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SOME STRUCTURE THEOREMS FOR INVERSE LIMITS WITH SET-VALUED FUNCTIONS

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Abstract. We investigate inverse limits with set-valued bonding functions. We generalize theorems of W. T. Ingram and William S. Mahavier, and of Van Nall, on the connectedness of the inverse limit space. We establish a fixed point theorem and show that under certain conditions, inverse limits with set-valued bonding functions can be realized as ordinary inverse limits. We also obtain some results that are useful in determining the existence of certain subcompacta of the inverse limit on a single space with a single set-valued bonding function.

All spaces considered in this paper will be metric. A *continuum* is a compact, connected metric space. A continuous function $f\colon X \to Y$ will be referred to as a *mapping*. We wish to consider inverse limits on inverse sequences $X_1, X_2, \ldots$ of compacta with upper semi-continuous bonding functions $G_n^{n+1}\colon X_{n+1} \to 2^{X_n}$. These inverse limits have been called generalized inverse limits and inverse limits with set-valued bonding functions. The literature on and interest in these inverse limits is growing fairly rapidly (see [3], [4], [5], [6], [7], [8], [10], [11], [12]). Perusing these papers, one notices that there are some commonly-used notations and terminology for important concepts related to a function $G\colon X \to 2^Y$ that are defined, in a natural way, relative to X, Y, and $X \times Y$. Ordinarily, the graph of G would lie in $X \times 2^Y$ with the product topology induced from the topology of X and the topology of 2^Y. However, we wish to view the graph of G as a subset of $X \times Y$, and for $x \in X$, we view $G(x)$ as a subset of Y rather than a point in 2^Y. For essentially all of the properties we are interested in, the topologies of X, Y, and $X \times Y$ will influence

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our terminology. With this in mind, we will write $G\colon X \to Y$ and say that G is a *set-valued function* from X to Y if and only if G is a function from X to 2^Y. We note that the set $\{(x,y) \in X \times Y \mid y \in G(x)\}$, which we call the *graph of* G and denote by $\overrightarrow{\mathrm{gr}}\, G$, coincides with the definition of G being a relation from X to Y. That is, $G \subset X \times Y$ and all of the notations $y \in G(x)$, $(x,y) \in G$, and xGy store the same meaning and are not uncommon. Nevertheless, in this paper, we will typically distinguish between G and the graph of G. Most of the structural theorems for the inverse limits studied in this paper are related to properties of the graphs of the set-valued bonding functions.

A set-valued function $G\colon X \to Y$ is *upper semi-continuous at the point* $x \in X$ if, for each open set V in Y containing the set $G(x)$, there is an open set U in X such that $x \in U$ and $G(p) \subset V$ for each $p \in U$. If $G\colon X \to Y$ is upper semi-continuous at each point of X, then G is said to be *upper semi-continuous* (usc). Suppose H is a subset of $X \times Y$ and the projection of H into X is A. It has been shown (see [6, Theorem 2.1]) that H is closed if and only if H is the graph of a usc set-valued function $h\colon A \to Y$. We will occasionally use this equivalence without further reference.

Let $X_1, X_2, \ldots$ be a sequence of compacta and for each $n \geq 1$, let $G_n^{n+1}\colon X_{n+1} \to X_n$ be a usc set-valued function. We say that $\{X_n, G_n^{n+1}\}$ is an *inverse sequence with usc set-valued bonding functions.* The *inverse limit* of the sequence, denoted $\varprojlim\{X_n, G_n^{n+1}\}$, is the set $\{(x_1, x_2, \ldots) \in \prod_{n\geq 1} X_n \mid x_n \in G_n^{n+1}(x_{n+1}) \text{ for } n \geq 1\}$. The topological structure of these inverse limits, as related to the topological structure of the graphs of the bonding functions, is the focus of the paper.

1. Graphs and Inverse Graphs

Important objects in this study will be generalized graphs as introduced by W. T. Ingram and William S. Mahavier [6, p. 123]. We change their notation and terminology slightly. Suppose that $\{X_i, G_i^{i+1}\}$ is an inverse sequence of compacta with usc set-valued bonding functions $G_i^{i+1}\colon X_{i+1} \to X_i$. For $m \geq 1$, we define the *inverse graph of* $(G_1^2, G_2^3, \ldots, G_m^{m+1})$ as follows.

$$\overleftarrow{\mathrm{gr}}\,(G_1^2, G_2^3, \ldots, G_m^{m+1}) = \{x \in \prod_{i=1}^{m+1} X_i \mid x_i \in G_i^{i+1}(x_{i+1}) \text{ for } 1 \leq i \leq m\}.$$

The set $G(G_1^2, G_2^3, \ldots, G_m^{m+1})$ in [6] is the same as our inverse graph of $(G_1^2, G_2^3, \ldots, G_m^{m+1})$.

For a collection of two compacta X_1 and X_2 and $G\colon X_2 \to X_1$ a usc set-valued function, let $D(G) = \{x \in X_2 \mid y \in G(x) \text{ for some } y \in X_1\}$

and $R(G) = \{y \in X_1 \mid y \in G(x) \text{ for some } x \in X_2\}$. If $A \subset X_2$, let $G|_A$ be the usc set-valued function such that $D(G|_A) = A$ and $G|_A(x) = G(x)$ for $x \in A$. We note that $\overleftarrow{\text{gr}}\,(G) = \overrightarrow{\text{gr}}\,(G^{-1})$, where $G^{-1}\colon R(G) \to X$ is defined by $x \in G^{-1}(y)$ if and only if $y \in G(x)$. Note also that the graph of G is homeomorphic to the inverse graph of G as subsets of $X_2 \times X_1$ and $X_1 \times X_2$, respectively. For homeomorphic spaces X and Y, we write $X \overset{T}{\approx} Y$. So we have that

(∗) $\overleftarrow{\text{gr}}\,G \overset{T}{\approx} \overrightarrow{\text{gr}}\,G$.

Since we will be using graphs and inverse graphs throughout this section and §2, it will be convenient to simplify the notation somewhat. We let $\overleftarrow{\text{gr}}\,G_1^{m+1} = \overleftarrow{\text{gr}}\,(G_1^2, G_2^3, \ldots, G_m^{m+1})$. For consistency of notation in theorems that follow, if $1 \le k \le m+1$, we let $\overleftarrow{\text{gr}}\,G_k^k = X_k$.

For the special case when, for each $1 \le i \le m$, $X_i = X = X_{i+1}$ and $G_i^{i+1} = G$, we call $G_{m+1}^{\leftarrow} = \overleftarrow{\text{gr}}\,(G, G, \ldots, G)$ the $(m+1)$-*inverse graph* of G. Note that the $(m+1)$-*inverse graph* of G sits in $\prod_{i=1}^{m+1} X$. So, according to this notation, $G_2^{\leftarrow} = \overleftarrow{\text{gr}}\,G$.

If each $X_i = [0,1]$, one can "see" what an inverse graph $\overleftarrow{\text{gr}}\,G_1^{m+1}$ looks like by doing the following. Draw the inverse graph of G_i^{i+1} for each $1 \le i \le m$. Start by applying $(G_2^3)^{-1}$ to the second coordinates of each point of $\overleftarrow{\text{gr}}\,(G_1^2)$, getting $\overleftarrow{\text{gr}}\,(G_1^2, G_2^3)$ sitting in $[0,1]^3$ "above" the inverse graph of G_1^2 in $[0,1]^2$. Continue this process through $m-1$ steps to get $\overleftarrow{\text{gr}}\,G_1^{m+1}$. We will discuss some examples later.

Fix $1 \le j < m+1$. It is clear that $\prod_{i=1}^{j} X_i \times \prod_{i=j+1}^{m+1} X_i \overset{T}{\approx} \prod_{i=1}^{m+1} X_i$ under the homeomorphism defined by $h((x_1, \ldots, x_j), (x_{j+1}, \ldots, x_{m+1})) = (x_1, \ldots, x_j, x_{j+1}, \ldots, x_{m+1})$. So we will make no distinction between these spaces or between subsets M and $h(M)$ of the two.

For $1 \le j < k < m+1$, let $G_j^{m+1}[k]\colon\ \overleftarrow{\text{gr}}\,G_{k+1}^{m+1} \to \overleftarrow{\text{gr}}\,G_j^k$ be the usc set-valued function defined by $(x_j, x_{j+1}, \ldots, x_k) \in G_j^{m+1}[k](x_{k+1}, \ldots, x_{m+1})$ if and only if $(x_j, \ldots, x_{m+1})$ is in $\overleftarrow{\text{gr}}\,G_j^{m+1}$. We note that $\overleftarrow{\text{gr}}\,G_j^{m+1}[k] \overset{T}{\approx} \overleftarrow{\text{gr}}\,G_j^{m+1}$.

For $j \ge 1$, let $\pi_j\colon \prod_{i=1}^{\infty} X_i \to X_j$ denote j^{th}-coordinate projection. It will be useful to also let $\pi_j\colon \prod_{i=k}^{m} X_i \to X_j$ denote j^{th}-coordinate projection for any finite subsequence $\{k, k+1, \ldots, m-1, m\}$ of $\mathbb{N}$ with $k < m$ and $k \le j \le m$.

Suppose for each $1 \le i \le m$, $G_i^{i+1}\colon X_{i+1} \to X_i$ is a usc set-valued function. In general, $\overleftarrow{\text{gr}}\,G_1^{m+1}$ may not be connected even if each $\overleftarrow{\text{gr}}\,(G_i^{i+1})$ is connected (see [6, Example 1] and consider the 3-inverse graph of M). We define a property for usc set-valued functions on continua that will ensure that the inverse graphs are connected. Our property is weaker than the properties in [6, Theorem 4.3 and Theorem 4.5] and equivalent to the properties in [12, Theorem 3.1].

A usc set-valued function $G\colon X_2 \to X_1$ is *continuum-valued* if, for each $x \in X_2$, the set $G(x)$ is connected in X_1.

Observation 1.1 follows from $(*)$ and [6, Theorem 4.1].

Observation 1.1. *If $G\colon X_2 \to X_1$ is a usc continuum-valued function and X_2 is connected, then $\overleftarrow{\mathrm{gr}}\, G$ is connected.*

We say that a usc set-valued function $G\colon X_2 \to X_1$ is a *union of usc continuum-valued functions* if, for each $x \in X_2$ and each $y \in G(x)$, there exists a usc continuum-valued function $g\colon X_2 \to X_1$ such that $y \in g(x)$ and $\overleftarrow{\mathrm{gr}}\, g \subset \overleftarrow{\mathrm{gr}}\, G$.

If the usc set-valued function $G\colon X_2 \to X_1$ is surjective, we say that $G^{-1}\colon X_1 \to X_2$ is a *union of usc continuum-valued functions* if, for each $y \in X_1$ and $x \in G^{-1}(y)$, there exists a usc continuum-valued function $f\colon X_1 \to X_2$ such that $x \in f(y)$ and $\overrightarrow{\mathrm{gr}}\, f \subset \overrightarrow{\mathrm{gr}}\, G^{-1} = \overleftarrow{\mathrm{gr}}\, G$.

Observation 1.2. *If the usc set-valued function $G\colon X_2 \to X_1$ is a union of usc continuum-valued functions, then for each closed subset K of X_2, $G|_K\colon K \to X_1$ is a union of usc continuum-valued functions.*

Observation 1.3 follows from [6, Theorem 4.3 and Theorem 4.5].

Observation 1.3. *Suppose $X_1, X_2, \ldots, X_{m+1}$ are continua and, for each $1 \le i \le m$, $G_i^{i+1}\colon X_{i+1} \to X_i$ is a surjective usc set-valued function. If each G_i^{i+1} is continuum-valued (or if each $(G_i^{i+1})^{-1}$ is continuum-valued), then $\overleftarrow{\mathrm{gr}}\,(G_1^{m+1})$ is connected.*

In the lemmas and theorems that follow in this section, we will be working in $\prod_{i=1}^{m+1} X_i$, where each X_i is a continuum.

Lemma 1.4. *Suppose that $X_1, X_2, \ldots, X_{m+1}$ are continua and, for each $1 \le i \le m$, $G_i^{i+1}\colon X_{i+1} \to X_i$ is a surjective usc set-valued function with a connected inverse graph. Suppose also that for each $1 \le i \le m$, $G_i^{i+1}\colon X_{i+1} \to X_i$ is a union of usc continuum-valued functions. Then the usc set-valued function $G_1^{m+1}[m]\colon X_{m+1} \to \overleftarrow{\mathrm{gr}}\, G_1^m$ is a union of usc continuum-valued functions.*

Proof. We use induction on the number of bonding functions. If $m = 1$, then $G_1^2[1] = G_1^2$ and $G_1^2\colon X_2 \to X_1$ is a union of usc continuum-valued functions by assumption.

Assume that $G_1^m[m-1]\colon X_m \to \overleftarrow{\mathrm{gr}}\, G_1^{m-1}$ is a union of usc continuum-valued functions.

Let $(x_1, x_2, \ldots, x_m) \in G_1^{m+1}[m](x_{m+1})$. Since G_m^{m+1} is a union of usc continuum-valued functions, there exists a usc continuum-valued function $h_m^{m+1}\colon X_{m+1} \to X_m$ such that $x_m \in h_m^{m+1}(x_{m+1})$ and $\overleftarrow{\mathrm{gr}}\ h_m^{m+1} \subset \overleftarrow{\mathrm{gr}}\ G_m^{m+1}$. Also, by inductive assumption, there exists a usc continuum-valued function $h_1^m\colon R(h_m^{m+1}) \to \overleftarrow{\mathrm{gr}}\ G_1^{m-1}$ such that $(x_1, \ldots, x_{m-1}) \in h_1^m(x_m)$ and $\overleftarrow{\mathrm{gr}}\ h_1^m \subset \overleftarrow{\mathrm{gr}}\ G_1^m[m-1]$. Since $R(h_m^{m+1}) = \pi_m(\overleftarrow{\mathrm{gr}}\ h_m^{m+1})$, it follows that $R(h_m^{m+1})$ is closed and connected in X_m.

Let $L = \{(z_1, \ldots, z_{m+1}) \in \overleftarrow{\mathrm{gr}}\ G_1^{m+1} \mid (z_1, \ldots, z_{m-1}) \in h_1^m(z_m) \text{ and } z_m \in h_m^{m+1}(z_{m+1})\}$. Note that $(x_1, x_2, \ldots, x_{m+1}) \in L$ and $L \overset{T}{\approx} \overleftarrow{\mathrm{gr}}\ (h_1^m, h_m^{m+1})$. By definition, $L \subset \overleftarrow{\mathrm{gr}}\ G_1^{m+1}$. We now view L as the inverse graph of a usc set-valued function ℓ from X_{m+1} to $\overleftarrow{\mathrm{gr}}\ G_1^m$. We have left only to see that ℓ is continuum-valued.

To see that for each $x \in X_{m+1}$, $\ell(x)$ is non-empty and connected, we note that $\ell(x) \overset{T}{\approx} \overleftarrow{\mathrm{gr}}\ h_1^m|_{h_m^{m+1}(x)}$, which is non-empty by the existence of h_1^m and h_m^{m+1}. Since both h_m^{m+1} and h_1^m are continuum-valued, it follows from Observation 1.1 that $\ell(x)$ is connected. □

Lemma 1.5. *Suppose that $X_1, X_2, \ldots, X_{m+1}$ are continua and for each $1 \leq i \leq m$, $G_i^{i+1}\colon X_{i+1} \to X_i$ is a surjective usc set-valued function whose inverse graph is connected. Suppose also that for each $1 \leq i \leq m$, $(G_i^{i+1})^{-1}\colon X_i \to X_{i+1}$ is a union of usc continuum-valued functions. Then the usc set-valued function $(G_1^{m+1}[1])^{-1}\colon X_1 \to \overleftarrow{\mathrm{gr}}\ G_2^{m+1}$ is a union of usc continuum-valued functions.*

Proof. The proof is similar to the proof of Lemma 1.4. □

The proof of Theorem 1.6 is similar to the proof of Theorem 4.5 in [6].

Theorem 1.6. *Suppose that $X_1, X_2, \ldots, X_{m+1}$ are continua and for each $1 \leq i \leq m$, $G_i^{i+1}\colon X_{i+1} \to X_i$ is a surjective usc set-valued function with a connected inverse graph. Suppose also that for each $1 \leq i \leq m$, $G_i^{i+1}\colon X_{i+1} \to X_i$ is a union of usc continuum-valued functions. Then $\overleftarrow{\mathrm{gr}}\ G_1^{m+1}$ is connected.*

Proof. We use induction on the number of bonding functions. For $m = 1$, $\overleftarrow{\mathrm{gr}}\ G_1^2$ is connected by assumption.

Assume that $\overleftarrow{\mathrm{gr}}\ G_1^m$ is connected. Let A and B be closed sets whose union is $\overleftarrow{\mathrm{gr}}\ G_1^{m+1}$, and let h be the projection from $\overleftarrow{\mathrm{gr}}\ G_1^{m+1}$ onto $\overleftarrow{\mathrm{gr}}\ G_1^m$. Since $h(A) \cup h(B) = \overleftarrow{\mathrm{gr}}\ G_1^m$, there exists a point $p = (p_1, \ldots, p_m) \in h(A) \cap h(B)$. Assume $a = (p_1, \ldots, p_m, y_{m+1}) \in A$ and $b = (p_1, \ldots, p_m, z_{m+1}) \in B$. By inductive assumption, $\overleftarrow{\mathrm{gr}}\ G_1^m$ is connected. Also, by Lemma 1.4, $G_1^m[m-1]\colon X_m \to \overleftarrow{\mathrm{gr}}\ G_1^{m-1}$ is a union of usc continuum-valued functions.

Let $g\colon X_m \to \overleftarrow{\mathrm{gr}}\ G_1^{m-1}$ be a usc continuum-valued function such that $(p_1, \ldots, p_{m-1}) \in g(p_m)$ and $\overleftarrow{\mathrm{gr}}\ g \subset \overleftarrow{\mathrm{gr}}\ G_1^m[m-1]$.

Let $S = \{(x_1, \ldots, x_{m+1}) \in \overleftarrow{\mathrm{gr}}\ G_1^{m+1} \mid (x_1, \ldots, x_{m-1}) \in g(x_m)$ and $x_m \in G_m^{m+1}(x_{m+1})\}$. We view S as the inverse graph of a usc set-valued function $s\colon\ \overleftarrow{\mathrm{gr}}\ G_m^{m+1} \to \prod_{i=1}^{m-1} X_i$ defined by $s(x_m, x_{m+1}) = g(x_m)$. Note that $\overleftarrow{\mathrm{gr}}\ G_m^{m+1}$ is connected by assumption and s is continuum-valued. It follows from Observation 1.1 that $S \overset{T}{\approx} \overleftarrow{\mathrm{gr}}\ s$ is connected. Note that a and b are in S; so S meets both A and B. It follows that A and B are not mutually separated. Hence, $\overleftarrow{\mathrm{gr}}\ G_1^{m+1}$ is connected. □

Theorem 1.7. *Suppose that $X_1, X_2, \ldots, X_{m+1}$ are continua and for each $1 \leq i \leq m$, $G_i^{i+1}\colon X_{i+1} \to X_i$ is a surjective usc set-valued function whose inverse graph is connected. Suppose also that for each $1 \leq i \leq m$, $(G_i^{i+1})^{-1}\colon X_i \to X_{i+1}$ is a union of usc continuum-valued functions. Then $\overleftarrow{\mathrm{gr}}\ G_1^{m+1}$ is connected.*

Proof. The proof is similar to the proof of Theorem 1.6. □

In [12, Theorem 3.1], Van Nall has a condition on a surjective relation $F\colon X \to X$ that ensures $\varprojlim\{X, F\}$ is a continuum. His condition that F be the union of a collection of closed subsets $\{F_\alpha\}_{\alpha\in\Gamma}$ with certain properties is equivalent to F being a union of usc continuum-valued functions. Via Nall's Theorem 3.3, having F^{-1} be a union of usc continuum-valued functions also leads to the connectedness of $\varprojlim\{X, F\}$. We are able to generalize his Theorem 3.1 (see Corollary 1.8) by additionally showing that for different factor spaces and different set-valued bonding functions, the inverse graphs and the inverse limit space $\varprojlim\{X_i, G_i^{i+1}\}$ will be continua.

Completely analogously to [6, theorems 4.4, 4.6, 4.7, and 4.8], we get the following corollary.

Corollary 1.8. *Let $X_1, X_2, \ldots$ be an inverse sequence of continua and suppose that for each $i \geq 1$, $G_i^{i+1}\colon X_{i+1} \to X_i$ is a surjective usc set-valued function whose inverse graph is connected. Suppose also that for each $i \geq 1$, $G_i^{i+1}\colon X_{i+1} \to X_i$ is a union of usc continuum-valued functions (or for each $i \geq 1$, $(G_i^{i+1})^{-1}\colon X_i \to X_{i+1}$ is a union of usc continuum-valued functions). Then $\varprojlim\{X_i, G_i^{i+1}\}$ is a continuum.*

Ingram and Mahavier [7, p. 90, Theorem 126] have shown that if $X = \varprojlim\{X_i, G_i^{i+1}\}$ is an inverse limit with usc set-valued bonding functions such that for each $i \geq 1$, one of G_i^{i+1} or $(G_i^{i+1})^{-1}$ is continuum-valued, then X is a continuum. Ingram has asked the author in correspondence if an analogous theorem is possible if "continuum-valued" is replaced with "a union of continuum-valued functions." We provide an example below to show that this, in general, is not the case.

Example 1.9. There exists an inverse limit space $X = \varprojlim\{[0,1], G_i^{i+1}\}$, where $(G_1^2)^{-1}$ is a continuum-valued function, G_2^3 is the union of two mappings, G_i^{i+1} is the identity mapping for each $i \geq 3$, and X is disconnected.

Proof. Let $\overrightarrow{\mathrm{gr}}\, G_1^2$ be the union of $[0,1] \times \{0\}$ and the convex arc from the point $(0,0)$ to the point $(\frac{3}{4}, 1)$ in $[0,1]^2$. Note that $(G_1^2)^{-1}$ is a continuum-valued function. Specifically, $(G_1^2)^{-1}(0) = [0,1]$ and $(G_1^2)^{-1}(t) = \frac{3}{4}t$ for $0 < t \leq 1$. Let $\overrightarrow{\mathrm{gr}}\, G_2^3$ be the union of the convex arc from $(0, \frac{3}{4})$ to $(1,1)$ in $[0,1]^2$ and the diagonal in $[0,1]^2$. Note that G_2^3 is the union of two mappings. Specifically, $f_1(t) = \frac{1}{4}t + \frac{3}{4}$ and $f_2(t) = t$. Let $G_i^{i+1} = \mathrm{id}$. Let $X = \varprojlim\{[0,1], G_i^{i+1}\}$. To see that X is not connected, note that the point $(0,0,0,\ldots) \in X$ and the point $(1, \frac{3}{4}, 0, 0, \ldots)$ is the only point of X in the open set $U = (\frac{7}{8}, 1] \times (\frac{5}{8}, \frac{7}{8}) \times [0, \frac{1}{8}) \times [0,1] \times [0,1] \times \ldots$. It is also easy to see that the 3-inverse graph $\overleftarrow{\mathrm{gr}}\, G_1^3$ is not connected. □

It would be useful to know what conditions on set-valued bonding functions on absolute retracts (or on $[0,1]$) would produce inverse graphs that are absolute retracts. We mention this again later after we establish a fixed point theorem related to absolute retracts. That the bonding functions be unions of usc continuum-valued functions whose graphs are absolute retracts is not enough. Ingram provides a counterexample [5, Example 4.2], even on $[0,1]$. Note that his bonding function f is a union of usc continuum-valued functions and the graph of f is an absolute retract, but the 3-inverse graph contains a simple closed curve.

Question 1.10. Let $X_1, \ldots, X_{m+1}$ be absolute retracts and for $1 \leq i \leq m$, let $G_i^{i+1}\colon X_{i+1} \to X_i$ be a usc set-valued function whose graph is an absolute retract. What additional conditions on G_i^{i+1} will ensure that $\overleftarrow{\mathrm{gr}}\, G_1^{m+1}$ is an absolute retract?

2. k-Tail Sequences in Inverse Sequences with Set-Valued Functions

Let $X = \varprojlim\{X_n, G_n^{n+1}\}$, where for each $n \geq 1$, X_n is a compactum with $\mathrm{diam}(X_n) = 1$, and $G_n^{n+1}\colon X_{n+1} \to X_n$ is a usc set-valued function. Let d denote the usual metric on $\prod_{n \geq 1} X_n$. Let $\rho_n\colon \prod_{i \geq 1} X_i \to \prod_{i=1}^n X_i$ denote projection onto the first n coordinates.

Let $k \in \mathbb{N}$ and for $i \geq k$, let Y_i be a compactum such that $Y_i \subset X_i$. Suppose that $\{Z_i^{i+1}\colon Y_{i+1} \to X_i\}_{i \geq k}$ is a sequence of usc set-valued functions such that for each $i \geq k$, $\overrightarrow{\text{gr}}\, Z_i^{i+1} \subset \overrightarrow{\text{gr}}\, G_i^{i+1}$. Suppose also that for $i \geq 0$,

(i) $Y_{k+i} \subset R(Z_{k+i}^{k+i+1})$, and
(ii) $(Z_{k+i}^{k+i+1})^{-1}$ is a mapping (from $R(Z_{k+i}^{k+i+1})$ into X_{k+i+1}).

Under these conditions, we say that $\{Z_i^{i+1}\}_{i\geq k}$ is a *k-tail sequence of inverse mappings* (with respect to the inverse sequence $\{X_n, G_n^{n+1}\}$). We use the k-tail sequence $\{Z_i^{i+1}\}_{i\geq k}$ to generate a subcompactum of X. Let $A_k = R(Z_k^{k+1})$. For $1 \leq i < k$, let $A_i = R(G_i^{i+1}|_{A_{i+1}})$ and let $g_i^{i+1} = G_i^{i+1}|_{A_{i+1}}$. For $i \geq 1$, let $A_{k+i} = (Z_{k+i-1}^{k+i})^{-1}(A_{k+i-1})$ and let $g_{k+i-1}^{k+i} = Z_{k+i-1}^{k+i}|_{A_{k+i}}$. Note that for $i \geq 1$, $A_{k+i} \subset Y_{k+i}$

Let $A(k) = \varprojlim\{A_n, g_n^{n+1}\}$. By [4, Theorem 2.4], $A(k)$ is a subcompactum of X. We say that $A(k)$ is the *subcompactum of X generated by the k-tail sequence* $\{Z_i^{i+1}\}_{i\geq k}$. We note that if $\{Z_i^{i+1}\}_{i\geq k}$ is a k-tail sequence (of inverse mappings), then $\{Z_i^{i+1}\}_{i\geq k+j}$ is a $(k+j)$-tail sequence for each $j \geq 1$. Furthermore, it is straightforward to see that the sequence $\{A(k+j)\}_{j\geq 0}$ of subcompacta of X generated by the $(k+j)$-tail sequences is nested. That is, $A(k) \subset A(k+1) \subset \ldots \subset A(k+j) \subset \ldots$.

Theorem 2.1. *Let $X = \varprojlim\{X_n, G_n^{n+1}\}$, where for each $n \geq 1$, X_n is a compactum and $G_n^{n+1}\colon X_{n+1} \to X_n$ is a usc set-valued function. Suppose $A(k)$ is the subcompactum of X generated by the k-tail sequence $\{Z_i^{i+1}\}_{i\geq k}$. Then the projection map $\rho_k|_{A(k)}\colon A(k) \to \overleftarrow{\text{gr}}\, G_1^k|_{A_k}$ is a homeomorphism.*

Proof. To see that $\rho_k|_{A(k)}$ is one-to-one, let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be points of $A(k)$ and suppose that $\rho_k(x) = \rho_k(y)$. So x and y agree in their first k coordinates. Also, $x_k \in Z_k^{k+1}(x_{k+1})$ and $y_k \in Z_k^{k+1}(y_{k+1})$ by definition of $A(k)$. By property (ii), $(Z_k^{k+1})^{-1}$ is a mapping. Since $x_k = y_k$, it follows that $x_{k+1} = y_{k+1}$. Similarly, $x_{k+1} \in Z_{k+1}^{k+2}(x_{k+2})$ and $y_{k+1} \in Z_{k+1}^{k+2}(y_{k+2})$. Since $(Z_{k+1}^{k+2})^{-1}$ is a mapping, it follows that $x_{k+2} = y_{k+2}$. So we get that $x_{k+i} = y_{k+i}$ for all $i \geq 1$. Hence, $x = y$.

Now we show that $\rho_k(A(k)) = \overleftarrow{\text{gr}}\, G_1^k|_{A_k}$.

Let $x = (x_1, x_2, \ldots, x_k, \ldots) \in A(k)$; so $\rho_k(x) = (x_1, x_2, \ldots, x_k)$. By definition of $A(k)$, $x_k \in R(Z_k^{k+1}) = A_k$, and for $1 \leq i < k$, $x_{k-i} \in G_{k-i}^{k-i+1}(x_{k-i+1})$. It follows that $(x_1, x_2, \ldots, x_k) \in \overleftarrow{\text{gr}}\, G_1^k|_{A_k}$.

Let $(x_1, x_2, \ldots, x_k) \in \overleftarrow{\text{gr}}\, G_1^k|_{A_k}$. By definition, $x_k \in A_k$. Since $(Z_k^{k+1})^{-1}$ is a mapping, there exists unique $x_{k+1} \in X_{k+1}$ such that $x_k \in Z_k^{k+1}(x_{k+1})$, and thus by definition of A_{k+1}, $x_{k+1} \in A_{k+1}$. Similarly, by properties (i) and (ii), for each $i \geq 2$, there exists unique $x_{k+i} \in A_{k+i}$ such that

$x_{k+i-1} \in Z_{k+i-1}^{k+i}(x_{k+i})$. Hence, $x = (x_1, x_2, \ldots, x_k, \ldots) \in A(k)$ and $\rho_k(x) = (x_1, x_2, \ldots, x_k)$. We have that $\rho_k|_{A(k)}$ is a homeomorphism onto $\overleftarrow{\mathrm{gr}}\ G_1^k|_{A_k}$. □

In the proof of Theorem 2.1, we see that for $x = (x_1, x_2, \ldots, x_k, \ldots) \in A(k)$, the k^{th} coordinate uniquely determines the i^{th} coordinate for all $i \geq k+1$. We refer to $(x_k, x_{k+1}, \ldots)$ as the *k-tail sequence generated by* x_k.

Theorem 2.2. *Let* $X = \varprojlim\{X_n, G_n^{n+1}\}$*, where for each* $n \geq 1$*,* X_n *is a compactum and* $G_n^{n+1}\colon X_{n+1} \to X_n$ *is a usc set-valued function. Suppose* $A(k)$ *is the subcompactum of* X *generated by the* k*-tail sequence* $\{Z_i^{i+1}\}_{i \geq k}$*. If* $R(Z_k^{k+1}) = X_k$*, then there exists a retraction* r_k *of* X *onto* $A(k)$ *and* $d(r_k, \mathrm{id}|_X) < \frac{1}{2^{k-1}}$*.*

Proof. From Theorem 2.1, we have that $\rho_k|_{A(k)}$ is a homeomorphism onto the inverse graph $\overleftarrow{\mathrm{gr}}\ G_1^k|_{A_k}$. By assumption, $A_k = R(Z_k^{k+1}) = X_k$. So $\overleftarrow{\mathrm{gr}}\ G_1^k|_{A_k} = \overleftarrow{\mathrm{gr}}\ G_1^k$. Let $\gamma_k\colon \overleftarrow{\mathrm{gr}}\ G_1^k \to A(k)$ be the inverse homeomorphism of $\rho_k|_{A(k)}$. Define $r_k\colon X \to A(k)$ by $r_k = \gamma_k \rho_k$.

If $x \in A(k)$, then $r_k(x) = \gamma_k \rho_k(x) = x$ by definition of γ_k. So r_k is a retraction onto $A(k)$. Let $x \in X$. Then $\rho_k(r_k(x)) = \rho_k(\gamma_k \rho_k(x)) = \rho_k(x)$. So $r_k(x)$ and x agree in their first k coordinates. It follows that $d(r_k, \mathrm{id}|_X) < \frac{1}{2^{k-1}}$. □

Corollary 2.3. *Let* $X = \varprojlim\{X_n, G_n^{n+1}\}$*, where for each* $n \geq 1$*,* X_n *is a compactum and* $G_n^{n+1}\colon X_{n+1} \to X_n$ *is a usc set-valued function. Suppose that* $\{Z_i^{i+1}\}_{i \geq k}$ *is a* k*-tail sequence and, for each* $i \geq k$*,* $R(Z_i^{i+1}) = X_i$*. Then* X *contains a monotonic increasing sequence of compacta* $\{A(n)\}_{n \geq k}$ *such that for each* $n \geq k$*,* $A(n)$ *is homeomorphic to the inverse graph* $\overleftarrow{\mathrm{gr}}\ G_1^n$ *under the projection map* $\rho_n|_{A(n)}$*. Furthermore, there exists a sequence of retractions* $\{r_n\colon X \to A(n)\}_{n \geq k}$ *that converges uniformly to the identity map on* X*.*

Proof. For each $n \geq k$, let $A(n)$ be the subcompactum of X generated by the n-tail sequence $\{Z_i^{i+1}\}_{i \geq n}$. It has previously been noted that $\{A(n)\}_{n \geq k}$ is a nested sequence of compacta. The remaining conditions follow from Theorem 2.1 and Theorem 2.2. □

In [2], C. A. Eberhart and J. B. Fugate make the following definition. Let X be a continuum (compactum) and let P be a topological property. Suppose $\{f_n\colon X \to X\}$ is a sequence of mappings that converges to the identity map on X. If for each $n \geq 1$, $f_n(X)$ has property P, then it is said that X *can be approximated from within by continua (compacta)* *with property* P. They proved (in the Hausdorff setting) that if X is a continuum that can be approximated from within by continua with the fixed point property, then X has the fixed point property.

So if each of the $A(n)$'s in Corollary 2.3 has the fixed point property, then we get the following corollary.

Corollary 2.4. *Let* $X = \varprojlim\{X_n, G_n^{n+1}\}$, *where for each* $n \geq 1$, X_n *is a continuum and* $G_n^{n+1}\colon X_{n+1} \to X_n$ *is a usc set-valued function. Suppose that* X *is a continuum and* $\{Z_i^{i+1}\}_{i\geq k}$ *is a* k*-tail sequence such that for each* $i \geq k$, $R(Z_i^{i+1}) = X_i$. *If each inverse graph* $\overleftarrow{\mathrm{gr}}\ G_1^n$ *has the fixed point property, then* X *has the fixed point property.*

In general, ordinary inverse limits do not necessarily have the fixed point property, even when the factor spaces are absolute retracts (or trees). In our setting, if the graph of each set-valued bonding function G_i^{i+1} contains a subcompactum that projects homeomorphically onto X_i in the i^{th}-coordinate and if each inverse graph $\overleftarrow{\mathrm{gr}}\ G_1^n$ is an absolute retract, the inverse limit space will have the fixed point property. Recall Question 1.10.

In order to understand the terminology in the next corollary, we provide the following definition, which appears in [9].

Let $\{K_n\}$ be a sequence of compact subsets of a compactum X and let $\{f_n\colon K_{n+1} \to K_n\}$ be a sequence of mappings.

We say that the inverse sequence $\{K_n, f_n\}$ *converges exactly in the space* X provided that

(i) the limits $\lim x_n$ in X exist for each $(x_1, x_2, \ldots) \in \varprojlim\{K_n, f_n\}$,

(ii) the function $\ell(\lim x_n) = (x_1, x_2, \ldots)$ is a homeomorphism between $\lim K_n$ and $\varprojlim\{K_n, f_n\}$, and

(iii) identifying $\lim K_n$ with $\varprojlim\{K_n, f_n\}$ by the homeomorphism ℓ, the projection mappings converge uniformly to the identity map on $\lim K_n$.

If $\{K_n, f_n\}$ converges in X and $\lim K_n = X$, we say that $\{K_n, f_n\}$ is an *internal inverse limit structure* on X.

Corollary 2.5 gives a condition under which we can realize an inverse limit with usc set-valued bonding functions as an ordinary inverse limit.

Corollary 2.5. *Let* $X = \varprojlim\{X_n, G_n^{n+1}\}$, *where, for each* $n \geq 1$, X_n *is a compactum and* $G_n^{n+1}\colon X_{n+1} \to X_n$ *is a usc set-valued function. Suppose that* $\{Z_i^{i+1}\}_{i\geq k}$ *is a* k*-tail sequence and, for each* $i \geq k$, $R(Z_i^{i+1}) = X_i$. *Then the inverse sequence* $\{A(n), r_n|_{A(n+1)}\}$ *is an internal inverse limit structure on* X *with retractions as bonding maps. Hence,* X *can be realized* *as an inverse limit on copies of the inverse graphs* $\overleftarrow{\mathrm{gr}}\, G_1^n$ *with retractions as bonding maps.*

Proof. In [9], M. M. Marsh and J. R. Prajs prove that if X is a compactum, $\{X_n\}$ is a nested increasing sequence of subcompacta of X, and there exists a sequence of retractions $\{f_n \colon X \to X_n\}$ that converges uniformly to id_X, then X admits an internal inverse limit structure on a subsequence of $\{X_n\}$ with retractions as bonding maps. They also show that if the retractions f_n satisfy $f_m \circ f_n = f_m$ for all $m < n$, then X admits an internal inverse limit structure on the sequence $\{X_n\}$ with retractions as bonding maps. So we need only to see that for each pair of natural numbers $k \leq m < n$, $r_m \circ r_n = r_m$.

Let $x = (x_1, x_2, \ldots) \in X$; so $x_i \in G_i^{i+1}(x_{i+1})$ for each $i \geq 1$. We get that

$$\begin{aligned} r_m(r_n(x)) &= \gamma_m \rho_m(\gamma_n \rho_n(x)) \\ &= \gamma_m \rho_m(\gamma_n(x_1, x_2, \ldots, x_n)) \\ &= \gamma_m(x_1, x_2, \ldots, x_m) \\ &= \gamma_m \rho_m(x) \\ &= r_m(x). \end{aligned}$$

Hence, $\varprojlim\{A(n), r_n|_{A(n+1)}\}$ is an internal inverse limit structure on X with retractions as bonding maps. □

3. Inverse Limits on One Compactum with a Single Set-Valued Bonding Function

We now turn our attention to inverse limits on one compactum M with one usc set-valued bonding function G. Even if M is a continuum, it follows from [12, Example 3.4] that the inverse limit $\varprojlim\{M, G\}$ may not be connected. Thus, the theorems in this section are, in general, about compacta. Let Δ denote the diagonal in $M \times M$; that is, $\Delta = \{(x, x) \mid x \in M\}$. For $X = \varprojlim\{M, G\}$, let $\sigma \colon X \to X$ be defined by $\sigma(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$. We call σ the (left) shift map. Also let $\Delta_\infty = \{(x, x, x, \ldots) \in \prod_{i \geq 1} M \mid x \in M\}$.

In this section and in §4, since G is the set-valued bonding function from one factor space to the previous one in the inverse sequence, we will think of $R(G)$ as being a subset of the first factor space and $D(G)$ as being the second factor space. With this in mind, if H is a closed subset of the graph of G, it will be convenient to define $D(H) = \{x_2 \mid (x_2, x_1) \in H\}$ and $R(H) = \{x_1 \mid (x_2, x_1) \in H\}$. So $D(H)$ is a subset of the second factor space and $R(H)$ is a subset of the first factor space.

Corollary 3.1. *Let* $X = \varprojlim\{M, G\}$. *Suppose* $H \subset \overrightarrow{\mathrm{gr}}\ G$ *and* H *is the graph of a usc set-valued function* h *such that* $h^{-1}\colon M \to M$ *is a mapping. Then* X *contains a monotonic increasing sequence of compacta* $\{A(n)\}_{n\geq 2}$ *such that, for each* $n \geq 2$, $A(n)$ *is homeomorphic to the* n*-inverse graph of* G *under the projection map* $\rho_n|_{A(n)}$. *Furthermore, there exists a sequence of retractions* $\{r_n\colon X \to A(n)\}_{n\geq 2}$ *that converges uniformly to the identity map on* X.

Proof. Note that $h = Z_i^{i+1}$ for $i \geq 2$ forms a 2-tail sequence of inverse mappings with $R(h) = M$ for $i \geq 2$. So the corollary follows immediately from Corollary 2.3. □

Corollary 3.2. *Let* $X = \varprojlim\{M, G\}$. *Suppose* $H \subset \overrightarrow{\mathrm{gr}}\ G$ *and* H *is the graph of a usc set-valued function* h *such that* $h^{-1}\colon M \to M$ *is a mapping. Then the inverse limit* X *contains a homeomorphic copy of* $\overrightarrow{\mathrm{gr}}\ G$.

Proof. Note that $A_2 = R(H) = M$; so $\overrightarrow{\mathrm{gr}}\ G_1^2|_{A_2} = \overrightarrow{\mathrm{gr}}\ G \overset{T}{\approx} \overleftarrow{\mathrm{gr}}\ G$. Thus, by Theorem 2.1, $A(2) \overset{T}{\approx} \overleftarrow{\mathrm{gr}}\ G$. So X contains a homeomorphic copy of $\overrightarrow{\mathrm{gr}}\ G$. □

Corollary 3.3. *Let* $X = \varprojlim\{M, G\}$. *Suppose* $H \subset \overrightarrow{\mathrm{gr}}\ G$ *and* H *is the graph of a usc set-valued function* h *such that* $h^{-1}\colon M \to M$ *is a mapping. Then the inverse sequence* $\{A(n), r_n|_{A(n+1)}\}$ *is an internal inverse limit structure on* X *with retractions as bonding maps. Hence,* X *can be realized as an inverse limit on copies of the inverse graphs* $G_n^{\leftarrow}$ *with retractions as bonding maps.*

Proof. Since $\{h\}_{i\geq 2}$ is a 2-tail sequence of inverse mappings, this corollary follows immediately from Corollary 2.5. □

For $X = \varprojlim\{M, G\}$, suppose $\overrightarrow{\mathrm{gr}}\ G$ contains subcompacta H and F such that $H - F \neq \emptyset \neq F - H$ and both H and F are graphs of usc set-valued functions h and f such that $h^{-1}\colon M \to M$ and $f^{-1}\colon M \to M$ are mappings. Then, for each $\alpha \in \prod_{i\geq 2}\{h, f\}$, we have a unique 2-tail sequence of inverse mappings with $\alpha(i) = Z_i^{i+1} \in \{h, f\}$ for each $i \geq 2$. By Theorem 2.1, each α gives rise to a subcompactum $A(\alpha)$ of X such that $A(\alpha) \overset{T}{\approx} \overleftarrow{\mathrm{gr}}\ G|_{A_2} = \overleftarrow{\mathrm{gr}}\ G$. So we have a Cantor set of copies of the graph of G in X. However, how pairs of copies intersect is not, in general, clear. Such intersections are related to the set $H \cap F$ and to the dynamics of $h^{-1}\colon M \to M$ and $f^{-1}\colon M \to M$, both separately and together.

Furthermore, if for each $i \geq 2$, we have a choice of two usc set-valued functions Z_i^{i+1} or W_i^{i+1} that generate 2-tail sequences and $N \subset \overrightarrow{\mathrm{gr}}$

$G|_{R(Z_2^3)} \cap \overrightarrow{\mathrm{gr}}\, G|_{R(W_2^3)}$, then we will have a Cantor set of copies of N in the inverse limit space. If these copies of N are disjoint, we might expect that they lie in composants of some indecomposable subcontinuum in X. We will see in Theorem 3.5 and Theorem 3.8 that if N has symmetry or "periodic symmetry" in $\overrightarrow{\mathrm{gr}}\, G$ with respect to its various 2-tail sequences, then the Cantor set of copies of N will typically not lie in an indecomposable continuum. Hence, we are more likely to see indecomposable continua in X if $\overrightarrow{\mathrm{gr}}\, G$ doesn't have symmetry.

Theorems 3.5, 3.8, 3.9, and 3.10 establish some conditions under which we expect to have infinitely many copies of $\overrightarrow{\mathrm{gr}}\, G$ or subcompacta of $\overrightarrow{\mathrm{gr}}\, G$ in X.

Lemma 3.4. *Let $X = \varprojlim\{M, G\}$. Suppose N is a subcompactum of $\overrightarrow{\mathrm{gr}}\, G$ such that N is symmetric; that is, $N = N^{-1}$. Then X contains a copy L of N on which $\sigma|_L$ is an involution.*

Proof. Let $L = \{(s, t, s, t, \ldots) \in X \mid (t, s) \in N\}$. Clearly, L is a subset of X since N is symmetric. We note that $\rho_2|_L$ is a homeomorphism onto N. Both one-to-one and onto are immediate. That $\sigma|_L$ is an involution is also immediate. □

Let N_0, N, and N_1 be compacta and let C be the standard middle-thirds Cantor set. We say that $\mathcal{L}$ is a *Cantor set shuffle of copies of N between a copy of N_0 and a copy of N_1* in the space X provided that

(1) for $\alpha \in C$, $L_\alpha \in \mathcal{L}$ and $L_\alpha \subset X$,
(2) $L_0 \overset{T}{\approx} N_0$, $L_1 \overset{T}{\approx} N_1$, and $L_\alpha \overset{T}{\approx} N$ for $\alpha \notin \{0, 1\}$, and
(3) there exists a continuous one-to-one selection $S\colon C \to \bigcup_{\alpha \in C} L_\alpha$.

If $X = \varprojlim\{M, G\}$ and N is a subcompactum of $\overrightarrow{\mathrm{gr}}\, G$, let $\Delta|_{D(N)} = \{(x, x) \in \Delta \mid x \in D(N)\}$ and $\Delta|_{R(N)} = \{(x, x) \in \Delta \mid x \in R(N)\}$.

One might note here that Theorem 3.5 applies to Example 4.3, but does not apply to examples 4.1, 4.2, or 4.4.

Theorem 3.5. *Let $X = \varprojlim\{M, G\}$. Suppose N is a subcompactum of $\overrightarrow{\mathrm{gr}}\, G$, $N \not\subset \Delta$, $\Delta|_{D(N)} \cup \Delta|_{R(N)} \subset G$, $N \cup N^{-1} \subset G$, and $N \cap N^{-1} \subset \Delta$. Then X contains a Cantor set shuffle of copies L_α of N between a copy of $R(N)$ and a copy of $D(N)$. Furthermore, for $\alpha \neq \beta$, $L_\alpha \cap L_\beta = \{(s, s, s, \ldots) \in \Delta_\infty \mid (s, s) \in N \cap \Delta\} \overset{T}{\approx} N \cap \Delta$.*

Proof. Let $\alpha_0 = (0, 0, 0, \ldots)$ and $\alpha_1 = (1, 1, 1, \ldots)$. Let $\mathcal{S} = \prod_{n \geq 1}\{0, 1\} - \{\alpha_0, \alpha_1\}$.

For $\alpha \in \mathcal{S}$, define $\ell_\alpha\colon N \to X$ by

$$\pi_i \ell_\alpha(t,s) = \begin{cases} s & \text{if } \alpha(i) = 0, \\ t & \text{if } \alpha(i) = 1. \end{cases}$$

Define $\ell_0\colon \Delta|_{R(N)} \to X$ by $\ell_0(s,s) = (s,s,s,\ldots)$ and $\ell_1\colon \Delta|_{D(N)} \to X$ by $\ell_1(t,t) = (t,t,t,\ldots)$.

Let $L_0 = \ell_0(\Delta|_{R(N)})$ and $L_1 = \ell_1(\Delta|_{D(N)})$. For $\alpha \in \mathcal{S}$, let $L_\alpha = \ell_\alpha(N)$. By hypothesis, L_0, L_1, and each $L_\alpha \subset X$.

Clearly, $L_0 = \{(s,s,s,\ldots) \mid s \in R(N)\}$ and $L_1 = \{(s,s,s,\ldots) \mid s \in D(N)\}$. Fix $\alpha \in \mathcal{S}$. Let n be a natural number such that $\alpha(n) \neq \alpha(n+1)$. Assume that $\alpha(n) = 0$ and $\alpha(n+1) = 1$. Let $h\colon L_\alpha \to M_n \times M_{n+1}$ be projection onto the $(n,n+1)$-coordinates. We claim that h is a homeomorphism onto N. If $\alpha(n) = 1$ and $\alpha(n+1) = 0$, then h will be a homeomorphism onto N^{-1}.

Suppose that $h(x_1, x_2, \ldots) = h(y_1, y_2, \ldots)$ for points x and y in $L_\alpha = \ell_\alpha(N)$. Then $(x_{n+1}, x_n) \in N$ by definition of ℓ_α, and $(x_n, x_{n+1}) = (y_n, y_{n+1})$. So $\ell_\alpha(x_n, x_{n+1}) = \ell_\alpha(y_n, y_{n+1})$, and hence $x = y$. Clearly, h is onto N. Thus, each L_α is a copy of N in X.

Fix $(t,s) \in N - \Delta$. Let δ be the distance from t to s in M. Identify the Cantor set with $C = \prod_{i \geq 1} \{0, \delta\}$ and $\mathcal{S}$ with $C - \{(0,0,\ldots), (\delta,\delta,\ldots)\}$ for the remainder of this proof. We now define a continuous one-to-one selection S from the Cantor set into $L_0 \cup L_1 \cup (\bigcup_{\alpha \in \mathcal{S}} L_\alpha)$. Define $S\colon \mathcal{S} \to \bigcup_{\alpha \in \mathcal{S}} L_\alpha$ by $S(\alpha) = \ell_\alpha(t,s)$. Let $S(0,0,\ldots) = (s,s,\ldots)$ and $S(\delta,\delta,\ldots) = (t,t,\ldots)$. Clearly, $S(\alpha) \in L_\alpha$ for each α. It is easy to see that S is an isometry from C into X. So it follows that S is continuous and one-to-one. Thus, S is the desired selection.

Suppose $\alpha \neq \beta$ and $x \in L_\alpha \cap L_\beta$. So $x \in \ell_\alpha(N) \cap \ell_\beta(N)$. Hence, there exists $(t,s) \in N$ and $(b,a) \in N$ such that $\pi_i(x) = \pi_i \ell_\alpha(t,s)$ and $\pi_i(x) = \pi_i \ell_\beta(b,a)$ for all $i \geq 1$. Since $\alpha \neq \beta$, for some $j \geq 1$, $\alpha(j) \neq \beta(j)$. So assume, without loss of generality, that $\alpha(j) = 0$ and $\beta(j) = 1$. So $\pi_j(x) = s$ and $\pi_j(x) = b$. Thus, $s = b$.

Suppose there exists $m \neq j$ such that $\alpha(m) = 0 = \beta(m)$. Then $\pi_m(x) = \pi_m \ell_\alpha(t,s) = s$ and $\pi_m(x) = \pi_m \ell_\beta(b,a) = a$. So $s = a = b$. Thus, it follows that $x = (s,s,s,\ldots) \in \Delta_\infty$ and $(s,s) \in N$. Similarly, if there exists $m \neq j$ such that $\alpha(m) = 1 = \beta(m)$, we get that $t = b = s$, and therefore $x = (s,s,s,\ldots)$. Otherwise, we have that for all $i \geq 1$, $\alpha(i) \neq \beta(i)$. So pick any m, where $\alpha(m) = 1$. So $\beta(m) = 0$. It follows that $\pi_m(x) = t$ and $\pi_m(x) = a$. So $t = a$. Thus, $(t,s) = (a,b) \in N \cap N^{-1}$. By hypothesis, $(t,s) \in \Delta$, and again we have that $x = (s,s,s,\ldots) \in \Delta_\infty$ and $(s,s) \in N$. $\square$

In [1], H. Cook defined clumps of continua. According to his terminology, our Cantor set shuffle of copies of N in Theorem 3.5 above is an upper semi-continuous clump of copies of N with center homeomorphic to $N \cap \Delta$. Our Theorem 3.9 also produces an upper semi-continuous clump of continua in the inverse limit space. Recently, Ingram [5] has shown that certain inverse limits with set-valued bonding functions are 1-dimensional upper semi-continuous clumps of tree-like continua, which, by Cook [1, Theorem 12], is a tree-like continuum. So perhaps clumps of continua occur frequently in inverse limits with set-valued bonding functions.

Lemma 3.6 (Insertion Lemma 1)**.** *Let* $X = \prod_{i\geq 1} M$ *and* $N \subset M \times M$. *Let* $g\colon N \to \prod_{i\geq 1} M$ *be a mapping such that for each* $(t,s) \in N$, *each coordinate of* $g(t,s)$ *is either* t *or* s. *Let* $f_1\colon D(N) \to \prod_{i=1}^{m} M$ *and* $f_2\colon R(N) \to \prod_{i=1}^{m} M$ *be mappings such that* $f_1(s) = f_2(s)$ *for* $(s,s) \in N \cap \Delta$. *Let* $u\colon \mathbb{N} \to \mathbb{N}$ *be an increasing sequence. Define* $g\langle f_1, f_2\rangle_u\colon N \to X$ *by*

$$g\langle f_1, f_2\rangle_u(t,s) = (a_1, \ldots, a_{u_1}, f_j(a_{u_1}), a_{u_1+1}, \ldots, a_{u_2}, f_j(a_{u_2}), a_{u_2+1}, \ldots),$$

where $g(t,s) = (a_1, a_2, \ldots)$, $j = 1$ *if* $a_{u_i} = t$, *and* $j = 2$ *if* $a_{u_i} = s$. *Then* $g\langle f_1, f_2\rangle_u$ *is continuous.*

Proof. The function $g\langle f_1, f_2\rangle_u$ is continuous if $\pi_n g\langle f_1, f_2\rangle_u\colon N \to M$ is continuous for each $n \geq 1$. Suppose $\{a^i\}$ is a sequence of points in N converging to the point $a \in N$. Suppose n is a coordinate where $\pi_n g\langle f_1, f_2\rangle_u(a)$ does not fall among the coordinates of $f_j(a_{u_i})$ for $i \geq 1$. Then $\pi_n g\langle f_1, f_2\rangle_u(a) = a_k$ for some $k \geq 1$. So, clearly, $\{\pi_n g\langle f_1, f_2\rangle_u(a^i) = a^i_k\}$ converges to $\pi_n g\langle f_1, f_2\rangle_u(a) = a_k$.

So suppose that n is a coordinate that corresponds to some coordinate of $f_j(a_{u_k})$ for some $k \geq 1$ and $j \in \{1,2\}$. Since $\{a^i\}$ converges to a, $\{g(a^i)\}$ converges to $g(a)$, and hence $\{a^i_{u_k}\}$ converges to a_{u_k}. Since f_j is continuous, $\{f_j(a^i_{u_k})\}$ converges to $f_j(a_{u_k})$. So, these points of $\prod_{i=1}^{m} M$ converge coordinate-wise. That is, $\{\pi_n g\langle f_1, f_2\rangle_u(a^i)\}$ converges to $\pi_n g\langle f_1, f_2\rangle_u(a)$.

Hence, $g\langle f_1, f_2\rangle_u$ is continuous. □

Lemma 3.7 (Insertion Lemma 2)**.** *Let* $X = \prod_{i\geq 1} M$, $m \in \mathbb{N}$, $s, t \in M$, $C = \prod_{i\geq 1}\{0,1\}$, *and* $\ell\colon C \to X$ *be a mapping defined by*

$$\pi_i \ell(\alpha) = \begin{cases} s & \text{if } \alpha(i) = 0, \\ t & \text{if } \alpha(i) = 1. \end{cases}$$

Let $\langle a\rangle = (a_1, a_2, \ldots, a_m)$ *and* $\langle b\rangle = (b_1, b_2, \ldots, b_m)$ *be finite sequences with each* $a_i, b_i \in M$. *Let* $\hat{\ell}\colon C \to X$ *be the mapping that for each* $\alpha \in C$, $\hat{\ell}$ *inserts* $\langle a\rangle$ *between successive coordinates of* $\ell(\alpha)$ *that are both* s *and* *inserts* $\langle b\rangle$ *between successive coordinates of* $\ell(\alpha)$ *that are both* t*. Then* $\hat{\ell}$ *is continuous.*

Proof. The proof is similar to the proof of Insertion Lemma 1. □

Let $X = \varprojlim\{M, G\}$, where M is a compactum and G is a usc set-valued function. Let $m \in \mathbb{N}$ and let $T \subset M$. We say that G is *inverse periodic of period* m for each point $t \in T$ provided that, for each point $t \in T$, there exists a sequence $(x_1, x_2, \ldots, x_{m+1})$ such that $x_1 = t = x_{m+1}$ and $x_i \in G(x_{i+1})$ for each $1 \le i \le m$. We call $(x_2, x_3, \ldots, x_m)$ an $m-1$ *inverse sequence between* t's. If there exists a continuous one-to-one function $f\colon T \to \prod_{i=1}^{m-1} M$ such that, for each $t \in T$, $f(t)$ is an $m-1$ inverse sequence between t's, we say that f is an *m-periodic inverse sequence for* T.

We note that if there exists a 2-tail sequence $\{Z_i^{i+1}\}$ of inverse mappings with $T = R(Z_2^3)$, and for $t \in R(Z_2^3)$, the $(m+2)^{\text{th}}$ coordinate of the 2-tail sequence generated by t is equal t, then the function that assigns t to the 3^{rd} through $(m+1)^{\text{th}}$ coordinates of the 2-tail sequence that t generates will be a periodic inverse sequence for T.

Theorem 3.8 is a generalization of Theorem 3.5. We provided a proof of Theorem 3.5 first so that the proof of Theorem 3.8, which involves messy notation, will be easier to follow. The proof of Theorem 3.8 is very similar to that of Theorem 3.5, and we use some of the same items defined therein.

Theorem 3.8 applies to Example 4.4.

Theorem 3.8. *Let* $X = \varprojlim\{M, G\}$*. Suppose* N *is a subcompactum of* $\overrightarrow{\text{gr}}\, G$*,* $N \not\subset \Delta$*,* $N \cup N^{-1} \subset G$*, and* $N \cap N^{-1} \subset \Delta$*. Suppose that* $f_2\colon R(N) \to \prod_{i=1}^{m-1} M$ *and* $f_1\colon D(N) \to \prod_{i=1}^{m-1} M$ *are periodic inverse sequences (of period* m*) such that* $f_1(s) = f_2(s)$ *for* $(s, s) \in N \cap \Delta$*. Suppose there exists* $(t, s) \in N$ *such that the first coordinate of* $f_1(t)$ *is not* s *and the first coordinate of* $f_2(s)$ *is not* t*. Then* X *contains a Cantor set shuffle of copies* $\hat{L}_\alpha$ *of* N *between a copy of* $R(N)$ *and a copy of* $D(N)$*.*

Proof. Let $\alpha_0 = (0, 0, 0, \ldots)$ and $\alpha_1 = (1, 1, 1, \ldots)$. Let $\mathcal{S} = \prod_{n\ge 1}\{0, 1\} - \{\alpha_0, \alpha_1\}$.

For $\alpha \in \mathcal{S}$, let $\ell_\alpha\colon N \to \prod_{i\ge 1} M$ be the same mapping defined in the proof of Theorem 3.5. Also, let ℓ_0 and ℓ_1 be defined as in the proof of Theorem 3.5. Note here that the images of ℓ_α may not be in X. For $\alpha \in \mathcal{S} \cup \{\alpha_0, \alpha_1\}$, define $u_i\colon \mathbb{N} \to \mathbb{N}$ inductively by letting u_1 be the first i where $\alpha(i) = \alpha(i+1)$. Then let u_n be the first i where $i > u_{n-1}$ and $\alpha(i) = \alpha(i+1)$.

Let $\ell_\alpha\langle f_1, f_2\rangle_u \colon N \to X$ be defined as in Insertion Lemma 1. For $(t,s) \in N$, we write $\ell_\alpha\langle t,s\rangle_u$ for $\ell_\alpha\langle f_1, f_2\rangle_u(t,s)$. By Insertion Lemma 1, $\ell_\alpha\langle f_1, f_2\rangle_u$ is continuous.

For a given point $(t,s) \in N$ and $\alpha \in \mathcal{S}$, it is fairly straightforward to see what the point $\ell_\alpha\langle t,s\rangle_u$ looks like. Say, for example, that the first few coordinates of α are $(0,1,0,0,0,1,1,0,\ldots)$. Then $\ell_\alpha\langle t,s\rangle_u = (s,t,s,f_2(s),s,f_2(s),s,t,f_1(t),t,s,\ldots)$. For simplicity of notation, we will hereafter let $\hat{\ell}_\alpha = \ell_\alpha\langle f_1, f_2\rangle_u$. It is clear that the images of $\hat{\ell}_\alpha$ are points of X. Let $\hat{\ell}_0 \colon \Delta|_{R(N)} \to X$ be given by $\hat{\ell}_0(s,s) = \ell_{\alpha_0}\langle s,s\rangle_u$ and $\hat{\ell}_1 \colon \Delta|_{D(N)} \to X$ be given by $\hat{\ell}_1(t,t) = \ell_{\alpha_0}\langle t,t\rangle_u$.

Let $\hat{L}_0 = \hat{\ell}_0(\Delta|_{R(N)})$ and $\hat{L}_1 = \hat{\ell}_1(\Delta|_{D(N)})$. For $\alpha \in \mathcal{S}$, let $\hat{L}_\alpha = \hat{\ell}_\alpha(N)$. So $\hat{L}_0$, $\hat{L}_1$, and each $\hat{L}_\alpha$ are subcompacta of X.

Clearly, $\hat{L}_0 \overset{T}{\approx} R(N)$ and $\hat{L}_1 \overset{T}{\approx} D(N)$. Fix $\alpha \in \mathcal{S}$. Let n be a natural number such that $\alpha(n) \neq \alpha(n+1)$. Assume that $\alpha(n) = 0$ and $\alpha(n+1) = 1$. Let $\hat{h} \colon \hat{L}_\alpha \to M_n \times M_{n+1}$ be projection onto the $(n,n+1)$-coordinates. Since f_1 and f_2 are one-to-one, the proof that $\hat{h}$ is one-to-one is analogous to the proof that h is one-to-one in Theorem 3.5. It is clear again that $\hat{h}$ is onto N.

Fix $(t,s) \in N - \Delta$ as given in the hypothesis. Let S be the selection from the Cantor set into $\bigcup_{\alpha\in\mathcal{S}} L_\alpha$ defined in the proof of Theorem 3.5. Recall that for $\alpha \in \mathcal{S}$, $S(\alpha) = \ell_\alpha(t,s)$. Define $\hat{S} \colon \mathcal{S} \to \bigcup_{\alpha\in\mathcal{S}} \hat{L}_\alpha$ as in Insertion Lemma 2 for the finite sequences $f_1(t)$ and $f_2(s)$. Also, let $\hat{S}(0,0,\ldots) = (s,f_2(s),s,f_2(s),s,\ldots)$; let $\hat{S}(1,1,\ldots) = (t,f_1(t),t,\ldots)$. The map $\hat{S}$ is continuous by Insertion Lemma 2.

Suppose $\hat{S}(\alpha) = x = \hat{S}(\beta)$ with $\alpha \neq \beta$ and $x = (x_1, x_2, \ldots)$. Let j be the first coordinate where $\alpha(j) \neq \beta(j)$. If $j = 1$, then $t = s$, contradicting the choice of (t,s). So assume that $j > 1$. By choice of j, $\alpha(j-1) = \beta(j-1)$. Assume, without loss of generality, that $\alpha(j-1) = 0 = \beta(j-1)$, $\alpha(j) = 0$ and $\beta(j) = 1$. So for some $n \geq 1$, $x_n = s$, $x_{n+1} = t$, and x_{n+1} is the first coordinate of $f_2(s)$, contradicting our hypothesis. So $\hat{S}$ is one-to-one.

Note that $\hat{S}(\alpha) = \ell_\alpha\langle t,s\rangle_u = \hat{\ell}_\alpha(t,s) \in \hat{\ell}_\alpha(N) = \hat{L}_\alpha$. So $\hat{S}$ is the desired selection. □

Theorem 3.9 applies to Example 4.1 and Example 4.2.

Theorem 3.9. *Let $X = \varprojlim\{M, G\}$. Suppose N is a subcompactum of $\overrightarrow{\mathrm{gr}}\, G$, $N \not\subset \Delta$, and $\Delta|_{D(N)} \cup \Delta|_{R(N)} \subset G$. Then X contains a sequence $\{L_n\}$, of distinct copies of N, that converges to a copy of $R(N)$ in Δ_∞. Furthermore, for $m \neq n$, $L_m \cap L_n = \{(s,s,s,\ldots) \mid (s,s) \in N \cap \Delta\} \overset{T}{\approx} N \cap \Delta$. For each $n \geq 1$, σ maps L_{n+1} homeomorphically onto L_n.*

Proof. For $n \geq 1$, let $L_n = \{(x_1, x_2, \ldots) \in X \mid x_i = s \text{ for } 1 \leq i \leq n,\ x_i = t \text{ for } i \geq n+1, \text{ and } (t,s) \in N\}$. By hypothesis, (s,s) and (t,t) are in $\overrightarrow{\text{gr}}\, G$, so each L_n is a subset of X. It is easy to see that for each $n \geq 1$, the projection of L_n onto the $(n, n+1)$-coordinates is a homeomorphism of L_n onto N^{-1}, which is homeomorphic to N.

Suppose for some $m < n$, $x \in L_m \cap L_n$. So there exists $(t,s) \in N$ and $(b,a) \in N$ such that $x = (s, s, \ldots, s, t, t, \ldots) \in L_m$ and $x = (a, a, \ldots, a, b, b, \ldots) \in L_n$. So $s = a$ since first coordinates are equal and $t = b$ since $(n+1)^{\text{th}}$ coordinates are equal. Since $m < n$ and the $(m+1)^{\text{th}}$ coordinates are equal, we get that $t = a$. It follows that $x = (s, s, s, \ldots) \in \Delta_\infty$ and that $(s,s) \in N \cap \Delta$. So, for $m \neq n$, L_m and L_n meet if and only if N meets Δ. Since $N \not\subset \Delta$, each L_n contains a point that is in no other L_m.

Let $s \in R(N)$ and consider the point $z = (s, s, s, \ldots) \in \Delta_\infty$. Let $(t,s) \in N$. For $n \geq 1$, let $y_n = (s, s, \ldots, s, t, t, \ldots) \in L_n$. Clearly, $\{y_n\}$ converges to z.

Suppose for each $n \geq 1$, $y_n \in L_n$ and $\{y_n\}$ converges to $z \in X$. Let s be the first coordinate of z. Now, $\{y_n\}$ must converge coordinate-wise to z. For $n \geq 1$ and $i \geq 1$, let y_{ni} be the i^{th}-coordinate of y_n. So $\{y_{n1}\}$ converges to s. For $n \geq 2$, $y_{n2} = y_{n1}$; so $\{y_{n2}\}$ converges to s. For $n \geq 3$, $y_{n3} = y_{n1}$; so $\{y_{n3}\}$ converges to s. Continuing, we get that each coordinate of z is s. It follows that $\{L_n\}$ converges to $\{(s, s, s, \ldots) \mid s \in R(N)\}$, which is a topological copy of $R(N)$.

Clearly, for each $n \geq 1$, $\sigma|_{L_{n+1}} \colon L_{n+1} \to L_n$ is a homeomorphism. □

Theorem 3.10. *Let $X = \varprojlim\{M, G\}$ and let N be a subcompactum of $\overrightarrow{\text{gr}}\, G$ with $N \not\subset \Delta$. Suppose that $f_2 \colon R(N) \to \prod_{i=1}^{m-1} M$ and $f_1 \colon D(N) \to \prod_{i=1}^{m-1} M$ are periodic inverse sequences (of period m) such that $f_1(s) = f_2(s)$ for $(s,s) \in N \cap \Delta$. Suppose also there exists $(t,s) \in N$ such that the first coordinate of $f_2(s)$ is not t. Then X contains a sequence $\{\hat{L}_n\}$, of distinct copies of N, that converges to a copy of $R(N)$. For each $n \geq 1$, σ^m maps $\hat{L}_{n+1}$ homeomorphically onto $\hat{L}_n$.*

Proof. The proof follows in a manner analogous to how the proof of Theorem 3.8 followed the proof of Theorem 3.5. □

4. Examples

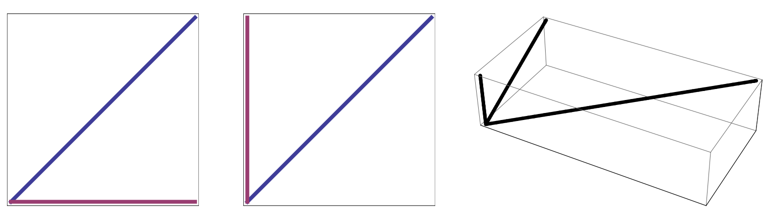
We revisit some well-known examples of inverse limits on $[0,1]$ with a single usc set-valued bonding function. We note in each case how certain properties about the inverse limit can be immediately deduced from our theorems. In each of these examples, the bonding function is a union of usc continuum-valued functions and so, by Theorem 1.6, the inverse limit will be connected. Also, the graph of each set-valued bonding function in these examples contains a subcompactum H that is the graph of a set-valued function h such that $h^{-1}\colon [0,1] \to [0,1]$ is a mapping. So, by Corollary 3.3, each inverse limit can be realized as an ordinary inverse limit on a nested sequence of copies of its inverse graphs $G_m^{\leftarrow}$ with retractions for bonding maps.

In each example, we show the graph of G, the inverse graph of G, and the 3-inverse graph of G. The 3-inverse graph is drawn as if it is sitting in the first three coordinates of the Hilbert cube.

Example 4.1 ([6, Example 2])**.** If we take $N = \overrightarrow{\mathrm{gr}}\, G$, then by Theorem 3.9, X contains a sequence of copies of $\overrightarrow{\mathrm{gr}}\, G$ (arcs), each pair intersecting in the set Δ_∞, converging to Δ_∞.

On the other hand, if we take $N = [0,1] \times \{0\}$, then by Theorem 3.9, X contains a sequence of arcs, each pair intersecting in the point $\{(0,0,0,\ldots)\}$, converging to a copy of $R(N)$, namely the point $(0,0,0,\ldots)$. Hence, the sequence must be a null sequence of arcs. Since each point of X that is not in Δ_∞ is of the form $(0,0,\ldots,0,t,t,\ldots)$ for $t \in (0,1]$, it follows that X is Δ_∞ union the null sequence of arcs.

See Figure 1.

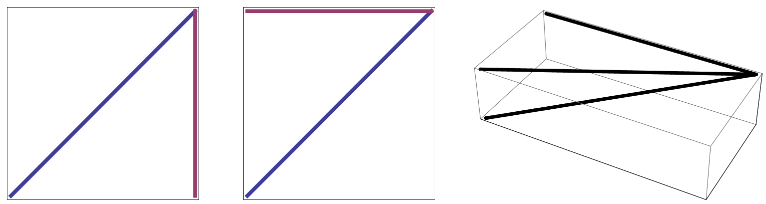


(a) $\overrightarrow{\mathrm{gr}}\, G$ (b) $\overleftarrow{\mathrm{gr}}\, G$ (c) $G_3^{\leftarrow}$

FIGURE 1

Example 4.2. In this example, letting $N = \{1\} \times [0,1]$ and applying Theorem 3.9, we again have a convergent sequence of arcs, each pair meeting at the point $(1,1,1,\ldots)$. However, since $R(N) = [0,1]$, the arcs converge to Δ_∞.

See Figure 2.

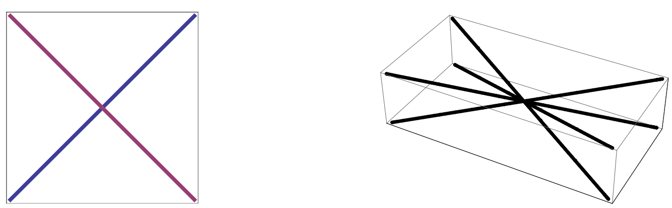


(a) $\overrightarrow{\text{gr}}\, G$ (b) $\overleftarrow{\text{gr}}\, G$ (c) $G_3^{\leftarrow}$

FIGURE 2

Example 4.3 ([8, Example 4])**.** Let N be the convex arc from $(\frac{1}{2}, \frac{1}{2})$ to $(1, 0)$. The conditions of Theorem 3.5 are satisfied, $R(N) = [0, \frac{1}{2}]$, and $D(N) = [\frac{1}{2}, 1]$. So X contains a Cantor set shuffle $\mathcal{L}$ of copies of N (arcs) between the lower half of Δ_∞ and the upper half of Δ_∞. Since $N \cap \Delta = \{(\frac{1}{2}, \frac{1}{2})\}$, each pair of arcs meets at the point $(\frac{1}{2}, \frac{1}{2}, \ldots)$. So $\cup\mathcal{L}$ is homeomorphic to the cone over the Cantor set. Applying Corollary 3.3, we see that X is an inverse limit on a nested sequence of copies of the n-inverse graphs of G with retractions as bonding maps. Each $G_n^{\leftarrow}$ is a 2^n-od, (see $G_3^{\leftarrow}$). So $X = \cup\mathcal{L}$.

See Figure 3.

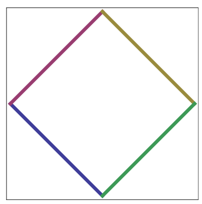


(a) $\overrightarrow{\text{gr}}\, G = \overleftarrow{\text{gr}}\, G$ (b) $G_3^{\leftarrow}$

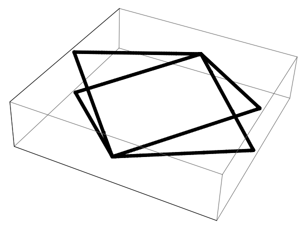
FIGURE 3

Example 4.4 ([6, Example 4])**.** Since $\overrightarrow{\text{gr}}\, G$ is symmetric, by Lemma 3.4, we immediately have that X contains a simple closed curve. Applying Theorem 3.8, let N be the convex arc from $(0, \frac{1}{2})$ to $(\frac{1}{2}, 1)$. Note that $N \cap \Delta = \emptyset$, $D(N) = [0, \frac{1}{2}]$, and $R(N) = [\frac{1}{2}, 1]$. Let $f_1(t) = \frac{1}{2} - t$ for $t \in D(N)$ and $f_2(s) = \frac{3}{2} - s$ for $s \in R(N)$. We see that f_1 and f_2 are periodic inverse sequences (of period 2) for $D(N)$ and $R(N)$, respectively. The point $(\frac{1}{4}, \frac{3}{4})$ will suffice as the special point (t, s) mentioned in the hypothesis. So X contains a Cantor set shuffle of arcs between a copy of $R(N)$ and a copy of $D(N)$. In this example, looking at $G_3^{\leftarrow}$ may give the impression that X will be the suspension of a Cantor set. However, if we consider $G_4^{\leftarrow}$, we see that "above" each of the two vertices in $G_3^{\leftarrow}$, we will have two points, namely $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, \frac{1}{2}, 1)$ above $(\frac{1}{2}, 0, \frac{1}{2})$, and $(\frac{1}{2}, 1, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 1, \frac{1}{2}, 1)$ above $(\frac{1}{2}, 1, \frac{1}{2})$.

See Figure 4.



(a) $\overrightarrow{\mathrm{gr}}\, G = \overleftarrow{\mathrm{gr}}\, G$



(b) $G_3^{\leftarrow}$

FIGURE 4

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