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A CHARACTERIZATION OF TREE-LIKE INVERSE LIMITS ON $[0, 1]$ WITH INTERVAL-VALUED FUNCTIONS

M. M. MARSH

Abstract. We provide a characterization of tree-likeness in inverse limits on $[0, 1]$ with interval-valued functions. We also show that flat spots, in certain inverse sequences, give rise to subcontinua of the inverse limit space that are either copies of subcontinua of the partial graphs in the inverse sequence or copies of products of subcontinua of the partial graphs and ordinary inverse limits.

In [15], the author provided necessary conditions and sufficient conditions for an inverse limit on $[0, 1]$ with interval-valued bonding functions to be a tree-like continuum. Corollaries 27 and 28 of [15] give sufficient conditions for such inverse limits to have dimension larger than one. We show that one of the conditions in each of these two corollaries can be eliminated, thus providing a simply-stated characterization of tree-likeness in this setting (see Corollary 3). Under the same conditions that characterize tree-likeness of the inverse limit, we characterize the partial graphs (definition to follow) in this setting as λ-dendroids.[1]

Additionally, we show that if one of the continuum-valued bonding functions, in certain inverse sequences, has a flat spot, then the inverse limit space must contain either a copy of a subcontinuum of some partial graph in the inverse sequence or a copy of a product of a subcontinuum of a partial graph and an ordinary inverse limit on subcontinua of some of the factor spaces (see Theorem 1). This result is critical for establishing a lower bound for the dimension of the inverse limit space. Other results

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[1]Results in this paper were presented at the 2016 Spring Topology and Dynamics Conference, Waco, TX.

and examples related to one dimensionality and tree-likeness in inverse limits with set-valued functions can be found in [9], [10], [11], and [15].

Since, even for inverse limits on $[0, 1]$ with set-valued functions, the dimension of the inverse limit space can be either finite or infinite, it is of importance to have conditions related to the factor spaces, the bonding functions, and the partial graphs that will determine the dimension of the inverse limit space. In the setting of this paper, it follows from [2] that all inverse limits have trivial shape. Since for continua with trivial shape, tree-likeness is equivalent to having dimension one, our results are about dimension one as well as about tree-likeness.

For results related to all dimensions in inverse limits with set-valued functions, see [1] and [12]. Hisao Kato has results in [12] that give both upper and lower bounds on the dimension of the inverse limit space. His lower bound result in Theorem 3.8, when specialized to the setting of this paper, follows from our Theorem 2.

A *compactum* is a compact metric space. All spaces considered in this paper will be compacta. A *continuum* is a connected compactum. A continuous function with be referred to as a *map* or *mapping*. For a compactum X, $\dim(X)$ will denote covering dimension.

A function $f\colon X \to 2^Y$ is *upper semi-continuous at the point* $x \in X$ if, for each open set V in Y containing the set $f(x)$, there is an open set U in X such that $x \in U$ and $f(p) \subset V$ for each $p \in U$. If $f\colon X \to 2^Y$ is upper semi-continuous at each point of X, then f is said to be *upper semi-continuous*. We refer to functions $f\colon X \to 2^Y$ as *set-valued functions* from X to Y and we write $f\colon X \to Y$ is a set-valued function.

A set-valued function $f\colon X \to Y$ is *continuum-valued* if, for each $x \in X$, the set $f(x)$ is a subcontinuum of Y. The *graph* of f, which we denote by $G(f)$, is the set of points in $X \times Y$ consisting of points (x, y) with $y \in f(x)$. For each product $X \times Y$ of compacta X and Y, let $c_1\colon X \times Y \to X$ and $c_2\colon X \times Y \to Y$ denote coordinate projection. The set-valued function $f\colon X \to Y$ is *surjective* if $c_2(G(f)) = Y$.

Let $X_1, X_2, \ldots$ be a sequence of compacta. Our setting will be the product space $\prod_{i \geq 1} X_i$ with the usual metric. Throughout, we let $\{X_i, f_i\}_{i \geq 1}$ denote an inverse sequence with upper semi-continuous set-valued bonding functions, and its inverse limit is given by

$$\varprojlim\{X_i, f_i\} = \{\mathbf{x} = (x_1, x_2, \ldots) \in \prod_{i \geq 1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } i \geq 1\}.$$

For $j, m \in \mathbb{N}$ with $j \leq m$, we define the set below.

$$G_j^{m+1} = G'(f_j, \ldots, f_m) = \{\mathbf{x} \in \prod_{i=j}^{m+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } j \leq i \leq m\}.$$

We refer to these sets as *partial graphs* in the inverse sequence. For consistency of notation, if $i \geq 1$, we let $G_i^i = X_i$. The notation $X \overset{T}{\approx} Y$ ($X \overset{T}{\subset} Y$) will indicate that X is homeomorphic to Y (X is homeomorphic to a subset of Y).

For $1 \leq j < k$, we denote the set-valued composition function $f_j \circ f_{j+1} \circ \ldots \circ f_k \colon X_{k+1} \to X_j$ by $f_{j,k+1}$. For $j \geq 1$, let $\pi_j \colon \prod_{i=1}^{\infty} X_i \to X_j$ denote j^{th}-coordinate projection. If $A \subset X_{i+1}$ for some $i \geq 1$, let $f_i|_A$ be the set-valued function with domain A such that $f_i|_A(x) = f_i(x)$ for $x \in A$. If $A \subset X_m$ for some $m \geq 1$, let $G_j^m|_A = \{z \in G_j^m \mid \pi_m(z) \in A\}$. It should be noted that we also have $G_j^m|_A = G'(f_j|_{f_{j+1,m}(A)}, \ldots, f_{m-2}|_{f_{m-1,m}(A)}, f_{m-1}|_A)$.

A set-valued function $f \colon X \to Y$ has a *flat spot* (at $p \in Y$) if there exists a point $p \in Y$ and a nondegenerate continuum $X' \subset X$ such that $X' \times \{p\} \subset G(f)$. We say that $X' \times \{p\}$ is a *flat spot of* f. Let $\{X_i, f_i\}_{i \geq 1}$ be an inverse sequence with set-valued functions and let $1 \leq i < j$. When f_i is a member of an inverse sequence, we write $\{x_i\} \times Y_{i+1}$ is a flat spot of f_i, where Y_{i+1} is a subcontinuum of X_{i+1}.

A *flat spot at x_j of f_j composes to a nondegenerate value of f_i* in the composition $f_i \circ f_{i+1} \circ \ldots \circ f_j$ if $f_i(x_j)$ is nondegenerate for $i = j-1$ and if there exists a point x_{i+1} in $f_{i+1,j}(x_j)$ such that $f_i(x_{i+1})$ is nondegenerate for $i < j-1$.

The notion of a k-tail sequence in an inverse sequence, which we call here an *n-tail sequence*, was introduced in [14]. We repeat the definition with different notation that should make the concept more accessible. Let $n \in \mathbb{N}$ and for $i \geq n$, let Y_i be a compactum such that $Y_i \subset X_i$. Suppose that $\{k_i \colon Y_{i+1} \to X_i\}_{i \geq n}$ is a sequence of set-valued functions such that for each $i \geq 1$, $G(k_i) \subset G(f_i)$. Suppose also that, for $i \geq 0$,

(i) $Y_{n+i} \subset k_{n+i}(Y_{n+i+1})$, and
(ii) $(k_{n+i})^{-1}$ is a map (from a subcompactum of X_{n+i} into X_{n+i+1}).

Under these conditions, we say that $\{k_i\}_{i \geq n}$ is an *n-tail sequence of inverse mappings* (with respect to the inverse sequence $\{X_i, f_i\}$). We use the n-tail sequence $\{k_i\}_{i \geq n}$ to generate a subcompactum of X. Let $A_n = k_n(Y_{n+1})$. For $1 \leq i < n$, let $A_i = f_i(A_{i+1})$ and let $g_i = f_i|_{A_{i+1}}$. For $i \geq 1$, let $A_{n+i} = (k_{n+i-1})^{-1}(A_{n+i-1})$ and let $g_{n+i-1} = k_{n+i-1}|_{A_{i+1}}$.

Let $A(n) = \varprojlim\{A_i, g_i\}$. By [7, Theorem 2.4], $A(n)$ is a subcompactum of X. We say that $A(n)$ is the *subcompactum of X generated by the n-tail sequence* $\{k_i\}_{i\geq n}$. By [14, Theorem 2.1], $A(n) \overset{T}{\approx} G_1^n|_{A_n}$.

A continuum is *hereditarily unicoherent* if the intersection of each pair of its subcontinua is connected. A mapping $g\colon X \to Y$ is *weakly confluent* if, for each subcontinuum K of Y, there exists a component H of $g^{-1}(K)$ such that $g(H) = K$. A continuum Y is in *Class*(W) if each surjective mapping of a continuum onto Y is weakly confluent. Some classes of continua that are contained in Class(W) are arclike, non-planar circle-like, atriodic tree-like, atriodic acyclic, and atriodic with symmetric span zero (see [3], [4], [5],and [13]).

Theorem 1 sets up the tools we need to have the dimension of the partial graphs not exceed the dimension of the inverse limit space. Since, for inverse limits with mappings, the dimension of the factor spaces can exceed the dimension of the inverse limit space, and each inverse limit with set-valued functions is also an ordinary inverse limit on its partial graphs [11, Theorem 4.1], Theorem 2 may be somewhat surprising. Corollary 2, which follows from Theorem 1, is the main tool for proving Theorem 2.

Theorem 1. *Let $X = \varprojlim\{X_i, f_i\}$, where, for each $i \geq 1$, X_i is a hereditarily unicoherent continuum, $f_i\colon X_{i+1} \to X_i$ is a surjective, continuum-valued function, and $c_2|_{G(f_i)}\colon G(f_i) \to X_i$ is weakly confluent. If there exists a flat spot $\{x_n\} \times Y_{n+1}$ of f_n for some $n \geq 1$ in the inverse sequence, then*

(1) *either X contains a copy of $G_1^m|_{Z_m}$, for some $m \geq n+1$ and some nondegenerate subcontinuum Z_m of X_m*
(2) *or there exists a subcontinuum Y of X such that Y is homeomorphic to $G_1^n|_{\{x_n\}} \times \varprojlim\{Y_i, g_i\}_{i\geq n+1}$, where each Y_i is a nondegenerate subcontinuum of X_i and each g_i is a surjective mapping.*

Proof. Let $n \geq 1$, where $\{x_n\}\times Y_{n+1}$ is a flat spot of f_n. Since $c_2|_{G(f_{n+1})}$ is weakly confluent, there exists a subcontinuum F_{n+1} of $G(f_{n+1})$ such that $c_2(F_{n+1}) = Y_{n+1}$. If either $c_1(F_{n+1}) = \{z\}$ or $c_1(F_{n+1})$ contains a point z in X_{n+2} where $Z_{n+1} = f_{n+1}(z) \cap Y_{n+1}$ is nondegenerate, then any point $\mathbf{x} = (x_1, x_2, \ldots)$ of X with $\pi_{n+2}(\mathbf{x}) = z$ creates an $(n+2)$-tail sequence. In particular, for $i \geq n+2$, we let $k_i(x_{i+1}) = x_i$ and $G(k_i) = \{(x_{i+1}, x_i)\}$. So, $\{k_i\}_{i\geq n+2}$ is an $(n+2)$-tail sequence with first coordinate z. It follows from [14, Theorem 2.1] that X contains a copy of $G_1^{n+2}|_{\{z\}}$.

Since $G_1^{n+1}|_{Z_{n+1}} \overset{T}{\approx} G_1^{n+1}|_{Z_{n+1}} \times \{z\} \overset{T}{\subset} G_1^{n+2}|_{\{z\}}$, (1) holds for $m = n+1$.

So, we assume that $c_1(F_{n+1})$ is nondegenerate and contains no z where $f_{n+1}(z)\cap Y_{n+1}$ is nondegenerate. Hence, $c_1(F_{n+1})$ is a nondegenerate subcontinuum Y_{n+2} of X_{n+2}, and $g_{n+1}\colon Y_{n+2}\to Y_{n+1}$, defined by $g_{n+1}(x) = f_{n+1}(x)\cap Y_{n+1}$, is a surjective mapping. Also, $G(g_{n+1})\subset G(f_{n+1})$. We repeat the process in the previous paragraph for $c_2|_{G(f_{n+2})}\colon G(f_{n+2})\to X_{n+2}$. By weak confluence, we pick a subcontinuum F_{n+2} of $G(f_{n+2})$ such that $c_2(F_{n+2}) = Y_{n+2}$. If either $c_1(F_{n+2}) = \{z\}$ or $c_1(F_{n+2})$ contains a point z in X_{n+3} where $Z_{n+2} = f_{n+2}(z)\cap Y_{n+2}$ is nondegenerate, then we have an $(n+3)$-tail sequence, giving a subcontinuum of X that is a copy of $G_1^{n+2}|_{Z_{n+2}} \overset{T}{\subset} G_1^{n+3}|_{\{z\}}$ and satisfying (1) for $m = n+2$.

We digress momentarily to make an observation that will be used in the proof of Theorem 2.

Observation. If Z_{n+2} is a flat spot in Y_{n+2}, then $g_{n+1}(Z_{n+2}) = \{y_{n+1}\}$ for some $y_{n+1}\in Y_{n+1}$. So, $x_n\in f_n(y_{n+1}) = f_n(g_{n+1}(Z_{n+1}))$. That is, $\{y_{n+1}\}\times Z_{n+2}$ is a flat spot that composes to $\{x_n\}$.

If no such z exists in $c_1(F_{n+2})\subset X_{n+3}$, we let $Y_{n+3} = c_1(F_{n+2})$, and again define a surjective mapping $g_{n+2}\colon Y_{n+3}\to Y_{n+2}$.

Continuing this process, we eventually find an $m\geq n+1$ for which (1) holds, or we have an inverse sequence $\{Y_i, g_i\}_{i\geq n+1}$, where each g_i is a surjective mapping. We claim that, ignoring the extra parentheses, $Y = G_1^n|_{\{x_n\}}\times \varprojlim\{Y_i, g_i\}_{i\geq n+1}$ is a subset of X. To see this, let $\mathbf{y} = (y_1,\ldots,y_n,y_{n+1},\ldots)$ be a point of Y. By definition of $G_1^n|_{\{x_n\}}$, we have that $y_n = x_n$, and for $1\leq i\leq n+1$, $y_i\in f_i(y_{i+1})$. For $i\geq n+2$, $g_i(y_{i+1}) = y_i$ and by definition of g_i, $y_i\in f_i(y_{i+1})$. So, $\mathbf{y}\in X$ and (2) holds. □

Corollary 1. Let $X = \varprojlim\{X_i, f_i\}$, where for each $i\geq 1$, X_i is hereditarily unicoherent and in Class(W), and $f_i\colon X_{i+1}\to X_i$ is a surjective, continuum-valued function. If there exists a flat spot $\{x_n\}\times Y_{n+1}$ of f_n for some $n\geq 1$ in the inverse sequence, then

(1) either X contains a copy of $G_1^m|_{Z_m}$, for some $m\geq n+1$ and some nondegenerate subcontinuum Z_m of X_m
(2) or there exists a subcontinuum Y of X such that Y is homeomorphic to $G_1^n|_{\{x_n\}}\times\varprojlim\{Y_i, g_i\}_{i\geq n+1}$, where each Y_i is a nondegenerate subcontinuum of X_i and each g_i is a surjective mapping.

Proof. Since each X_i is in Class(W), it follows that each $c_2|_{G(f_i)}\colon G(f_i)\to X_i$ is weakly confluent. The corollary follows. □

Corollary 2. Let $X = \varprojlim\{[0,1], f_i\}$, where for each $i \geq 1$, $f_i\colon [0,1] \to [0,1]$ is a surjective, interval-valued function. If there exists a flat spot $\{x_n\} \times Y_{n+1}$ of f_n for some $n \geq 1$ in the inverse sequence, then

(1) either X contains a copy of $G_1^m|_{Z_m}$, for some $m \geq n+1$ and some nondegenerate subinterval Z_m of $[0,1]$

(2) or there exists a subcontinuum Y of X such that Y is topologically the product of $G_1^n|_{\{x_n\}}$ and a nondegenerate arclike continuum $\varprojlim\{Y_i, g_i\}_{i \geq n+1}$.

Proof. We only need to observe that $[0,1]$ is in Class(W), and the inverse limit in (2) of Theorem 1 is an inverse limit on nondegenerate arcs with surjective bonding mappings. □

Theorem 2 generalizes [15, corollaries 27 and 28].

Theorem 2. *Let* $X = \varprojlim\{[0,1], f_i\}$*, where, for each* $i \geq 1$*,* $f_i\colon [0,1] \to [0,1]$ *is a surjective, interval-valued function. If either there exists an* $i \geq 1$ *where* $\dim(G(f_i)) = 2$ *or there exists a flat spot that composes to a nondegenerate interval in the inverse sequence, then* $\dim(X) \geq 2$.

Proof. If $\dim(G(f_n)) = 2$ for some $n \geq 1$, then it follows from [16, Theorem 10.2] that $G(f_n) \overset{T}{\approx} G(f_n^{-1}) = G_n^{n+1}$ contains a closed two cell of the form $[s_n, t_n] \times [s_{n+1}, t_{n+1}]$. In this case, each point of $[s_{n+1}, t_{n+1}]$ is a nondegenerate value of f_n that contains the interval $[s_n, t_n]$. Also, $\{s_n\} \times [s_{n+1}, t_{n+1}]$ is a flat spot of f_n.

Let $n \geq 1$. Suppose either $\dim(G(f_n)) = 2$ or $\{x_n\} \times [s_{n+1}, t_{n+1}]$ is a flat spot of f_n that composes to x_{j+1} with $j < n$, and $f_j(x_{j+1}) = [s_j, t_j]$, a nondegenerate interval. In either case, we can apply Corollary 2. We consider two cases.

Case A. Suppose (1) in Corollary 2 holds. So, X contains a copy of $G_1^m|_{Z_m}$ for some $m \geq n+1$ and for some nondegenerate subinterval Z_m of $[0,1]$. Recall, in the proof of Theorem 1, that in this case there exists z in Y_{m+1} such that $f_m(z) \cap Y_m$ is a nondegenerate subcontinuum, say $[s_m, t_m]$, of Y_m.

If $m = n+1$, then either $[s_n, t_n] \times [s_m, t_m] \subset G_n^{n+1} = G_n^m$, giving us that $\dim(G_n^m) = 2$, or the flat spot $\{x_n\} \times [s_m, t_m]$ composes to the nondegenerate value $f_j(x_{j+1})$, and by [15, Theorem 24], $\dim(G_j^m) \geq 2$.

If $m \geq n+2$, by construction and choice of m, we have that for $n+1 \leq i < m$, g_i is a mapping of Y_{i+1} onto a nondegenerate continuum Y_i.

Suppose that $\dim(G(f_n)) = 2$ with $[s_n, t_n] \times [s_{n+1}, t_{n+1}] \subset G_n^{n+1}$. Let $\hat{f}_n\colon G'(g_{n+1}, \ldots, g_{m-1}) \to [s_n, t_n]$ be the set-valued function defined by

$\hat{f}_n(x_{n+1},\ldots,x_m) = [s_n,t_n]$. We observe that

$$G(\hat{f}_n^{-1}) = [s_n,t_n] \times G'(g_{n+1},\ldots,g_{m-1}) \overset{T}{\subset} G_n^m|_{[s_m,t_m]}.$$

Since each g_i is a surjective mapping, $G'(g_{n+1},\ldots,g_{m-1})$ is a nondegenerate arc. It follows that $\dim(G_n^m|_{[s_m,t_m]}) \geq 2$.

Suppose that $\{x_n\} \times [s_{n+1},t_{n+1}]$ is a flat spot of f_n that composes to x_{j+1} with $j < n$. We consider two subcases.

Subcase i. Suppose that $\{x_{m-1}\} \times [s_m,t_m]$ is a flat spot of g_{m-1}. By the Observation in the proof of Theorem 1, $\{x_{m-1}\} \times [s_m,t_m]$ is a flat spot that composes to $\{x_n\}$. By assumption, x_n composes to x_{j+1}, where $f_j(x_{j+1})$ is nondegenerate, giving us that the flat spot $\{x_{m-1}\} \times [s_m,t_m]$ composes to the nondegenerate value $f_j(x_{j+1})$, in which case, by [15, Theorem 24], we have that the partial graph

$$G'' = G'(f_j|_{f_{j+1,n}(x_n)},\ldots,f_n|_{[s_{n+1},t_{n+1}]},\\ g_{n+1}|_{g_{n+2,m-1}(x_{m-1})},\ldots,g_{m-1}|_{[s_m,t_m]})$$

has dimension greater than one.

Since $G'' \subset G_j^m|_{[s_m,t_m]}$, we have that $\dim(G_j^m|_{[s_m,t_m]}) \geq 2$.

Subcase ii. Suppose that $[s_m,t_m]$ is not a flat spot of f_{m-1}. By assumption, $f_j(x_{j+1})$ is a nondegenerate arc. Also, the partial graph (with mappings) $G'(g_{n+1}|_{g_{n+2,m}([s_m,t_m])},\ldots,g_{m-1}|_{[s_m,t_m]})$ is a nondegenerate arc. Since $\{x_n\}$ composes to $\{x_{j+1}\}$, let $(x_{j+1},\ldots,x_n)$ be a point of G_{j+1}^n. Then

$$f_j(x_{j+1}) \times \{x_{j+1}\} \times \ldots \times \{x_n\} \times G'(g_{n+1}|_{g_{n+2,m}([s_m,t_m])},\ldots,g_{m-1}|_{[s_m,t_m]})$$

is topologically a subset of $G_j^m|_{[s_m,t_m]}$. It follows that $\dim(G_j^m|_{[s_m,t_m]}) \geq 2$.

So, in all subcases of Case A, we have, by [15, Corollary 6], that $\dim(G_1^m|_{[s_m,t_m]}) \geq 2$. Since X contains a copy of $G_1^m|_{[s_m,t_m]}$, $\dim(X) \geq 2$.

Case B. Suppose (2) in Corollary 2 holds. Since $\varprojlim\{Y_i,g_i\}_{i\geq n+1}$ is a nondegenerate arclike continuum, it has dimension one. If $\{x_n\}$ composes to the nondegenerate value x_{j+1} of f_j, we have that $G_j^n|_{\{x_n\}}$ is a nondegenerate continuum and hence has dimension at least one. So, by [15, Theorem 24], $\dim(G_1^n|_{\{x_n\}}) \geq 1$. It follows from the Hurewicz Product Theorem that $\dim(Y) \geq 2$. Hence, $\dim(X) \geq 2$.

Suppose that $\dim(G(f_n)) = 2$ with $[s_n,t_n] \times [s_{n+1},t_{n+1}] \subset G_n^{n+1}$. Let $\hat{X} = \varprojlim\{[s_i,t_i],g_i\}_{i\geq n+1}$. Let $\hat{f}_n\colon \hat{X} \to G_1^n|_{[s_n,t_n]}$ be the continuum-valued function defined by $\hat{f}_n(x_{n+1},\ldots) = G_1^n|_{[s_n,t_n]}$. Note that

$$G_1^n|_{[s_n,t_n]} \times \varprojlim\{Y_i,g_i\}_{i\geq n+1} = G(\hat{f}_n^{-1}) \overset{T}{\subset} X.$$

Each of $G_1^n|_{[s_n,t_n]}$ and $\varprojlim\{Y_i, g_i\}_{i\geq n+1}$ is a nondegenerate continuum. So, it follows from the Hurewicz Product Theorem that $\dim(X) \geq 2$. □

Corollary 3 generalizes [15, corollaries 29 and 30]. As mentioned at the beginning of the paper, by [2], Corollary 3 is also a characterization of dimension one of the inverse limit space in this setting.

Corollary 3. Let $X = \varprojlim\{[0,1], f_i\}$, where, for each $i \geq 1$, $f_i\colon [0,1] \to [0,1]$ is a surjective, interval-valued function. Then X is tree-like if and only if $\dim(G(f_i)) = 1$ for each $i \geq 1$, and no flat spot composes to a nondegenerate value in the inverse sequence.

Proof. $\Rightarrow$: Since X is tree-like, $\dim(X) = 1$. So, it follows from Theorem 2 that $\dim(G(f_i)) = 1$ for each $i \geq 1$, and no flat spot composes to a nondegenerate value in the inverse sequence.

$\Leftarrow$: This implication is established in [15, Theorem 26]. □

Corollary 4 generalizes [15, Theorem 17].

Corollary 4. Let $\{[0,1], f_i\}_{i=1}^n$ be a finite inverse sequence, where, for each $1 \leq i \leq n$, $f_i\colon [0,1] \to [0,1]$ is a surjective, interval-valued function. The following are equivalent.

(1) G_1^{n+1} is a λ-dendroid.
(2) For each $1 \leq i \leq n$, $\dim(G(f_i)) = 1$, and no flat spot composes to a nondegenerate value in the finite inverse sequence.
(3) G_i^j is a λ-dendroid for each $1 \leq i \leq j \leq n+1$.

Proof. For $1 \leq k \leq n$, let $X_{k+1} = \varprojlim\{[0,1], g_i\}$, where, for each $1 \leq i \leq k$, $g_i = f_i$, and for $i > k$, g_i is the identity map on $[0,1]$. It is easy to see that, for each $1 \leq k \leq n$, $X_{k+1} \overset{T}{\approx} G_1^{k+1}$.

(1) $\Rightarrow$ (2): Suppose that G_1^{n+1} is a λ-dendroid. So, X_{n+1} is a λ-dendroid, and is, therefore, tree-like. By Corollary 3, $\dim(G(g_i)) = 1$ for each $i \geq 1$, and no flat spot composes to a nondegenerate value in the inverse sequence. The implication follows.

(2) $\Rightarrow$ (3): By either [15, Theorem 26] or Corollary 3, we get that each X_{k+1} is tree-like. So, each G_1^{k+1} is tree-like and has dimension one. It follows from [15, Theorem 17] that each G_i^j is a λ-dendroid for $1 \leq i \leq j \leq n+1$.

(3) $\Rightarrow$ (1): This implication is obvious. □

References

[1] Iztok Banic and Sina Greenwood, *On the dimension of Mahavier products and inverse limits with set-valued functions on intervals*. Abstract. 50th Spring Topology and Dynamical Systems Conference, Waco, TX, 2016.

[2] Włodzimierz J. Charatonik and Robert P. Roe, *Inverse limits of continua having trivial shape*, Houston J. Math. **38** (2012), no. 4, 1307–1312.

[3] James F. Davis, *Atriodic acyclic continua and class W*, Proc. Amer. Math. Soc. **90** (1984), no. 3, 477–482.

[4] Gary Allan Feuerbacher, *Weakly chainable circle-like continua*. Ph.D Thesis. University of Houston, 1974.

[5] J. Grispolakis and E. D. Tymchatyn, *Continua which admit only certain classes of onto mappings*, Topology Proc. **3** (1978), no. 2, 347–362 (1979).

[6] W. Hurewicz, *Sur la dimension des produits Cartésiens*, Ann. of Math. (2) **36** (1935), no. 1, 194–197.

[7] W. T. Ingram, *Inverse limits with upper semi-continuous bonding functions: Problems and some partial solutions*, Topology Proc. **36** (2010), 353–373.

[8] ———, *An Introduction to Inverse Limits with Set-valued Functions*. Springer Briefs in Mathematics. New York: Springer, 2012.

[9] ———, *Tree-likeness of certain inverse limits with set-valued functions*, Topology Proc. **42** (2013), 17–24.

[10] ———, *Concerning dimension and tree-likeness of inverse limits with set-valued functions*, Houston J. Math. **40** (2014), no. 2, 621–631.

[11] ———, *One-dimensional inverse limits with set-valued functions*, Topology Proc. **46** (2015), 243–253.

[12] Hisao Kato, *On dimension and shape of inverse limits with set-valued functions*. To appear in Fundamenta Mathematicae.

[13] M. M. Marsh, *Some generalizations of universal mappings*, Rocky Mountain J. Math. **27** (1997), no. 4, 1187–1198.

[14] ———, *Some structure theorems for inverse limits with set-valued functions*, Topology Proc. **42** (2013), 237–258.

[15] ———, *Tree-like inverse limits on* $[0,1]$ *with interval-valued functions*, Topology Proc. **48** (2016), 215–232.

[16] Sam B. Nadler, Jr. *Dimension Theory: An Introduction with Exercises*. Aportaciones Matemáticas: Textos, 18. México: Sociedad Matemática Mexicana, 2002.

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