

FIXED POINT THEOREMS FOR PARTIALLY OUTWARD MAPPINGS

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ABSTRACT. We generalize some classical theorems related to dimension. We extend Brouwer's fixed point theorem to a class of mappings whose images are not necessarily a subset of the domain. These results also generalize theorems of B.R. Halpern and G.M. Bergman. As applications, we prove some theorems for maps that pull absolute retracts outward into attached sphere collars. We note relationships to the relative Nielsen theory and show that certain of our applications can also be obtained using results of H. Schirmer.

Let X and Y be topological spaces. By a *map* or *mapping* $f: X \rightarrow Y$ we mean a continuous function. We investigate fixed points for mappings $f: X \rightarrow Y$, where $f(X)$ need not be a subset of X . Clearly, if X and $f(X)$ are disjoint, f will have no fixed points. So, our study will consider mappings where $X \cap f(X) \neq \emptyset$ and $f(X) - X \neq \emptyset$. We refer to such mappings as *partially outward* mappings in the spirit of B. R. Halpern and G. M. Bergman, who defined *inward* and *outward* maps in [7]. Our methods are entirely topological, but we discuss connections to results obtained from the Lefschetz or Nielsen theories and we provide an alternative proof, which uses the relative Nielsen theory [20], of a corollary to our main theorem. In the process, we generalize (in the compact Hausdorff setting) theorems related to dimension of S. Eilenberg and E. Otto, of K. Morita, and of W. Holsztyński. We generalize (in Euclidean n -space and Hilbert square summable sequence space) the Brouwer fixed point theorem and theorems of Halpern and Bergman. Applications of our main result yield some fixed point theorems that, heretofore, have only been obtainable using algebraic invariants associated with the mapping or its homotopy class. In particular, we show that maps that pull absolute retracts outward into an attached S^k -collar have fixed points. The author acknowledges discussion of initial ideas for this paper in a seminar with Charles Hagopian, Janusz Prajs, Alejandro Illanes, and Verónica Martínez-de-la-Vega.

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1. DEFINITIONS AND PRELIMINARY THEOREMS

Let X be a compact Hausdorff space with subsets A , B , and F (possibly empty), and suppose that A and B are closed and disjoint. We say that F *separates* X *between* A *and* B if $X - F$ is not connected between A and B . That is, if there exist mutually separated sets M and N such that $X - F = M \cup N$, $A \subseteq M$, and $B \subseteq N$. We point out that if F separates X between A and B , then F is disjoint from $A \cup B$. We say that the closed set F (possibly empty) *weakly cuts* A *from* B in X (or F *weakly cuts* X *between* A *and* B) if each closed connected set in X that intersects both A and B also intersects F . The closed set F need not be disjoint from $A \cup B$ to weakly cut A from B .

Following terminology used by W. Holsztyński in [8, 9], define an n -system in the topological space X as a finite sequence $\{(A_{-i}, A_i)\}_{i=1}^n$ of pairs of closed sets such that $A_{-i} \cap A_i = \emptyset$ for $1 \leq i \leq n$. The n -system $\{(A_{-i}, A_i)\}_{i=1}^n$ is *separable* if there exist closed sets $\{F_i\}_{i=1}^n$ such that $\bigcap_{i=1}^n F_i = \emptyset$ and for each $1 \leq i \leq n$, F_i separates A_{-i} from A_i . We say the n -system is *weakly separable* if there exist closed sets $\{F_i\}_{i=1}^n$ such that $\bigcap_{i=1}^n F_i = \emptyset$ and for each $1 \leq i \leq n$, F_i weakly cuts A_{-i} from A_i in X .

In 1938, S. Eilenberg and E. Otto [5] proved that, for separable metric spaces X , the covering dimension of X is greater than or equal n ($\dim X \geq n$) if and only if there exists a non-separable n -system in X . In 1950, K. Morita [13] established this equivalence for normal spaces, see [16, Th.9-9, p.51]. In 1964, W. Holsztyński [8] proved that, for normal spaces X , $\dim X \geq n$ if and only if there exists a universal mapping of X onto I^n , where $I^n = \prod_{i=1}^n I$ and $I = [-1, 1]$. A mapping $f: X \rightarrow Y$ of topological spaces is said to be *universal* if for each mapping $g: X \rightarrow Y$, there is a point $x \in X$ such that $f(x) = g(x)$. Thus, together these results establish the following theorem.

Theorem 1. *Let X be a normal space. The following are equivalent.*

- (1) $\dim X \geq n$,
- (2) *there exists a non-separable n -system in X , and*
- (3) *there exists a universal mapping of X onto I^n .*

We will show that for compact Hausdorff spaces X , the same equivalences can be established with weakly separable n -systems replacing separable n -systems. First we need two lemmas. An analogous form of the first lemma was proved in the setting of metric continua in [11, Lemma 1]. Essentially the same proof works for the lemma below in compact Hausdorff spaces. Since the proof is short, we include it for completeness. The author wishes to thank Eldon Vought for helpful discussions concerning the proof of this lemma. We recall the statement of the Wire Cutting Theorem (see [12, Th44, p15]), which we use in the proof of Lemma 1 below.

Wire Cutting Theorem. *If H and K are mutually exclusive closed subsets of the perfectly compact closed point set M but M contains no continuum*

intersecting both H and K , then M is the sum of two mutually exclusive closed point sets one containing H and the other containing K .

Lemma 1. *Suppose F is a closed set in the compact Hausdorff space X that weakly cuts X between the disjoint closed sets A and B . If U is an open set containing F , then there is a closed set $E \subseteq U$ such that E separates X between A and B .*

Proof. If F is empty, let U be any open set in X . By the definition of weak cutting, no closed connected set in X intersects both A and B . Thus, by the Wire Cutting theorem $X = X - F$ is not connected between A and B . So, $F = \emptyset \subseteq U$ separates A from B and we are done. Hence, we assume that $F \neq \emptyset$.

Suppose one of A or B is a subset of U . Suppose $A \subseteq U$. Let V be an open set such that $A \subseteq V \subseteq \overline{V} \subseteq U$ and $\overline{V} \cap B = \emptyset$. Then the boundary of V is a closed set in U that separates X between A and B . Thus, we may assume that neither A nor B is a subset of U .

First we show that $U - (A \cup B)$ separates X between A and B . Let A' be the set of all $x \in X - U$ such that there exists a closed connected set $C \subseteq X - U$ with $x \in C$ and $C \cap (A - U) \neq \emptyset$. Analogously, let B' be the set of all $x \in X - U$ such that there exists a closed connected set $C \subseteq X - U$ with $x \in C$ and $C \cap (B - U) \neq \emptyset$. Clearly, A' and B' are closed, $A - U \subseteq A'$, and $B - U \subseteq B'$. Since F weakly cuts X between A and B , no closed connected subset of $X - U$ intersects both A' and B' . Again, by the Wire Cutting theorem, U separates X between A' and B' , and hence between $A - U$ and $B - U$. Since $A \cap B = \emptyset$, it follows that $U - (A \cup B)$ separates X between A and B .

By [10, Th3, p155], it follows that $U - (A \cup B)$ contains a closed set E that separates X between A and B . \square

Lemma 2. *Let X be a compact Hausdorff space. The n -system $\{(A_{-i}, A_i)\}_{i=1}^n$ in X is separable if and only if it is weakly separable.*

Proof. \Rightarrow : Obvious from the definitions.

\Leftarrow : For $1 \leq i \leq n$, let F_i be a closed set that weakly cuts between A_{-i} and A_i in X . Suppose that $\bigcap_{i=1}^n F_i = \emptyset$. For $1 \leq i \leq n$, there exists an open set U_i in X such that $F_i \subseteq U_i$ and $\bigcap_{i=1}^n U_i = \emptyset$. By Lemma 1, for each $1 \leq i \leq n$, there exists a closed set $E_i \subseteq U_i$ such that E_i separates X between A_{-i} and A_i . Since $\bigcap_{i=1}^n U_i = \emptyset$, it follows that $\bigcap_{i=1}^n E_i = \emptyset$. Therefore the n -system $\{(A_{-i}, A_i)\}_{i=1}^n$ is separable. \square

We have the following theorem.

Theorem 2. *Let X be a compact Hausdorff space. The following are equivalent.*

- (1) $\dim X \geq n$,
- (2) there exists a non-weakly separable n -system in X , and
- (3) there exists a universal mapping of X onto I^n .

For $n \geq 1$, let \mathbb{E}^n denote Euclidean n -space, B^n the unit ball in \mathbb{E}^n , and S^{n-1} the unit sphere in \mathbb{E}^n . For $1 \leq i \leq n$, let $\pi_i: \mathbb{E}^n \rightarrow \mathbb{R}$ be coordinate projection. For $n \geq 1$, let $I_{-i}^n = \pi_i^{-1}(-1) \cap I^n$ and $I_i^n = \pi_i^{-1}(1) \cap I^n$ be opposite sides of I^n .

W. Holsztyński showed in [8, Lemma] that, for normal spaces X , the mapping $f: X \rightarrow I^n$ is universal if and only if the n -system $\{(f^{-1}(I_{-i}^n), f^{-1}(I_i^n))\}_{i=1}^n$ is not separable. By Lemma 2, we have the following theorem.

Theorem 3. *Let X be a compact Hausdorff space. The mapping $f: X \rightarrow I^n$ is universal if and only if the n -system $\{(f^{-1}(I_{-i}^n), f^{-1}(I_i^n))\}_{i=1}^n$ is not weakly separable.*

2. A GENERALIZATION OF THE BROUWER FIXED POINT THEOREM

Suppose that $f, g: I^n \rightarrow \mathbb{E}^n$ are mappings. Suppose for some $1 \leq i \leq n$, $\pi_i f(x) \leq \pi_i g(x)$ for all $x \in I_{-i}^n$ and $\pi_i f(x) \geq \pi_i g(x)$ for all $x \in I_i^n$ (or both inequalities are reversed), then we say that f moves a pair of opposite sides of I^n in opposite directions relative to g . In particular, we say that f moves I_{-i}^n and I_i^n in opposite outward directions relative to g for the inequalities given, and inward relative to g if they are reversed. If g is the identity map on I^n , then we simply say that f moves a pair of opposite sides of I^n in opposite directions, outward or inward as the case may be.

Theorem 4. *If $f, g: I^n \rightarrow \mathbb{E}^n$ are mappings and f moves each pair of opposite sides of I^n in opposite directions relative to g , then f and g have a coincidence point.*

Proof. Fix i such that $1 \leq i \leq n$. Define the sets

$$\begin{aligned} L_i &= \{x \in I^n \mid \pi_i f(x) < \pi_i g(x)\}, \\ R_i &= \{x \in I^n \mid \pi_i f(x) > \pi_i g(x)\}, \\ H_i &= \{x \in I^n \mid \pi_i f(x) = \pi_i g(x)\}. \end{aligned}$$

We claim that H_i is not empty and that H_i weakly cuts I^n between I_{-i}^n and I_i^n . Suppose H_i is empty. Then $L_i \cup R_i = I^n$. By assumption, either $I_{-i}^n \subseteq L_i$ and $I_i^n \subseteq R_i$ or vice versa. In either case, $L_i \neq \emptyset \neq R_i$. But also L_i and R_i are disjoint open sets, contradicting the connectedness of I^n .

To see that H_i cuts weakly between I_{-i}^n and I_i^n in I^n , let K be a continuum in I^n intersecting both I_{-i}^n and I_i^n . Repeating the argument above for the sets $L_i \cap K$ and $R_i \cap K$ gives us that $H_i \cap K$ is not empty.

Hence, we have sets $\{H_i\}_{i=1}^n$, each weakly cutting I_{-i}^n from I_i^n in I^n . Note that for $1 \leq i \leq n$, $I_{-i}^n = id^{-1}(I_{-i}^n)$ and $I_i^n = id^{-1}(I_i^n)$, where $id: I^n \rightarrow I^n$ is the identity mapping. Since $id: I^n \rightarrow I^n$ is universal, it follows from Theorem 3 that the n -system $\{I_{-i}^n, I_i^n\}_{i=1}^n$ is not weakly separable. Therefore, $\bigcap_{i=1}^n H_i \neq \emptyset$. It follows that f and g have a coincidence point. \square

Corollary 1. *If $f: I^n \rightarrow \mathbb{E}^n$ is a mapping and f moves each pair of opposite sides of I^n in opposite directions, then f has a fixed point.*

Note that Corollary 1 is a generalization of the Brouwer fixed point theorem in the sense that it applies to a larger class of mappings whose images are not necessarily contained in I^n . Also, Brouwer's theorem follows immediately from Corollary 1. Furthermore, Theorem 4 and Corollary 1 give sufficient conditions for the existence of coincidence points, respectively fixed points, based only on behavior of the maps on the boundary of I^n . The interior of I^n may be mapped anywhere in \mathbb{E}^n .

Let H be the Hilbert cube in the Hilbert square summable sequence space ℓ_2 . That is, $H = \{(x_1, x_2, \dots) \in \ell_2 \mid |x_i| \leq \frac{1}{i} \text{ for all } i \geq 1\}$. For each $n \geq 1$, identify \mathbb{E}^n with $\{(x_1, x_2, \dots) \in \ell_2 \mid x_i = 0 \text{ for } i > n\}$ and I^n with $\mathbb{E}^n \cap H$. For $i \geq 1$, let $I_{-i}^\infty = \{(x_1, x_2, \dots) \in H \mid x_i = -\frac{1}{i}\}$ and $I_i^\infty = \{(x_1, x_2, \dots) \in H \mid x_i = \frac{1}{i}\}$ be opposite sides of H . Note that, with our identifications, for each $n \geq 1$ and each $1 \leq i \leq n$, $I_{-i}^\infty \cap \mathbb{E}^n = I_{-i}^n$ and $I_i^\infty \cap \mathbb{E}^n = I_i^n$.

Theorem 5. *If $f, g: H \rightarrow \ell_2$ are mappings and f moves each pair of opposite sides of H in opposite directions relative to g , then f and g have a coincidence point.*

Proof. Let d denote the metric on ℓ_2 induced by the usual inner product. Suppose f and g have no coincidence point and let ϵ be a positive number such that $d(g(x), f(x)) \geq \epsilon$ for each $x \in H$. For each $n \geq 1$, let $p_n: \ell_2 \rightarrow \mathbb{E}^n$ denote the natural projection. For each $n \geq 1$, the mapping $p_n f|_{I^n}: I^n \rightarrow \mathbb{E}^n$ moves each pair of opposite sides of I^n in opposite directions relative to $p_n g|_{I^n}: I^n \rightarrow \mathbb{E}^n$. To see this, fix $n \geq 1$, $1 \leq i \leq n$, and suppose that f moves I_{-i}^∞ and I_i^∞ in opposite outward directions relative to g . Let $x \in I_i^n = I_i^\infty \cap \mathbb{E}^n$. Since $x \in I_i^\infty$, by assumption, $f(x)_i \geq g(x)_i$. Since $n \geq i$, $p_n f(x)_i = f(x)_i \geq g(x)_i = p_n g(x)_i$. Similarly, for $x \in I_{-i}^n$, $p_n f(x)_i = f(x)_i \leq g(x)_i = p_n g(x)_i$.

So, Theorem 4 applies to the mappings $p_n f|_{I^n}$ and $p_n g|_{I^n}$ for each $n \geq 1$. For $n \geq 1$, let x^n be a coincidence point of $p_n f|_{I^n}$ and $p_n g|_{I^n}$; so, $p_n f(x^n) = p_n g(x^n)$. It follows that $g(x^n)$ and $f(x^n)$ agree on their first n coordinates. Assume without loss of generality that $\{x^n\}_{n \geq 1}$ converges to x in H . By continuity of f and g , $\{f(x^n)\}_{n \geq 1}$ converges to $f(x)$, $\{g(x^n)\}_{n \geq 1}$ converges to $g(x)$ and it follows that $f(x) = g(x)$. \square

Corollary 2. *If $f: H \rightarrow \ell_2$ is a mapping and f moves each pair of opposite sides of H in opposite directions, then f has a fixed point.*

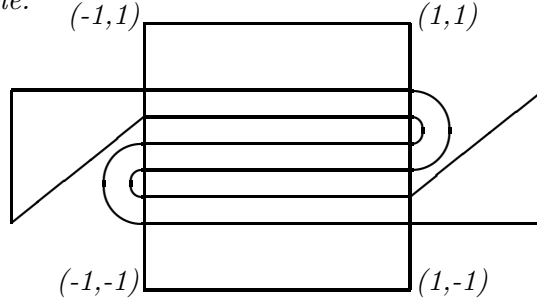
In 1968, B. R. Halpern and G. M. Bergman [7] defined *inward*, *weakly inward*, *outward*, *weakly outward*, and *nowhere outward normal maps* on compact, convex sets in topological vector spaces. Without repeating their definitions, the idea is as follows. Let K be a compact convex set in the topological vector space X . For each $x \in K$, associate three sets: the *inward set* I_x , which is the union of all "rays" from x through other points of K ; the *outward set* O_x , which is $-I_x + 2x$ (vector space operations); and

the *outward normal set* N_x , which is the preimage of x under the nearest point mapping n of X onto K (that is, $N_x = n^{-1}(x)$). They prove that

- (1) in a strictly convex normed linear space X , if $f: K \rightarrow X$ is a map such that no point x maps into N_x , then f has a fixed point, and
- (2) in a topological vector space X whose continuous linear functionals distinguish points, if $f: K \rightarrow X$ is a weakly inward (weakly outward) map meaning that for each $x \in K$, $f(x)$ is in the closure of I_x (for each x , $f(x)$ is in the closure of O_x), then f has a fixed point.

Our Corollaries 1 and 2 generalize the second result in \mathbb{E}^n and ℓ_2 . Any weakly inward (or weakly outward) map on I^n (or H) satisfies the hypothesis of our Theorem 4 (or Theorem 5). Consider also the following example.

Example 1. In \mathbb{E}^2 , let $f: I^2 \rightarrow [-2, 2] \times I$ be any mapping such that $f(I_{-1}^2) = \{-2\} \times [-\frac{1}{2}, \frac{1}{2}]$ and $f(I_1^2) = \{2\} \times [-\frac{1}{2}, \frac{1}{2}]$. See the figure below for one such example.



By Corollary 1, f has a fixed point in I^2 . We cannot apply Halpern's and Bergman's results to get that f has a fixed point. For we observe that the image of the point $(-1, 1)$ in I^2 is neither in the closure of I_x nor in the closure of O_x . Furthermore, since the map $t: I_{-1}^2 \rightarrow \{-2\} \times I$ defined by $t(x) = x - (1, 0)$ is universal, t and $f|_{I_{-1}^2}$ have a coincidence point x . But then x is a point of I^2 that is mapped by f into its outward normal set N_x .

3. APPLICATIONS

Let A , N , and Y be compact absolute neighborhood retracts (ANRs), $A \subseteq N$, and $g: A \rightarrow Y$ a mapping. If g is an imbedding, we refer to the adjunction space $X = N \cup_g Y$ (see [22, p.165]) as Y with N attached at A . If $f: Y \rightarrow X$ is a mapping such that $f(A) \subseteq N$, we say that f pulls A (or Y) outward into N (or f is an outward pulling map on A). We look at a few examples of such maps for specific A , N , and Y , and for g an imbedding. For $m \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, we refer to any homeomorph of $S^k \times \prod_{i=1}^m I$ as an (S^k, m) -collar. We refer to an $(S^k, 1)$ -collar as simply an S^k -collar. Fix $n \geq 1$ and $0 \leq k \leq n - 1$, and let $g: S^k \times \{0\} \rightarrow B^n$ be an imbedding whose image bounds a $(k + 1)$ -ball in B^n . Let $X = (S^k \times [0, 1]) \cup_g B^n$. Then X is an n -ball with an attached S^k -collar. If the image of g lies in the boundary of B^n , denoted ∂B^n , we say that X is an n -ball with an S^k -collar attached to its boundary. Throughout it will be convenient to identify S^k ,

$S^k \times \{0\}$, and $g(S^k \times \{0\})$. Also, let π_1 and π_2 denote coordinate projection on $S^k \times [0, 1]$ with images lying in X . If f pulls S^k outward into $S^k \times [0, 1]$ and $\pi_1 f|_{S^k} = id|_{S^k}$, then we say that f pulls S^k *straight outward* into $S^k \times [0, 1]$.

Theorem 6. *Let $n \geq 1$ and suppose that X is an n -ball with an S^k -collar attached to its boundary for some $1 \leq k < n$. If $f: B^n \rightarrow X$ is a mapping that pulls S^k straight outward into $S^k \times [0, 1]$, then f has a fixed point.*

Proof. We identify B^n with I^n , $S^k \times [0, 1]$ with $\{(x_1, x_2, \dots, x_n) \mid 1 \leq |x_i| \leq 2 \text{ for } 1 \leq i \leq k+1 \text{ and } x_i = 0 \text{ for } i > k+1\}$, and $S^k \times \{0\}$ with the subset of $S^k \times [0, 1]$ such that $|x_i| = 1$ for $1 \leq i \leq k+1$. We assume that f is fixed point free. From the identifications above, we have that

$$S^k \times \{0\} = \bigcup_{i=1}^{k+1} (I_{-i}^{k+1} \cup I_i^{k+1})$$

and for each $1 \leq i \leq k+1$, $I_{-i}^{k+1} \subseteq I_{-i}^n$ and $I_i^{k+1} \subseteq I_i^n$. Since f is fixed point free and $\pi_1 f = id$ on $S^k \times \{0\}$, each point x of $S^k \times \{0\}$ has a neighborhood U_x in ∂I^n whose closure \overline{U}_x is homeomorphic to $I_i^n \approx I^{n-1}$ and such that $f(\overline{U}_x) \cap I^n = \emptyset$. By compactness of $S^k \times \{0\}$, there are finitely many of these closed neighborhoods $\overline{U}_{x_1}, \dots, \overline{U}_{x_m}$ covering $S^k \times \{0\}$. So, there exists $0 < t < 1$ such that the $(S^k, (n-k-1)$ -collar $T = \bigcup_{i=1}^{k+1} (I_{-i}^{k+1} \cup I_i^{k+1}) \times \Pi_{j=k+2}^n [-t, t]$ is a subset of $\bigcup_{j=1}^m \overline{U}_{x_j}$ (One might view this as a type of collaring theorem for $k < n-1$, see [23, Th.6.23]); and therefore $f(T) \cap I^n = \emptyset$. Now, there is a homeomorphism of the pair $(I^n, \bigcup_{i=1}^{k+1} (I_{-i}^n \cup I_i^n))$ and $(I^{k+1} \times \Pi_{j=k+2}^n [-t, t], T)$, so there is no loss of generality to assume that $\bigcup_{i=1}^{k+1} (I_{-i}^n \cup I_i^n) = T$ and that $f(\bigcup_{i=1}^{k+1} (I_{-i}^n \cup I_i^n)) \cap I^n = \emptyset$. Thus, we have that for $1 \leq i < k+1$, f maps the opposite sides I_{-i}^n and I_i^n of I^n in opposite outward directions, and for $k+1 < i \leq n$, f maps the opposite sides I_{-i}^n and I_i^n of I^n in opposite inward directions. By Corollary 1, f has a fixed point. \square

Since we only need to know the behavior of the mapping f on ∂I^n in Corollary 1, we point out that the proof of Theorem 6 above, with little modification, proves the following stronger result which doesn't require the image of f to be a subset of X . However, the statement of the theorem is somewhat cumbersome, so we will not point out similar extensions of other applications. Note, nevertheless, the "partial outwardness" of the map in the sense of Halpern and Bergman.

Theorem 7. *Let k and n be integers with $0 \leq k < n$. Suppose that $f: B^n \rightarrow \mathbb{E}^n$ is a mapping, B is a $(k+1)$ -ball in B^n whose boundary $S \approx S^k$ lies in ∂B^n . If there exists a neighborhood T of S lying in ∂B^n such that T is a $(S^k, n-k-1)$ -collar, $f(p, y)$ is in the closure of the outward set of $\{p\} \times I^{n-k-1}$ for each $(p, y) \in T$ (where $p \in S^k$ and $y \in I^{n-k-1}$), and $f(x)$ is in the closure of the inward set of x for each $x \notin T$, then f has a fixed point.*

Now we generalize Theorem 6 to an n -ball with arbitrary attached S^k -collar.

Theorem 8. *Let $n \geq 1$ and suppose that X is an n -ball with an attached S^k -collar for some $1 \leq k < n$. If $f: B^n \rightarrow X$ is a mapping that pulls S^k straight outward into $S^k \times [0, 1]$, then f has a fixed point.*

Proof. Identify B^{n+1} with $B^n \times [-1, 0]$ and B^n with $B^n \times \{0\}$. Now the imbedding $g: S^k \times \{0\} \rightarrow B^n$ can be considered to have its image lying in ∂B^{n+1} . So, $X' = (S^k \times \{0\}) \cup_g B^{n+1}$ is an $(n+1)$ -ball with an S^k -collar attached to its boundary. Let $\pi: B^n \times [-1, 0] \rightarrow B^n \times \{0\}$ be projection. The mapping $f\pi: B^{n+1} \rightarrow X \subseteq X'$ satisfies the hypothesis of Theorem 6, and thus has a fixed point x in $B^n \times [-1, 0]$. So, $f\pi(x) = x$. Since the image of $f\pi$ is contained in X , $x \notin B^n \times [-1, 0)$. So, $x \in B^n \times \{0\} = B^n$. Hence, $\pi(x) = x$ and we have that $f(x) = x$ with $x \in B^n$. \square

Theorem 9. *Let X be the Hilbert cube H with an attached S^k -collar for some $k \geq 0$. If $f: H \rightarrow X$ is a mapping that pulls S^k straight outward into $S^k \times [0, 1]$, then f has a fixed point.*

Proof. Suppose that f is fixed point free and ϵ is a positive number such that $d(x, f(x)) \geq \epsilon$ for all $x \in H$. Let n be large enough so that $n > k$, $S^k \times \{0\} \subseteq I^n$, and if $r: H \rightarrow I^n$ is the natural projection, then $d(x, r(x)) < \epsilon$ for all $x \in H$. Let $\hat{r}: X \rightarrow (S^k \times [0, 1]) \cup_g I^n$ be the mapping such that $\hat{r}|_{S^k \times [0, 1]} = id$ and $\hat{r}|_H = r$. The mapping $\hat{r}f|_{I^n}: I^n \rightarrow (S^k \times [0, 1]) \cup_g I^n$ satisfies the conditions of Theorem 8 and thus has a fixed point $x \in I^n$. So, $\hat{r}f(x) = x$. If $f(x) \in H - I^n$, then $x = \hat{r}f(x) = rf(x)$ and $d(x, f(x)) < \epsilon$ since r is an ϵ -mapping, a contradiction. If $f(x) \in I^n$, then $x = \hat{r}f(x) = rf(x) = f(x)$, a contradiction. If $f(x) \in S^k \times (0, 1]$, then $x = \hat{r}f(x) = f(x)$, a contradiction. It follows that f has a fixed point. \square

We now extend our results to compact absolute retracts (ARs) with attached S^k -collars.

Theorem 10. *Let M be an AR and let X be M with an attached S^k -collar for some $k \geq 0$. If $f: M \rightarrow X$ is a mapping that pulls S^k straight outward into $S^k \times [0, 1]$, then f has a fixed point.*

Proof. We assume that M is a subset of H . Let $r: H \rightarrow M$ be a retraction. Let $X' = H \cup X$. The mapping $fr: H \rightarrow X'$ satisfies the conditions of Theorem 9 above and thus has a fixed point $x \in H$. So, $fr(x) = x$. If $x \notin M$, then $fr(x) \notin M$. It follows that $fr(x) \in S^k \times (0, 1]$. Since $S^k \times (0, 1]$ is disjoint from H , $fr(x) \notin H$. So, $x \notin H$, a contradiction. Hence, $x \in M$ and $x = fr(x) = f(x)$. \square

It is reasonable to ask if similar results hold for straight outward pulling maps on spheres with attached S^k -collars. However, unless additional conditions are placed on the mapping, in general, there will not be fixed points. We consider some examples in the next few paragraphs.

Consider the case when $k = 0$. Suppose that X is an n -sphere with an attached S^0 -collar. It is preferable, in this case, to refer to X as an n -sphere with 2 stickers attached. Furthermore, we say that the stickers are attached at points p and q if $X \approx S^n \cup \{tp \mid 1 \leq t \leq 2\} \cup \{tq \mid 1 \leq t \leq 2\}$. Identify S^n with the suspension, ΣS^{n-1} , of S^{n-1} so that the stickers are attached at the two vertices, v and $-v$. Let $g: S^{n-1} \rightarrow S^{n-1}$ be a fixed point free map and let $\Sigma g: S^n \rightarrow S^n$ be its suspension. Let f be a map that pulls S^n outward into the two stickers leaving only S^{n-1} invariant. The composition map $f \circ \Sigma g$ is fixed point free.

Let $k \geq 1$. Suppose that X is an n -sphere with an attached S^k -collar. If $n = k$, then X is an S^k -collar and the map which sends $S^k \times \{0\}$ radially outward onto $S^k \times \{1\}$ has no fixed point in $S^k = S^k \times \{0\}$. If $n = k + 1$, then $S^n \approx \Sigma S^k$. Let $F: \Sigma S^k \rightarrow \Sigma S^k$ be a map that homeomorphically interchanges the upper half and lower half of ΣS^k and is the identity on S^k . Let $g: S^n \rightarrow X$ be a map that pulls S^k straight outward, pulls each point of $\Sigma S^k - \{v, -v\}$ away from v and $-v$, and leaves v and $-v$ fixed. Then $g \circ F$ is a fixed point free outward pulling map on S^k .

For S^k -collars attached to S^n for $n > k + 1$, by taking repeated suspensions of S^k (until we reach S^n) and using maps similar to F and g above, we can build fixed point free outward pulling maps on S^k .

4. ALGEBRAIC METHODS

We now turn our attention to similar results that have been established using numerical invariants of mappings (e.g., index theory, Lefschetz theory, Nielsen theory).

In 1957, D.G. Bourgin [1, Th.1] proved the following theorem.

Theorem 11 (Bourgin). *Suppose X is an absolute retract and $\{Y_i\}_{i=1}^n$, $n \neq 1$, are open subsets of X whose closures \overline{Y}_i are pairwise disjoint absolute retracts. Let $G = \bigcup_{i=1}^n Y_i$ and let f map $Z = X - G$ to X subject to $f(\text{bd } Y_i) \subseteq \overline{Y}_i$ for each $1 \leq i \leq n$. Then f has a fixed point.*

Bourgin used a notion of the index, which he defined in previous papers, of a map f on an open set U in an ANR X , denoted $\text{index}(f, U)$, and established the usual additive relationship between the ‘‘local’’ index and the Lefschetz number of f . In this case, $\text{index}(f, U) + \text{index}(f, X - \overline{U}) = L(f)$. His result then follows almost immediately since $L(f) = 1$, $\text{index}(f, Y_i) = 1$ for each $1 \leq i \leq n$, and the index is additive on disjoint open sets. We note that our Theorem 10 (with $k = 0$) is similar to and implied by Bourgin’s result for $n = 2$. Although, in the applications above, it is not immediate how to extend to get Bourgin’s result, we note that Bourgin’s result does not imply Theorem 10 for $k \geq 1$.

In 1968, C. Bowszyc [2] defined a relative Lefschetz number for a map of pairs (X, A) to (X, A) and established a relative Lefschetz fixed point theorem. Namely,

Theorem 12 (Bowszyc). *Given a compact mapping $f: (X, A) \rightarrow (X, A)$, where X and A are ANRs and A is closed in X , the condition $L(f) \neq 0$ implies there exists a fixed point for f in $\overline{X - A}$.*

Bowszyc used this theorem to establish some fixed point results and to reprove Bourgin's results. In 1985, L. Górniewicz and A. Granas [6] proved that Bowszyc's theorem is valid for a larger class of mappings, namely, for maps of compact attraction.

In 1986, H. Schirmer [18] developed a relative Nielsen number. For definitions and discussion of Nielsen theory, see [3, Chapters 6 & 7]. Briefly, the Nielsen number for a map $f: (X, A) \rightarrow (X, A)$ of pairs of compact ANRs is a lower bound (in some cases, a sharp lower bound) for the number of fixed points of each map $g: (X, A) \rightarrow (X, A)$ that is homotopic to f as a map of pairs. Our point of view has been to consider mappings f whose range is not contained in the domain and such that f pulls the domain (or part of it) outward into the range. In the applications we have considered, it is possible to pass back and forth from this point of view to one of mappings of pairs.

Suppose M is an AR, X is M with an attached S^k -collar, and $f: M \rightarrow X$ is an outward pulling map on $S^k \times \{0\}$. We can extend f to a map of pairs $\hat{f}: (X, A) \rightarrow (X, A)$, where $A = S^k \times [0, 1]$, by defining

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in M \\ f\pi_1(x) & \text{if } x \in A. \end{cases}$$

Of course, this technique may create fixed points for \hat{f} in A . In fact, if f is a straight outward pulling map, then $f(S^k \times \{0\})$ is a set of fixed points of \hat{f} . We are interested in fixed points for \hat{f} that lie in $\overline{X - A} = M$. Furthermore, in any situation where ∂A is a retract of A , we can similarly extend f from M to X .

Conversely, if $\hat{f}: (X, A) \rightarrow (X, A)$ is a map of ANR pairs, we can view X as $\overline{X - A}$ with A attached at ∂A . Since $\hat{f}(A) \subseteq A$, it follows that $f = \hat{f}|_{\overline{X - A}}$ is an outward pulling map on ∂A . Thus, a fixed point theorem that locates a fixed point of \hat{f} in $\overline{X - A}$ will also be a fixed point theorem for the outward pulling map f .

In [18], Schirmer developed a relative Nielsen theory, proving the necessary theorems to establish that it is a viable theory that extends the general Nielsen theory. In most of our applications, since X is simply connected and $S^k \times [0, 1]$ admits a fixed point free map, the relative Nielsen number $N(f; X, S^k \times [0, 1])$ of $f: (X, S^k \times [0, 1]) \rightarrow (X, S^k \times [0, 1])$ coincides with the usual Nielsen number $N(f)$ of $f: X \rightarrow X$ (see [18, Th.2.6]). So, we gain no additional information by considering the relative theory.

In [19], Schirmer uses the relative Nielsen theory to study deformations of the pair (X, A) . In particular, she relates the Euler characteristic of the components of X and A to the size and location of minimal fixed point sets for deformations of (X, A) . In [20], using techniques of K. Scholz [21],

she extends the relative theory to maps of noncompact, metrizable ANRs. She also defines Nielsen numbers for f restricted to the boundary of A , $\tilde{n}(f; X, A)$, and for f restricted to the complement of A , $\tilde{N}(f; X, A)$. Again, many of the results in [20] are concerned with the validity of the theory and relationships between the various Nielsen numbers. Theorems 3.7, 4.3, and 4.4, however, are very nice fixed point results which can also be used to establish our Corollary 1 and Theorem 10. The restricted Nielsen numbers are particularly useful in determining the location of fixed points of the map $f: (X, A) \rightarrow (X, A)$. Having passed from our point of view to an extended map of pairs as discussed earlier, we are interested in fixed points that lie in $\overline{X - A}$. If $\tilde{N}(f; X, A) > 0$, this will be the case. First, we provide an alternative proof for Corollary 1 that uses the relative Nielsen theory.

Alternative Proof of Corollary 1. Fix $n \in \mathbb{N}$ and $0 \leq k \leq n$. Assume, without loss of generality, that f moves the first k pairs of opposite sides of I^n in opposite outward directions and the last $n - k$ pairs of opposite sides of I^n in opposite inward directions. Let t be a real number large enough so that $f(I^n) \subseteq \prod_{i=1}^n [-t, t]$. Let $X = \prod_{i=1}^n [-t, t]$. For notational convenience, if $1 \leq i \leq n$, we denote the natural retraction of $[-t, t]$ onto $I = [-1, 1]$ by r_i . Let $r: X \rightarrow I^n$ be the product retraction; that is, $r = \prod_{i=1}^n r_i$.

It will be helpful to note that

$$(1) \quad \text{if } x \in X \text{ and } |\pi_j(x)| \geq 1 \text{ for some } 1 \leq j \leq n, \text{ then } r(x) \in I_{-j}^n \cup I_j^n.$$

To see this, note that since r is a product map, $|\pi_j r(x)| = |r_j \pi_j(x)| = 1$.

Let $A = \{x \in X \mid |\pi_i(x)| \geq 1 \text{ for some } 1 \leq i \leq k\}$. Note that if $k = 0$, then $A = \emptyset$. Otherwise, A is an ANR; in fact, A is an $(S^{k-1}, n - k + 1)$ -collar.

If $A = \emptyset$, then $fr: X \rightarrow f(I^n) \subseteq X$ has a fixed point $x \in X$ by Brouwer's Theorem, since X is a closed n -cell. Suppose that, for some $1 \leq j \leq n$, $|\pi_j(x)| > 1$. By (1), $r(x) \in I_{-j}^n \cup I_j^n$. Since f maps each pair of opposite sides of I^n inward, it follows that $|\pi_j fr(x)| \leq 1$. But since x is a fixed point of fr , $1 < |\pi_j(x)| = |\pi_j fr(x)| \leq 1$, a contradiction. Hence, for each $1 \leq i \leq n$, $|\pi_i(x)| \leq 1$. Thus, $x \in I^n$ and $x = r(x)$. So, $x = fr(x) = f(x)$. That is, x is a fixed point of f and we are done. So, we assume that $A \neq \emptyset$; hence, $k \geq 1$.

We claim that the map fr is a map of the pair (X, A) to itself. We need to show that $fr(A) \subseteq A$. Let $x \in A$. There exists $1 \leq j \leq k$ such that $|\pi_j(x)| \geq 1$. By (1), $r(x) \in I_{-j}^n \cup I_j^n$. Since f maps I_{-j}^n and I_j^n outward, it follows that $|\pi_j fr(x)| \geq 1$. Hence, $fr(x) \in A$.

Now we calculate the relative Nielsen number of fr , $\tilde{N}(fr; X, A)$, on the complement of A .

By Schirmer [20, Theorem 3.7], since X is simply connected, $\tilde{N}(fr; X, A)$ is determined by considering $L(fr)$ and $L(fr|_A)$. Since X is contractible, $L(fr) = 1$. It is clear that $fr|_A$ is homotopic to the identity map on A , so $L(fr|_A)$ is equal to the Euler characteristic of A , $\chi(A)$. Since A is an $(S^{k-1}, n - k + 1)$ -collar, $\chi(A) = \chi(S^{k-1}) \cdot \chi(I^{n-k+1}) = \chi(S^{k-1})$. It follows

that $L(fr|_A)$ is 0 or 2 as k is even or odd. In either case, $L(fr) \neq L(fr|_A)$, and it follows from Schirmer's theorem that $\tilde{N}(fr; X, A) = 1$. This implies that fr has a fixed point in $\overline{X - A}$. Let $x \in \overline{X - A}$ with $fr(x) = x$.

If $x \in A \cap (\overline{X - A})$, then for some $1 \leq j \leq k$, $|\pi_j(x)| \geq 1$. But also, $|\pi_j(x)| \leq 1$. So, $|\pi_j(x)| = 1$. Hence, $x \in I_{-j}^n \cup I_j^n$ and $x = r(x)$. So, x is a fixed point of f .

If $x \notin A$, then for each $1 \leq i \leq n$, $|\pi_i(x)| < 1$. Again, $x = r(x)$ and x is a fixed point of f . \square

We can also apply Theorem 3.7 in [20] to establish our Theorem 10. Assume the hypothesis of Theorem 10, let $A = S^k \times [0, 1]$, and extend $f: M \rightarrow X$ to $\hat{f}: (X, A) \rightarrow (X, A)$ as in the earlier discussion. As in the proof of Corollary 1 above, $L(\hat{f}) = 1$ and $L(f|_A)$ is either 0 or 2 as k is either odd or even. Thus, $L(\hat{f}) \neq L(f|_A)$ and it follows that $\tilde{N}(\hat{f}; X, A) = 1$. This implies that \hat{f} has a fixed point in $\overline{X - A} = M$. So, our original outward pulling map $f: M \rightarrow X$ has a fixed point. Schirmer shows in Example 3.8 of [20] that Bourgin's (and Bowszyc's) result follows from her Theorem 3.7.

In [20], if we have a pair (X, A) for which $\overline{X - A} = X$, then $\tilde{N}(f; X, A)$ coincides with $N(f; X, A)$ and provides no additional information. In 1989, R.F. Brown, R.E. Greene, and H. Schirmer [4] continued this program for maps of pairs (X, A) where $\overline{X - A} = X$. In particular, they investigated applications for X a manifold with boundary A and considered both continuous and smooth extensions of maps $f: A \rightarrow A$.

Finally, we call attention to several results of S.B. Nadler, Jr. related to partially outward maps. In [15], Nadler extends a result of I. Rosenholtz [17, Th 3.0] from open self maps of a continuum (a compact, connected metric space) to maps whose domain may not be the entire continuum. He proves that for Y a compact subset of a continuum X and $f: Y \rightarrow X$ an ϵ -expansive open map, f must have a fixed point. In [14], Nadler provides some examples of maps of n -balls ($n \geq 2$) onto larger n -balls, establishing that there are not always fixed points in this situation. He also proves that if K is a proper subcontinuum of S^2 that doesn't separate S^2 and $f: K \rightarrow S^2$ is a monotone surjective map taking a point x of ∂K outward (i.e., $f(x) \notin K$), then f has a fixed point.

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