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INVERSE LIMITS OF PROJECTIVE SPACES AND THE FIXED POINT PROPERTY

M.M. MARSH

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Abstract. We consider inverse limits of even dimensional projective spaces with essential bonding mappings. J. Segal and T. Watanabe [14] have shown that such inverse limits on complex projective space will have the fixed point property. We obtain some positive fixed point results for both the real and quaternionic projective spaces and we correct an error related to the real projective case in [9].

1. Introduction

Let $\mathbb{E}^n$ denote Euclidean n-space and let $S^n \subseteq \mathbb{E}^{n+1}$ denote the unit sphere. Let $\mathbb{R}P^n$, $\mathbb{C}P^n$, and $\mathbb{H}P^n$ denote, respectively, real, complex, and quaternionic projective space (for definitions, see [15, pages 40, 41, & 42]). A topological space X is said to have the *fixed point property* (fpp) if each mapping (continuous function) f of X to itself has a fixed point; that is, a point $x \in X$ such that $f(x) = x$. It is known that $\mathbb{R}P^n$ and $\mathbb{C}P^n$ have the fpp for even $n \in \mathbb{N}$ and that $\mathbb{H}P^n$ has the fpp for all $n \geq 2$ (see [1, Corollary 15.19]).

We consider inverse limits on even dimensional projective spaces and under what conditions the inverse limit space will have the fpp. A theorem of W. Holsztynski's [4, Corollary 1] says that each inverse limit on ANR's with universal bonding maps (compositions must be universal also) has the fpp. A map $f\colon X \to Y$ is *universal* if it has a coincidence point with each map $g\colon X \to Y$. So, we consider what conditions on the bonding maps and their compositions will make them universal. We review previous positive results for inverse limits on $\mathbb{R}P^{2n}$ [9] and $\mathbb{C}P^{2n}$ [14]. We obtain a positive result for inverse limits on $\mathbb{H}P^n$ when

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$n = 2$. A complete answer for the $\mathbb{R}P^{2n}$ and $\mathbb{H}P^{2n}$ cases, however, remains open since the "degree" of self maps (as bonding maps in the inverse system) on $\mathbb{R}P^{2n}$ and on $\mathbb{H}P^{2n}$ is not well-behaved with respect to the homotopy classes of the maps, as it is in the $\mathbb{C}P^{2n}$ case. In fact, we still do not have a complete answer to D. Bellamy's question [7], "Do inverse limits of the real projective plane with essential bonding maps have the fpp?". We claimed in [9] to have a complete answer to Bellamy's question as well as the general $\mathbb{R}P^{2n}$ case, however, the proof doesn't work for one homotopy class of essential bonding maps. We discuss this in more detail later and prove that if we use one specific representative map from this homotopy class as the bonding map, the inverse limit space will have the fixed point property in this case as well.

A theorem of R. Russo/C. Hagopian (see [12] and [9, Theorem 6]) shows that not all inverse limits of projective spaces will have the fpp, even for inverse limits of even dimensional projective spaces.

2. Some Inverse Limit Theorems

We will need a few general results about inverse limits. This section is included since the author was not able to locate references for the lemmas and theorems herein. Theorems 2.2 and 2.3 may be new results.

Lemma 2.1. *Suppose* $X = \varprojlim\{X_n, b_n^{n+1}\}$, $Y = \varprojlim\{Y_n, g_n^{n+1}\}$, *each with surjective bonding mappings. Let* $f_n \colon X_n \to Y_n$ *be a sequence of surjective maps such that* $g_n^{n+1} \circ f_{n+1} = f_n \circ b_n^{n+1}$ *for all* $n \geq 1$.
(1) The induced map $f \colon X \to Y$ *is surjective.*
(2) If, for all $n \geq 1$, b_n^{n+1} *is constant on the set* $f_{n+1}^{-1}(y)$ *for each* $y \in Y_{n+1}$, *then* f *is injective.*

Proof. (1) Let b_n and g_n denote respectively the projection maps of X and Y onto X_n and Y_n. Let $y \in Y$. Since for each $n \geq 1$, b_n and f_n are surjective, we can pick a sequence of points $\{z_n\}$ such that $z_n \in b_n^{-1} f_n^{-1} g_n(y)$. So, $f_n b_n(z_n) = g_n(y)$ for $n \geq 1$. We note that for $1 \leq i \leq n$,
$(*)$ $\quad f_i b_i(z_n) = f_i b_i^n b_n(z_n) = g_i^n f_n b_n(z_n) = g_i^n g_n(y) = g_i(y)$.

Let $x = \lim_{n\to\infty} z_{u_n}$, for some convergent subsequence of $\{z_n\}$. So, $x \in X$. We will show that $f(x) = y$. Fix $n \geq 1$ and consider the point $f(z_{u_n})$. For $1 \leq i \leq u_n$, we get that $g_i f(z_{u_n}) = f_i b_i(z_{u_n}) = g_i(y)$ from $(*)$. So, $f(z_{u_n})$ and y agree in all coordinates $1 \leq i \leq u_n$. It follows that the distance from $f(z_{u_n})$ to y is less than $\frac{1}{2^{u_n}}$. Hence, $\lim_{n\to\infty} f(z_{u_n}) = y$. Also, $\lim_{n\to\infty} f(z_{u_n}) = f(x)$. Therefore, $f(x) = y$.

(2) Let $x, y \in X$ and suppose that $f(x) = f(y)$. Then $g_n f(x) = g_n f(y)$ for all $n \geq 1$. Since f is induced, we have that $f_n b_n(x) = f_n b_n(y)$ for all $n \geq 1$. Hence, for $n+1 \geq 2$, $b_{n+1}(x), b_{n+1}(y) \in f_{n+1}^{-1}(f_{n+1} b_{n+1}(x))$. By hypothesis, we get that for all $n \geq 1$, $b_n(x) = b_n^{n+1} b_{n+1}(x) = b_n^{n+1} b_{n+1}(y) = b_n(y)$. Therefore, $x = y$. □

Theorem 2.2. *For X, Y, and $\{f_n\}$ satisfying the conditions of Lemma 1, the induced map f is a homeomorphism from X onto Y.*

Let M be a continuum and $g \colon M \to M$ be a surjective map. If N is a subset of M such that $g(N) \subseteq N$, then N is said to be *invariant* (with respect to g). We will say that N is *outvariant* (with respect to g) if $N \subseteq g(N)$. Furthermore, if N is outvariant and there exists a subset N' of N such that $g(N') = N$ and $g|_{N'}$ is a homeomorphism, we will say that (N', N) is an *expanding pair* (with respect to g). W.T. Ingram [5] and others have investigated the nature of inverse limits on a single continuum M with a single bonding mapping g when g leaves certain subsets invariant. In particular in [5], Ingram studies the inverse limit space when the invariant subsets are finite and the bonding mapping is Markov. We will show that, for inverse limits on a single factor space with one bonding map, outvariant sets can also be helpful in understanding the inverse limit space. In fact, the existence of an expanding pair of subcontinua (N', N) in M gives rise to a monotonic increasing sequence of homeomorphic copies of N in the inverse limit space. If the k-fold composition $g^k(N) = M$ for some $k \geq 1$, the closure of the union of the sequence of subcontinua is the inverse limit space. In addition, if there is a retraction of M onto N that behaves nicely with respect to g, then there is a retraction close to the identity map of the inverse limit space onto a homeomorphic copy of N. We will see an example of this in the proof of Theorem 4.1.

Theorem 2.3. *Let $X = \varprojlim\{M, g\}$, where M is a continuum and g is a surjective map. Suppose that (N', N) is an expanding pair of subcontinua with respect to g. Then*
(1) X contains a monotonic increasing sequence of subcontinua $\{N_n\}_{n \geq 1}$, each member of which is homeomorphic to N under the restriction of the projection mapping g_n, and
(2) if there is a $k \geq 1$ such that $g^k(N) = M$, then $\overline{\bigcup_{n \geq 1} N_n} = X$.

Proof. (1) Let $n \geq 1$. Let $L_n^n = N$, $L_i^n = g^{n-i}(N)$ for $1 \leq i \leq n$, and $L_i^n = (g|_{N'})^{n-i}(N)$ for $i > n$. Note that $L_{n+1}^n = N'$ and that for $i > n$, $(g|_{N'})^{n-i}$ is a homeomorphism. It follows that L_i^n is homeomorphic to $L_n^n = N$ for all $i > n$. We also note that $L_{i+1}^n \subseteq L_i^n$ for all $i \geq 1$.

Define the subset N_n of X by $N_n = \varprojlim\{L_i^n, g|_{L_i^n}\}$. We observe that $g_n|_{N_n}\colon N_n \to N$ is a homeomorphism for each $n \geq 1$. To see this, first notice that $g_n(N_n) = L_n^n = N$. Suppose $x, y \in N_n$ and $g_n(x) = g_n(y)$. For $1 \leq i \leq n$, $g_i(x) = g^{n-i}g_n(x) = g^{n-i}g_n(y) = g_i(y)$, and for $i > n$, $(g|_{L_i^n})^{i-n}(g_i(x)) = g_n(x) = g_n(y) = (g|_{L_i^n})^{i-n}(g_i(y))$. Since $(g|_{L_i^n})^{i-n}$ is a homeomorphism, it follows that $g_i(x) = g_i(y)$ for $i > n$ also. Thus, we have that $x = y$, establishing that g_n maps N_n homeomorphically onto N. It is easy to see that $N_n \subseteq N_{n+1}$ for each $n \geq 1$ since $L_{i+1}^n \subseteq L_i^n$ for all $i \geq 1$.

(2) We let $\beta_n\colon N \to N_n$ be the inverse homeomorphism of $g_n|_{N_n}$. Let $x \in X$ and $\epsilon > 0$. Let n be large enough so that g_{n-k} is an ϵ-map. There is a point $y_n \in N$ such that $g^k(y_n) = g_{n-k}(x)$. Let $y = \beta_n(y_n)$. Then $g_{n-k}(y) = g^k g_n(y) = g^k(y_n) = g_{n-k}(x)$. Since g_{n-k} is an ϵ-map, the distance from x to y is less than ϵ. Thus, x is a limit point of $\bigcup_{i\geq 1} N_i$. The equality follows. □

3. Projective Spaces, Inverse Limits, and Fixed Points

For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, we recall some facts about the homology/cohomology of $\mathbb{F}P^n$. If $\mathbb{F} = \mathbb{R}$, we use $\mathbb{Z}_2$ coefficients. If $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{H}$, we use $\mathbb{Z}$ coefficients and pass via the universal coefficient theorem to rational coefficients $\mathbb{Q}$ when we consider the Lefschetz coincidence number of a pair of mappings.

First, the homology/cohomology groups are (see [16, page 90])

$$H_i(\mathbb{R}P^n;\mathbb{Z}_2) \approx H^i(\mathbb{R}P^n;\mathbb{Z}_2) \approx \begin{cases} \mathbb{Z}_2 & \text{for } 0 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Refering to [15, page 40] and using Poincaré-Lefschetz duality,

$$H_i(\mathbb{C}P^n;\mathbb{Z}) \approx H^i(\mathbb{C}P^n;\mathbb{Z}) \approx \begin{cases} \mathbb{Z} & \text{for } i \equiv 0 (\mathrm{mod} 2) \text{ and } 0 \leq i \leq 2n, \\ 0 & \text{otherwise.} \end{cases}$$

$$H_i(\mathbb{H}P^n;\mathbb{Z}) \approx H^i(\mathbb{H}P^n;\mathbb{Z}) \approx \begin{cases} \mathbb{Z} & \text{for } i \equiv 0 (\mathrm{mod} 4) \text{ and } 0 \leq i \leq 4n, \\ 0 & \text{otherwise.} \end{cases}$$

Let α be a generator of $H_1(\mathbb{R}P^n;\mathbb{Z}_2)$ (respectively, of $H_2(\mathbb{C}P^n;\mathbb{Z})$ and of $H_4(\mathbb{H}P^n;\mathbb{Z})$), then α^k is a generator of $H_k(\mathbb{R}P^n;\mathbb{Z}_2)$ (respectively, of $H_{2k}(\mathbb{C}P^n;\mathbb{Z})$ and of $H_{4k}(\mathbb{H}P^n;\mathbb{Z})$) for $1 \leq k \leq n$. The image $\overline{\alpha}$ of α under the duality isomorphism is a generator of the corresponding cohomology groups; and similarly for $\overline{\alpha}^k$. It follows that the cohomology rings of $\mathbb{R}P^n$, $\mathbb{C}P^n$, and $\mathbb{H}P^n$ are truncated polynomial rings given by

$$H^*(\mathbb{R}P^n;\mathbb{Z}_2) \approx \mathbb{Z}_2[\overline{\alpha}]/(\overline{\alpha}^{n+1}),$$

$$H^*(\mathbb{C}P^n;\mathbb{Z}) \approx \mathbb{Z}[\overline{\alpha}]/(\overline{\alpha}^{n+1}),\ \text{and}$$
$$H^*(\mathbb{H}P^n;\mathbb{Z}) \approx \mathbb{Z}[\overline{\alpha}]/(\overline{\alpha}^{n+1})\ ,$$

(see [15, Corollaries 6.29, 6.32, & 6.33]).

For $\mathbb{C}$ and $\mathbb{H}$, we use the universal coefficient theorem to pass to rational coefficients. If β is the image of $\overline{\alpha}$ under the coefficient homomorphism, then we have that, for $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$, $H^*(\mathbb{F}P^n;\mathbb{Q}) \approx \mathbb{Q}[\beta]/(\beta^{n+1})$, see [15, page 202].

Given a map $f\colon \mathbb{F}P^n \to \mathbb{F}P^n$, define $\deg f$ to be the unique integer m (integer "mod 2" for $\mathbb{F} = \mathbb{R}$) such that $f_*(\alpha) = m\alpha$, where $f_*\colon H_j(\mathbb{F}P^n;\mathbb{Z}) \to H_j(\mathbb{F}P^n;\mathbb{Z})$ is the induced homomorphism ($j = 1, 2, 4$ for the three cases, and $\mathbb{Z}$ replaced with $\mathbb{Z}_2$ when $j = 1$).

From the ring structure and the comments above, it follows that $f^*(\overline{\alpha}^k) = m^k\overline{\alpha}^k$, where $f^*\colon H^k(\mathbb{R}P^n;\mathbb{Z}_2) \to H^k(\mathbb{R}P^n;\mathbb{Z}_2)$ is the induced homomorphism for $1 \le k \le n$, and $f^*(\beta^k) = m^k\beta^k$, where $f^*\colon H^{2k}(\mathbb{C}P^n;\mathbb{Q}) \to H^{2k}(\mathbb{C}P^n;\mathbb{Q})$ or $f^*\colon H^{4k}(\mathbb{H}P^n;\mathbb{Q}) \to H^{4k}(\mathbb{H}P^n;\mathbb{Q})$ for $1 \le k \le n$.

For a pair of maps $f, g\colon M_1 \to M_2$, where M_1 and M_2 are closed, oriented n-manifolds, we define the Lefschetz coincidence number of f and g (see [15, page 188]) by

$$L(f,g) = \Sigma_{q=0}^n (-1)^q \mathrm{tr}\ \theta_q,$$

where $\theta_q = \mu g^* \nu^{-1} f_*$, and μ and ν are Poincaré-Lefschetz duality isomorphisms (see diagram below).

$$\begin{array}{ccc} H_q(M_1;\mathbb{Q}) & \xrightarrow{f_*} & H_q(M_2;\mathbb{Q}) \\ {\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle\nu} \\ H^{n-q}(M_1;\mathbb{Q}) & \xleftarrow{g^*} & H^{n-q}(M_2;\mathbb{Q}) \end{array}$$

Since $\mathbb{C}P^n$ and $\mathbb{H}P^n$ are finite dimensional closed oriented manifolds, $L(f,g)$ is defined for maps on these projective spaces. For $\mathbb{R}P^n$, if we use $\mathbb{Z}_2$ coefficients, then Poincaré-Lefschetz duality still applies and the "mod 2" Lefschetz coincidence number $L(f,g)$ is defined analogously.

Hence, for maps $f, g\colon \mathbb{F}P^n \to \mathbb{F}P^n$ with $\deg f = m$ and $\deg g = k$, we calculate $L(f,g)$ getting that

$$L(f,g) = k^n + k^{n-1}m + k^{n-2}m^2 + \ldots + km^{n-1} + m^n.$$

In the case of $\mathbb{F} = \mathbb{R}$, all terms are non-negative since k and m are "mod 2" integers. It follows that

$$L(f,g) = \begin{cases} \frac{k^{n+1}-m^{n+1}}{k-m} & \text{if } k \neq m, \\ (n+1)m^n & \text{if } k = m. \end{cases}$$

Note that for $m \neq 0$ and n even, $L(f, g) \neq 0$; for n odd, $L(f, g)$ will be non-zero only in case $m \neq -k$.

Theorem 3.1. *If X is an inverse limit on $\mathbb{F}P^n$ with n even and the bonding maps having non-zero degree, then X has the fpp.*

Proof. Suppose $f\colon \mathbb{F}P^n \to \mathbb{F}P^n$ is either a bonding map or a composition of bonding maps in the inverse limit sequence. Since each bonding map has non-zero degree and degree of compositions is product of degrees, it follows that $\deg f = m \neq 0$. If $g\colon \mathbb{F}P^n \to \mathbb{F}P^n$ is a mapping, from the comments above concerning $L(f, g)$ and by [15, Corollary 7.14], f and g have a coincidence point. So, we have that f is universal. Thus, Holsztynski's result applies, and the theorem follows. □

If the degree function $\deg\colon [\mathbb{F}P^n; \mathbb{F}P^n] \to \mathbb{Z}$ ($\mathbb{Z}_2$ for $\mathbb{F} = \mathbb{R}$) is injective, then in the hypothesis of Theorem 3.1, we can replace the assumption that the bonding maps have non-zero degree with the assumption that they are essential (not homotopic to a constant). This is the case for $\mathbb{C}$, but not for $\mathbb{R}$ and $\mathbb{H}$. Segal and Watanabe [14] proved in 1992 that the degree function is injective on $[\mathbb{C}P^n; \mathbb{C}P^n]$ and used exactly the ideas above to establish

Theorem 3.2 (Segal and Watanabe)**.** *If X is an inverse limit on an even dimensional complex projective space with essential bonding maps, then X has the fpp.*

With regard to $\mathbb{H}P^n$, Marcum and Randall [8] have shown that there exist (for $n = 3, 4, 5$) essential self maps of $\mathbb{H}P^n$ having degree zero. They conjecture that such self maps of $\mathbb{H}P^n$ exist for all $n \geq 3$. Corollary 2 of [8] states that if $f\colon \mathbb{H}P^2 \to \mathbb{H}P^2$ is essential, then $\deg f \neq 0$. Thus, by Theorem 3.1, for $n = 2$ we have the analogous result of Theorem 3.2 for $\mathbb{H}P^n$.

The author acknowledges helpful conversations about the $\mathbb{H}P^n$ case with Michael Colvin.

Theorem 3.3. *If X is an inverse limit on $\mathbb{H}P^2$ with essential bonding maps, then X has the fpp.*

In [9, Theorem 5], we claimed to have proven that each inverse limit on an even dimensional real projective space with essential bonding maps has the fpp. The proof was based on a theorem in [10, Theorem 2.1] which states that there are only two homotopy classes of self maps on $\mathbb{R}P^n$ for n even. However, this is not the case; so the proof is incorrect. If Theorem 5 in [9] is replaced with Theorem 3.4 below, then the error is corrected. Theorem 3.4 provides a positive answer to Bellamy's question for all but one homotopy class of essential maps on $\mathbb{R}P^{2n}$. The proof of Theorem 3.4 follows from Theorem 4 and Corollary 1 in [9], and from Holsztynski's result.

Theorem 3.4. *Let X be an inverse limit on $\mathbb{R}P^{2n}$ with bonding mappings b_i^{i+1} that have essential covering lifts $g_i^{i+1}\colon S^{2n} \to S^{2n}$. Then X has the fixed point property.*

Let $p\colon S^n \to \mathbb{R}P^n$ denote covering projection and let $T\colon S^n \to S^n$ denote the antipodal map. Using vector space notation, we may also write $-x$ for $T(x)$. If $f\colon S^n \to S^n$ is a mapping, we say that f *induces a map* $\tilde{f}\colon \mathbb{R}P^n \to \mathbb{R}P^n$ if there exists a map $\tilde{f}$ such that $p \circ f = \tilde{f} \circ p$. We say that $f\colon S^n \to S^n$ *collapses antipodes (preserves antipodes)* if $f = f \circ T$ ($f \circ T = T \circ f$). It is easy to see that if $f\colon S^n \to S^n$ is a mapping that either collapses or preserves antipodes, then f induces a map $\tilde{f}\colon \mathbb{R}P^n \to \mathbb{R}P^n$. Furthermore, if $\tilde{f}\colon \mathbb{R}P^n \to \mathbb{R}P^n$ is a map, then $\tilde{f}$ has a covering lift f so that $p \circ f = \tilde{f} \circ p$ and f is either antipode preserving or antipode collapsing.

The set $[\mathbb{R}P^{2n}; \mathbb{R}P^{2n}]$ of homotopy classes of self mappings of $\mathbb{R}P^{2n}$ is, in fact, infinite (see [6] for the projective plane and [11] for the general case). We thank Robert Edwards for providing these two references. For the moment, we restrict our attention to $\mathbb{R}P^2$. According to B. Jiang in [6] (see top of page 2), the homotopy classification of self maps $\tilde{f}$ of $\mathbb{R}P^2$ is determined by considering lifts of $\tilde{f}$ and $\tilde{f} \circ p$ as $\tilde{f}_*\colon H_1(\mathbb{R}P^2) \to H_1(\mathbb{R}P^2)$ is trivial or not. Specifically, if $\tilde{f}_* \neq 0$ and $f\colon S^2 \to S^2$ is a lift of $\tilde{f} \circ p\colon S^2 \to \mathbb{R}P^2$, then the homotopy class of $\tilde{f}$ is classified by the degree of f, which will be an odd natural number. If $\tilde{f}_* = 0$ and $f'\colon \mathbb{R}P^2 \to S^2$ is a lift of $\tilde{f}\colon \mathbb{R}P^2 \to \mathbb{R}P^2$, then the homotopy class of $\tilde{f}$ is classified by the "mod 2" degree of f'. The situation is analogous for higher even dimensional real projective spaces. If the lift f is antipode preserving, then the degree of f is odd (see [17, Th.3]) and if f is antipode collapsing, then the map $f'\colon \mathbb{R}P^{2n} \to S^{2n}$ is defined by the equation $f' \circ p = f$ and is a lift of $\tilde{f}$, that is, $p \circ f' = \tilde{f}$. In this case, there are two homotopy classes determined by the "mod 2" degree of f', see Corollary to Theorem IIIb and Remark 9.1 in [11].

If we consider the general question, "For which homotopy classes $[g]$ of self maps g on $\mathbb{R}P^{2n}$ will the inverse limit space on $\mathbb{R}P^{2n}$ with bonding mappings from $[g]$ have the fpp?", then Theorems 3.1 and 3.4 provide a positive answer for all but two homotopy classes. In fact, the bonding maps may be from different homotopy classes as long as no more than finitely many are from these remaining two. They are the null-homotopic (inessential) maps and maps $g\colon \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ which are essential, yet induce the trivial homomorphism on $H_1(\mathbb{R}P^{2n})$. As stated earlier, inverse limits on $\mathbb{R}P^{2n}$ with inessential bonding mappings will, in general, not have the fpp. So, hereafter we will primarily be concerned with the second homotopy class mentioned above.

Specifically, if $\tilde{f}\colon \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ is a map such that $\tilde{f}_* = 0$ and $\tilde{f}$ lifts to $f'\colon \mathbb{R}P^{2n} \to S^{2n}$ where the "mod 2" degree of f' is non-trivial, then this case is unsettled as to the fpp of an inverse limit on $\mathbb{R}P^{2n}$ with maps from this homotopy class. We will provide a positive answer for this case as well if only one specific representative map is used as the bonding map or if each of the bonding maps are of a certain form.

4. The $\mathbb{R}P^2$ Case

In [6, page 224], Jiang uses the longitude/latitude parametrization $e(\phi,\theta) = (\cos\phi\cos\theta, \sin\phi\cos\theta, \sin\theta)$ of $S^2 \subseteq \mathbb{E}^3$ to define the map $F_2\colon S^2 \to S^2$ by $F_2(\phi,\theta) = (\phi, 2\theta + \frac{\pi}{2})$. Hereafter, we will denote the map F_2 by g. In Cartesian coordinates, the map g is given by $g(x,y,z) = (-2xz, -2yz, 1-2z^2)$. It is easy to check that g collapses antipodes. Note also that g is inessential, yet induces an essential map $\tilde{g}\colon \mathbb{R}P^2 \to \mathbb{R}P^2$. As pointed out by Jiang [6] (see top of page 224 and the definitions of the maps f_m in "Construction of good representatives"), $\tilde{g}$ lifts to a map $g'\colon \mathbb{R}P^2 \to S^2$ which has "mod 2" degree equal 1 and hence, $\tilde{g}$ is not null-homotopic (that is, $\tilde{g}$ is essential).

Let $X = \varprojlim\{S^2, g\}$ and $Y = \varprojlim\{\mathbb{R}P^2, \tilde{g}\}$. Let $\hat{p}\colon X \to Y$ be the map induced by the covering projection $p\colon S^2 \to \mathbb{R}P^2$. Since g collapses antipodes, it follows from Theorem 2.2 that $\hat{p}$ is a homeomorphism. Hence, we will show that Y has the fpp by showing that X has the fpp. First we wish to understand what the inverse limit space X (and Y) looks like.

For $0 \le \phi \le \pi$, let $S_\phi = \{e(\phi,\theta) \mid \frac{-\pi}{2} \le \theta \le \frac{3\pi}{2}\}$. Then S_ϕ is a great circle on S^2 passing through the points $(0,0,1)$ and $(0,0,-1)$ given in Cartesian coordinates. We observe that g leaves S_ϕ invariant for each ϕ, and in fact, $g^{-1}(S_\phi) = S_\phi$. Furthermore, if we identify S_ϕ with the unit circle S^1 through the homeomorphism $\alpha(\phi,\theta) = \exp(i(\theta + \frac{\pi}{2}))$, it is easy to check that $g|_{S_\phi}$ is the squaring map. Hence, for each S_ϕ, $\varprojlim\{S_\phi, g|_{S_\phi}\}$ is a dyadic solenoid in X. These dyadic solenoids have the point $\langle (0,\frac{-\pi}{2}), (0,\frac{-\pi}{2}), (0,\frac{-\pi}{2}), \ldots\rangle$ of X in common.

Theorem 4.1. $X = \varprojlim\{S^2, g\}$ *has the fixed point property.*

Proof. Suppose that $f\colon X \to X$ is a fixed point free mapping on X. Let ρ denote the geodesic metric on S^2 and d denote the metric on X inherited from the product space. Let $\epsilon > 0$ be chosen so that $d(x, f(x)) \ge \epsilon$ for all $x \in X$.

Let D be the "southern hemisphere" of S^2. That is, $D = \{(\phi, \theta) \mid \frac{-\pi}{2} \leq \theta \leq 0\}$ and let $D' = \{(\phi, \theta) \mid \frac{-\pi}{2} \leq \theta \leq \frac{-\pi}{4}\}$. We note that (D', D) is an expanding pair with respect to g. Let D_n be the sequence of subcontinua of X as given by (1) in Theorem 2.3. Let n be large enough so that g_{n-1} is an ϵ-map. Choose $\frac{\pi}{2} > \gamma > 0$ so that $d(x, y) \geq \epsilon$ implies that $\rho(g_{n-1}(x), g_{n-1}(y)) \geq \gamma$.

Let $E = \{(\phi, \theta) \mid \frac{-\gamma}{8} \leq \theta \leq \frac{\gamma}{8}\}$. Then E is a $\frac{\gamma}{4}$-annulus on S^2 with the equator as centerline. Suppose that $x = (\phi, \theta)$ and $y = (\phi', \theta')$ are points of E. Then $g(\phi, \theta) = (\phi, 2\theta + \frac{\pi}{2})$ and $g(\phi', \theta') = (\phi', 2\theta' + \frac{\pi}{2})$. Now,

$$\frac{\pi}{2} - \frac{\gamma}{4} \leq 2\theta + \frac{\pi}{2} \leq \frac{\pi}{2} + \frac{\gamma}{4} \quad \text{and} \quad \frac{\pi}{2} - \frac{\gamma}{4} \leq 2\theta' + \frac{\pi}{2} \leq \frac{\pi}{2} + \frac{\gamma}{4}.$$

Thus,

$$\rho(g(\phi, \theta), (\phi, \frac{\pi}{2})) \leq \frac{\gamma}{4} \quad \text{and} \quad \rho(g(\phi', \theta'), (\phi, \frac{\pi}{2})) \leq \frac{\gamma}{4}.$$

By the triangle inequality, $\rho(g(\phi, \theta), g(\phi', \theta')) \leq \frac{\gamma}{2} < \gamma$. We have shown that

$(**)$ for points $x, y \in E$, $\rho(g(x), g(y)) < \gamma$.

We now define a retraction of S^2 onto D that behaves nicely with respect to the bonding mapping g.

Let $r\colon S^2 \to D$ be defined by

$$r(\phi, \theta) = \begin{cases} (\phi, \theta) & \text{for } \frac{-\pi}{2} \leq \theta \leq 0, \\ (\frac{8\theta\pi}{\gamma} + \phi, -\theta) & \text{for } 0 \leq \theta \leq \frac{\gamma}{8}, \\ (\phi + \pi, -\theta) & \text{for } \frac{\gamma}{8} \leq \theta \leq \frac{\pi}{2}. \end{cases}$$

For $\theta = 0$, $r(\phi, 0) = (0 + \phi, 0) = (\phi, 0)$; so, the two rules agree. For $\theta = \frac{\gamma}{8}$, $r(\phi, \frac{\gamma}{8}) = (\pi + \phi, -\frac{\gamma}{8})$; again, the two rules agree.

Let $\beta_n\colon D \to D_n$ be the inverse homeomorphism of $g_n|_{D_n}$. Define $r_n\colon X \to D_n$ by $r_n = \beta_n r g_n$. We claim that r_n is a retraction and that $d(r_n, id) < \epsilon$. Suppose that $x \in D_n$. Then $g_n(x) \in D$ and $r_n(x) = \beta_n r g_n(x) = \beta_n g_n(x) = x$. So, r_n is a retraction. Let $x \in X$ and suppose that $g_n(x) = (\phi, \theta)$. There are three cases to consider according to the definition of r.

Case *i)* If $g_n(x) \in D$, then $r_n(x) = \beta_n r g_n(x) = \beta_n g_n(x)$. So, $g_n r_n(x) = g_n \beta_n g_n(x) = g_n(x)$. Since g_n is an ϵ-map, $d(r_n(x), x) < \epsilon$.

Case *ii)* If $g_n(x) \in (E - D)$ with $0 < \theta \leq \frac{\gamma}{8}$, then $r_n(x) = \beta_n r g_n(x) = \beta_n r(\phi, \theta) = \beta_n(\frac{8\theta\pi}{\gamma} + \phi, -\theta)$. So, $g_n r_n(x) = g_n \beta_n(\frac{8\theta\pi}{\gamma} + \phi, -\theta) = (\frac{8\theta\pi}{\gamma} + \phi, -\theta)$ since $(\frac{8\theta\pi}{\gamma} + \phi, -\theta)$ lies in D. So, $g_n(x)$ and $g_n r_n(x)$ are two points lying in E. By $(**)$, $\rho(g g_n(x), g g_n r_n(x)) < \gamma$; so $\rho(g_{n-1}(x), g_{n-1} r_n(x)) < \gamma$. Hence, $d(x, r_n(x)) < \epsilon$.

Case *iii)* If $g_n(x) \in (S^2 - D)$ with $\frac{\gamma}{8} \leq \theta \leq \frac{\pi}{2}$, then $r_n(x) = \beta_n r g_n(x) = \beta_n r(\phi, \theta) = \beta_n(\phi + \pi, -\theta)$. We note that $(\phi + \pi, -\theta)$ is antipodal to (ϕ, θ). So,

$gg_nr_n(x) = gg_n\beta_n(\phi+\pi,-\theta) = g(\phi+\pi,-\theta) = gg_n(x)$ since g collapses antipodal points. Thus, $g_{n-1}(x) = g_{n-1}r_n(x)$ and therefore, $d(x, r_n(x)) < \epsilon$.

Our claim is proven.

Since D_n is a disk, the map $r_nf|_{D_n}: D_n \to D_n$ must have a fixed point $x \in D_n$. But since $d(r_n, id) < \epsilon$, it follows that $d(x, f(x)) < \epsilon$, a contradiction. □

We note that X is disk-like since for each $\epsilon > 0$, there exists an $n \geq 1$ such that r_n is an ϵ-map onto a disk. Since X is homeomorphic to Y, we have that Y is a disk-like continuum which is also an inverse limit of projective planes with essential bonding maps, somewhat surprising to this author.

It is also of interest to note that X can be realized as an inverse limit of disks with universal bonding maps. Thus, the fact that X (and Y) has the fixed point property also follows from W. Holsztynski's result mentioned in the introduction. If Z, W, and Q are metric spaces and $f: Z \to W$ is a mapping, we say that f *can be almost factored through* Q if for each $\gamma > 0$, there exist maps $f_1: Z \to Q$ and $f_2: Q \to W$ such that $d(f(z), f_2f_1(z)) < \gamma$ for each $z \in Z$.

Theorem 4.2. *X can be realized as an inverse limit of topological disks with universal bonding mappings.*

Proof. Let D be the topological disk defined in the proof of Theorem 4.1. We will first show that $g: S^2 \to S^2$ can be almost factored through D. Let $\frac{\pi}{2} > \gamma > 0$. Define the annulus $E \subseteq S^2$ and the retraction $r: S^2 \to D$ as in the proof of Theorem 4.1. Let $\hat{g} = g|_D: D \to S^2$. We point out that g and $\hat{g} \circ r$ are $\gamma - close$. If $(\phi,\theta) \in D$, then $\hat{g}r(\phi,\theta) = \hat{g}(\phi,\theta) = g(\phi,\theta)$. If $(\phi,\theta) \in S^2 - (D \cup E)$, then $\hat{g}r(\phi,\theta) = \hat{g}(\phi+\pi,-\theta) = g(\phi+\pi,-\theta) = g(\phi,\theta)$ since g collapses antipodes. Finally, if $(\phi,\theta) \in E$, then $r(\phi,\theta) \in E \cap D$ and by $(**)$ in the proof of Theorem 4.1, $\rho(\hat{g}r(\phi,\theta), g(\phi,\theta)) < \gamma$. Hence, g can be almost factored through D.

It follows from M. Brown's [2, Th.3] approximation theorem that X is homeomorphic to $\varprojlim\{S^2, \hat{g}r_i\}$, where the r_i retractions of S^2 onto D are chosen so that each pair g and $\hat{g}r_i$ is γ_i-close and each γ_i is sufficiently small to satisfy the conditions of Brown's theorem. So, X is homeomorphic to the inverse limit of

$$S^2 \xleftarrow{\hat{g}} D \xleftarrow{r_1} S^2 \xleftarrow{\hat{g}} D \xleftarrow{r_2} S^2 \xleftarrow{\hat{g}} \ldots\ldots$$

It is well known and straightforward to prove that the inverse limit above is homeomorphic to the following inverse limit.

$$D \xleftarrow{r_1\hat{g}} D \xleftarrow{r_2\hat{g}} D \xleftarrow{r_3\hat{g}} D \xleftarrow{r_4\hat{g}} \ldots\ldots$$

There remains only to show that the maps $r_i\hat{g}$ (and their compositions) are universal. A theorem of Holsztynski's [4, Prop.10] gives us that each $r_i\hat{g}$ is universal if its restriction to the preimage of the boundary of D is essential. We consider $(r_i\hat{g})^{-1}(\partial D) = \hat{g}^{-1}r_i^{-1}(\partial D)$. Since r_i^{-1} is the identity map on ∂D for each $i \geq 1$, and $\hat{g}^{-1}$ is a homeomorphism on D, it follows that the maps $r_i\hat{g}$ (and their compositions) are universal. □

For $n \in \mathbb{N}$, consider maps k_n of the form $(\phi, \theta) \mapsto (\phi, 2n\theta + \frac{(2n-1)\pi}{2})$. Note that $k_1 = g$. Since each such map collapses the equator of S^2 to either the north or south pole, an argument similar to the proof of Theorem 4.2 will show that an inverse limit on S^2 having bonding maps from the collection $\{k_n\}_{n=1}^{\infty}$ will also produce an inverse limit space with the fpp.

5. The $\mathbb{R}P^{2n}$ Case

For $n \geq 1$, we represent S^{2n} as ΣS^{2n-1} (the suspension of S^{2n-1}), where $(x, \theta) \in \Sigma S^{2n-1} = S^{2n}$ for $x \in S^{2n-1}$ and $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$. So, $(x, \frac{\pi}{2}) = (y, \frac{\pi}{2})$ and $(x, \frac{-\pi}{2}) = (y, \frac{-\pi}{2})$ for all $x, y \in S^{2n-1}$. We allow for points (x, θ), where θ is any real number through the identifications $(x, \theta) = (x, \theta+2\pi)$ and $(x, \theta) = (-x, \pi-\theta)$. Define the map $g\colon S^{2n} \to S^{2n}$ by $g(x, \theta) = (x, 2\theta + \frac{\pi}{2})$. For $n = 1$, this is the same g as defined in §4. It is easy to check that g collapses antipodes and thus induces a map $\tilde{g}\colon \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$. We now show that g and $\tilde{g}$ have the same properties as the maps defined in §4.

That g is inessential can be seen by defining the homotopy $H\colon S^{2n} \times [0, 1] \to S^{2n}$ by $H((x, \theta), t) = (x, 2\theta t + \frac{\pi}{2})$. Note that $H((x, \theta), 0) = (x, \frac{\pi}{2})$ and $H((x, \theta), 1) = (x, 2\theta + \frac{\pi}{2}) = g(x, \theta)$ for all $(x, \theta) \in S^{2n}$.

Below we make use of the "degree" theory for maps of manifolds, using both geometric degree (sometimes called local index) and algebraic degree. Connections between the two and with the degree defined in [11] are established in [3]. It is easy to see that $\tilde{g}\colon \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a 2-sheeted covering at each point $p(x, 0)$ with sheets of opposite orientation centered at the points $p(x, \frac{\pi}{4})$ and $p(x, \frac{-\pi}{4})$. Hence, the degree of $\tilde{g}$ is zero and $\tilde{g}_*\colon H_{2n}(\mathbb{R}P^{2n}) \to H_{2n}(\mathbb{R}P^{2n})$ is trivial. If $\tilde{g}_*$ is multiplication by k on $H_1(\mathbb{R}P^{2n})$, then $\tilde{g}_*$ is multiplication by k^{2n} on $H_{2n}(\mathbb{R}P^{2n})$ as noted earlier in §3. So, $\tilde{g}_*\colon H_1(\mathbb{R}P^{2n}) \to H_1(\mathbb{R}P^{2n})$ is also trivial.

Let $g'\colon \mathbb{R}P^{2n} \to S^{2n}$ be a lift of $\tilde{g}$ so that $p \circ g' = \tilde{g}$. We also have that $g' \circ p = g$. For a point $(x, \frac{\pi}{4})$ in S^{2n}, $g(x, \frac{\pi}{4}) = (x, \pi) = (-x, 0)$ and $g^{-1}(-x, 0) = \{(x, \frac{\pi}{4}), (-x, -\frac{\pi}{4})\}$. Since $g' \circ p = g$, g' has a 1-sheeted covering at the point $(-x, 0)$ and hence, the "mod 2" degree of g' is one. By the homotopy lifting theorem, it follows that $\tilde{g}$ is essential.

As in the $\mathbb{R}P^2$ case, the inverse limits $X = \varprojlim\{S^{2n}, g\}$ and $Y = \varprojlim\{\mathbb{R}P^{2n}, \tilde{g}\}$ are homeomorphic.

Theorem 5.1. *$X = \varprojlim\{S^{2n}, g\}$ has the fixed point property.*

Proof. The proof is the same as in the $\mathbb{R}P^2$ case with the following slight modification to the definition of the retraction $r\colon S^{2n} \to D$, where D is the lower cone of ΣS^{2n-1}. Since $2n-1$ is odd, there is a homotopy $\{h_t \mid 0 \le t \le 1\}$ such that for $x \in S^{2n-1}$, $h_0(x) = x$ and $h_1(x) = -x$. Define

$$r(x,\theta) = \begin{cases} (x,\theta) & \text{for } \frac{-\pi}{2} \le \theta \le 0, \\ (h_{\frac{8\theta}{\gamma}}(x), -\theta) & \text{for } 0 \le \theta \le \frac{\gamma}{8}, \\ (-x,-\theta) & \text{for } \frac{\gamma}{8} \le \theta \le \frac{\pi}{2}. \end{cases}$$

The theorem follows. □

In a manner analogous to the $\mathbb{R}P^2$ case, we have the following theorem.

Theorem 5.2. *$X = \varprojlim\{S^{2n}, g\}$ can be realized as an inverse limit of topological $2n$-balls with universal bonding mappings.*

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Department of Mathematics & Statistics, California State University, Sacramento, Sacramento,CA 95819-6051

E-mail address: mmarsh@csus.edu