TOPOLOGY PROCEEDINGS

Volume 41, 2013

Pages 333–336

http://topology.auburn.edu/tp/

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Electronically published on October 30, 2012

Topology Proceedings

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ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS
Volume 41 (2013)
Pages 333-336

E-Published on October 30, 2012

NONCONNECTEDNESS OF INVERSE LIMIT SEQUENCES

W. T. INGRAM AND M. M. MARSH

Abstract. In this short paper we provide an example of an inverse limit sequence having a connected inverse limit but the inverse limit of the sequence of inverses of the bonding functions is not connected.

1. Introduction

In [3, Theorem 3.3] Van Nall proves that if X is a Hausdorff continuum and $f : X \to 2^X$ is a surjective upper semi-continuous set-valued function then $\varprojlim \boldsymbol{f}$ is connected if and only if $\varprojlim \boldsymbol{f^{-1}}$ is connected. This theorem, stated for inverse limits on $[0, 1]$ with set-valued functions, may be found in [1, Theorem 2.3, p. 16]. This result prompted the question whether Nall's theorem holds if the single bonding function f is replaced by a sequence of bonding functions [1, Problem 6.7]. In this paper we provide an example of a sequence of set-valued functions having a connected inverse limit for which the inverse limit with the sequence of inverses of the functions as bonding functions is not connected.

If X is a Hausdorff space, we denote the closed subsets of X by 2^X. If each of X and Y is a compact Hausdorff space, $f : X \to 2^Y$ being upper semi-continuous is equivalent to the graph of f, $G(f) = \{(x, y) \in X \times Y \mid y \in f(x)\}$, being a closed subset of $X \times Y$. If $\boldsymbol{s} = s_1, s_2, s_3, \ldots$ is a sequence, we denote the sequence in boldface type and its terms in italics. If $\boldsymbol{X}$ is a sequence of compact Hausdorff spaces and $f_n : X_{n+1} \to 2^{X_n}$ is an upper semi-continuous function for each positive integer n, the pair of sequences $\{\boldsymbol{X}, \boldsymbol{f}\}$ is called an *inverse sequence*.

2010 *Mathematics Subject Classification.* 54F15, 54H20.
Key words and phrases. Inverse limits, connected.

By the *inverse limit* of the inverse sequence $\{\boldsymbol{X}, \boldsymbol{f}\}$, denoted $\varprojlim \boldsymbol{f}$, is meant $\{\boldsymbol{x} \in \prod_{n>0} X_n \mid x_i \in f_i(x_{i+1})$ for each positive integer $i\}$. This definition differs from the definition of inverse limit with a sequence of mappings only in that $x_i = f_i(x_{i+1})$ is replaced by $x_i \in f_i(x_{i+1})$. Numerous articles and books are now in print that contain the background material needed to read this article so we refer the reader to our references for additional information.

2. Main Result

In our proofs we use two theorems that we state here for completeness. Both statements for inverse limits on $[0,1]$ can be found in [1]. We make use of some notation. Suppose $\boldsymbol{X}$ is a sequence of compact Hausdorff spaces and $f_n : X_{n+1} \to 2^{X_n}$ is an upper semi-continuous function for each $n \in \mathbb{N}$. Let $G_n = \{\boldsymbol{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1})$ for $1 \leq i \leq n\}$; we denote by G'_n its finite dimensional projection $\{\boldsymbol{x} \in [0,1]^{n+1} \mid x_i \in f_i(x_{i+1})$ for $1 \leq i \leq n\}$. The sequence $\boldsymbol{G}$ of approximations to the inverse limit is used in showing that the inverse limit is a nonempty, compact Hausdorff space. The sequence can also be used in arguing connectedness of the inverse limit, see [2, Theorem 116, p.85] or, for $[0,1]$,[1, Theorem 2.1, p. 14].

Theorem 2.1. *Suppose $\boldsymbol{X}$ is a sequence of Hausdorff continua and $f_n : X_{n+1} \to 2^{X_n}$ is an upper semi-continuous function for each positive integer n. Then $\varprojlim \boldsymbol{f}$ is connected if and only if G_n is connected for each positive integer n.*

The other theorem that we use is an observation made by Nall. In its statement, we use the notation that if $\boldsymbol{X}$ is a sequence of spaces and $f_n : X_{n+1} \to X_n$ is a set-valued function for each n, then, for $m < n$, $f_{m\,n} = f_m \circ f_{m+1} \circ \cdots \circ f_{n-1}$. Theorem 2.2 is explicitly stated on the interval $[0,1]$ (as needed here) in [1, Theorem 2.2].

Theorem 2.2 (**Nall**)**.** *Suppose $\boldsymbol{X}$ is a sequence of Hausdorff continua and $f_n : X_{n+1} \to 2^{X_n}$ is an upper semi-continuous function for each positive integer n. If there exist integers m and n with $m < n$ such that $G(f_{m\,n})$ is not connected, then $\varprojlim \boldsymbol{f}$ is not connected.*

Example 2.3. Let $f_1 : [0,1] \to 2^{[0,1]}$ be given by $f_1(t) = \{t, (t+3)/4\}$. Let $f_2(t) = t$ for $0 \leq t < 1/4$, $f_2(t) = \{t, (4t-1)/2\}$ for $1/4 \leq t \leq 3/4$, and $f_2(t) = t$ for $3/4 < t \leq 1$. For $n > 2$, let $f_n(t) = t$ for each $t \in [0,1]$. Let $\boldsymbol{g}$ be the sequence such that $g_n = f_n^{-1}$ for each positive integer n. Then, $\varprojlim \boldsymbol{f}$ is connected; $\varprojlim \boldsymbol{g}$ is not connected.

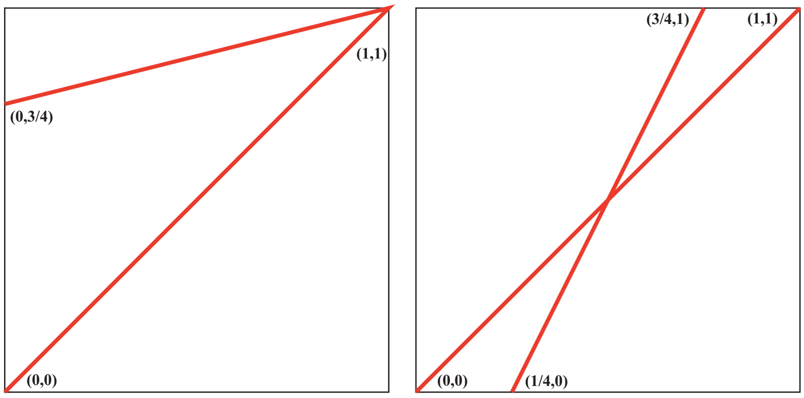

FIGURE 1. The functions f_1 and f_2 in Example 2.3

Proof. Let $\Delta = \{\boldsymbol{x} \in [0,1]^\infty \mid x_j = x_1 \text{ for } j = 2,3,\dots\}$. Let $G'(f_1, f_2) = \{(x_1, x_2, x_3) \in [0,1]^3 \mid x_1 \in f_1(x_2) \text{ and } x_2 \in f_2(x_3)\}$. Let $A_1 = \{(x,x,x) \in [0,1]^3 \mid x \in [0,1]\}$, $A_2 = \{(x,x,(2x+1)/4) \mid x \in [0,1]\}$, $A_3 = \{(t, 4t-3, 4t-3) \mid t \in [3/4,1]\}$, and $A_4 = \{(t, 4t-3, (8t-5)/4 \mid t \in [3/4,1]\}$. It is not difficult to see that $G'(f_1, f_2)$ is the union of the four arcs A_1, A_2, A_3, and A_4. Moreover, $(1/2,1/2,1/2) \in A_1 \cap A_2$, $(1,1,3/4) \in A_2 \cap A_4$ and $(1,1,1) \in A_1 \cap A_3$. It follows that $G'(f_1, f_2)$ is connected. Then, by Theorem 2.1, $\varprojlim \boldsymbol{f}$ is connected because $G_1 = G(f_1^{-1}) \times \Delta$ and $G_n = G'(f_1, f_2) \times \Delta$ for $n \geq 2$ are connected.

On the other hand, $G(g_{1\,3})$ is not connected because the point $(1,0)$ is an isolated point of $G(g_{1\,3})$. It follows from Theorem 2.2 that $\varprojlim \boldsymbol{g}$ is not connected. □

We close with a couple of remarks. (1) The terms of the sequence $\boldsymbol{f}$ in Example 2.3 could not have been chosen to be surjective mappings because the inverse limit with a sequence of inverses of mappings is an arc.

The authors are indebted to the referee for pointing out the following: (2) If $\boldsymbol{f}$ is the sequence of functions in Example 2.3, by letting $h_1 = f_2, h_2 = f_1$, and $h_i = f_i$ for $i > 2$, $\varprojlim \boldsymbol{h}$ is not connected by Theorem 2.2 because $(0,1)$ is an isolated point of $G(h_{1\,3})$. Thus, an inverse limit with set-valued functions can be connected but by merely switching the order of the bonding functions the inverse limit may no longer be connected.

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