2(A') = S^1$. So, the constant hc-sequence $\{A'\}$ generates a nested sequence $\{L_n\}$ of arcs with $\overline{\bigcup_{n\geq 1} L_n} = X$. If we let $r\colon S^1 \to A$ be the retraction that reflects the left side of S^1 across the y-axis onto A, we see that $g \circ r = g$. So, in fact, by Theorem 6(4), X is retractably arclike at each point of $\bigcup_{n\geq 1} L_n$.

As in the second example of Example 5, $X \overset{T}{\approx} \varprojlim\{A, f\}$, where $f = r \circ g|_A$. The graph of f is given in Figure 3, where it is easy to see that X is the union of two "tent map" Horseshoe continua (recall Example 1) with a common endpoint.

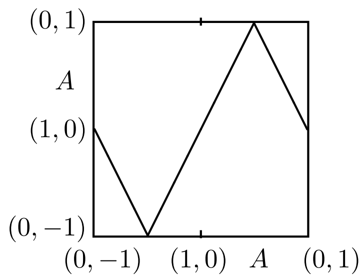

Figure 3. The graph of f

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