Some fixed point theorems for tree-like continua

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Abstract

We show that certain tree-like continua, that can be realized as inverse limits on $[0, 1]$ with set-valued bonding functions, must have the fixed point property. In this class of tree-like continua, one result involves inverse limits for which the bonding functions are interval-valued and their graphs are trees, and another result involves inverse limits whose partial graphs are dendrites.

Keywords: fixed point property, inverse limit, tree-like, set-valued function, universal mapping
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1. Introduction

A *compactum* is a compact metric space. A *continuum* is a connected compactum. A continuous function will be referred to as a *map* or *mapping*. A continuum X has *the fixed point property* (fpp) provided that whenever f is a self-mapping on X, there is a point $x \in X$ such that $f(x) = x$. If $\epsilon > 0$, a mapping $f\colon X \to Y$ is an ϵ *mapping* if $\operatorname{diam}(f^{-1}(y)) < \epsilon$ for each $y \in Y$. A continuum X is *tree-like* (*arclike*) if for each $\epsilon > 0$, there exists an ϵ-mapping of X onto a tree (an arc).

In 1979, D. Bellamy [1] published the first example of a tree-like continuum admitting a fixed-point-free mapping, thus answering a question posed by R.H. Bing [2] ten years earlier. Bellamy starts with the 6-adic solenoid, replacing an arc with the suspension over a 0-dimensional set, and recompactifying. Then, he identifies points with their algebraic inverses (the suspension

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is identified across its base to a fan) getting a modified 6-adic Knaster continuum where the fan has replaced an arc that contained the endpoint. The fixed-point-free mapping on the resulting continuum may be thought of as a modification of a mapping on the 6-adic Knaster continuum that is induced by the squaring map on the 6-adic solenoid. Bellamy's construction was very insightful, and very technical. Nevertheless, the underlying idea of the behavior of Bellamy's fixed-point-free mapping is relatively straightforward, and over the last forty years, the basic idea has been used by others to produce a number of tree-like continua admitting fixed-point-free mappings and homeomorphisms, some with additional properties. See, for example, [5], [6], [7], [19], [20], [21], [22], [23], [26], and [27]. Even so, basic questions remain unanswered in this area.

1. Is there a simple inverse limit representation of a tree-like continuum admitting a fixed-point-free mapping?

2. Is there a planar tree-like continuum without the fpp?

3. Can we eliminate the possibility of simply described examples by obtaining affirmative answers to questions of the following nature?

   (a) Fix $n \in \mathbb{N}$. Do inverse limits on trees with fewer than n branchpoints have the fpp?

   (b) Do inverse limits on a single tree (a simple triod) have the fpp?

   (c) Must an inverse limit on a simple triod with a single bonding mapping have the fpp?

   (d) What additional properties placed on the bonding mappings in (a) through (c) will produce a fixed point result?

In regards to Question 1, L.G. Oversteegen and J.T. Rogers, Jr. [26, 27] gave the first inverse limit descriptions of tree-like continua admitting fixed-point-free mappings. Recently, R. Hernández-Gutiérrez and L.C. Hoehn [8] have produced the "simplist" inverse limit description, thus far, of a tree-like continuum without the fpp. Their fixed-point-free mapping is induced from maps on the factor spaces that commute with the bonding maps. There is a bound on the number of components of point-inverses under both the bonding maps and the maps that commute with them, but the number of branchpoints of the factor spaces increases exponentially. A positive answer to Question

3(a) would indicate that we cannot expect anything much simpler. There have been some positive partial answers to questions of the type 3(a) through 3(d). See, for example, [3], [13], [14], and [15].

In the last fifteen years, since the introduction of "generalized" inverse limits, it has been shown that some tree-like continua can be realized as inverse limits on $[0, 1]$ with set-valued bonding functions. It would be of interest to know which tree-like continua in this class must have the fpp. Perhaps there are ones admitting fixed-point-free mappings, giving new tools in this study. This paper is a first step in such an investigation. Our fixed point results involve tree-like continuua X realized as inverse limits on $[0, 1]$ with set-valued functions, and our proofs additionally rely on expressing X as an ordinary inverse limit on its partial graphs, and then applying an "old" technique. Namely, we use W. Holsztynski's [9] theorem that inverse limits on absolute neighborhood retracts, whose bonding mappings and their compositions are universal, have the fpp.

2. Definitions

Let X and Y be compacta. We refer to functions $f : X \to 2^Y$ as *set-valued functions* from X to Y and we write $f : X \to Y$ is a set-valued function. Note that throughout, we are assuming that, for $x \in X$, the value $f(x)$ of a set-valued function is a closed set. The *graph* of f, which we denote by $G(f)$, is the set in $X \times Y$ consisting of all points (x, y) with $y \in f(x)$. If $B \subset G(f)$, we let $B^{-1} = \{(y, x) \mid (x, y) \in B\}$.

A set-valued function $f : X \to Y$ is *upper semi-continuous at the point* $x \in X$ if for each open set V in Y containing the closed set $f(x)$, there is an open set U in X such that $x \in U$ and $f(p) \subset V$ for each $p \in U$. If $f : X \to Y$ is upper semi-continuous at each point of X, then f is said to be *upper semi-continuous.*

The set-valued function $f : X \to Y$ is *surjective* if for each $y \in Y$, there exists $x \in X$ such that $y \in f(x)$. If the set-valued function $f : X \to Y$ is surjective, we let $f^{-1} : Y \to X$ be the set-valued function such that $x \in f^{-1}(y)$ if and only if $y \in f(x)$. Clearly, $G(f^{-1})$ is homeomorphic to $G(f)$. A set-valued function $f : X \to Y$ is *continuum-valued* if for each $x \in X$, the set $f(x)$ is a subcontinuum of Y.

For $f : X \to Y$ a set-valued function, and $A \subset X$, we let $f|_A$ be the set-valued function whose domain is A, and $f|_A(x) = f(x)$ for $x \in A$. If $x \in X$ and $f(x)$ is degenerate, we will sometimes treat $f(x)$ as a point of Y.

For $i \geq 1$, let X_i be a compactum, and let $f_i\colon X_{i+1} \to X_i$ be a surjective upper semi-continuous set-valued function. Throughout, we let $\{X_i, f_i\}_{i\geq 1}$ denote an inverse sequence, and its inverse limit is given by

$$\varprojlim\{X_i, f_i\} = \{x = (x_1, x_2, \ldots) \in \prod_{i\geq 1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } i \geq 1\}.$$

For $n \in \mathbb{N}$, we define the set below.

$$G_{n+1} = G'(f_1, \ldots, f_n) = \{x \in \prod_{i=1}^{n+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}.$$

We refer to these sets as *partial graphs* in the inverse sequence. For consistency of notation, we let $G_1 = X_1$. The notation $X \overset{T}{\approx} Y$ will indicate that X is homeomorphic to Y.

A set-valued function $f\colon X \to Y$ has a *flat spot* if there exists a point $p \in Y$ and a nondegenerate continuum $X' \subset X$ such that $X' \times \{p\} \subset G(f)$. We say that p is a *flat spot for* f.

For $1 \leq i < j$, we denote the set-valued composition function $f_i \circ f_{i+1} \circ \ldots \circ f_j\colon X_{j+1} \to X_i$ by $f_{i,j+1}$. A *flat spot at* x_j *for* f_j *composes to a nondegenerate value of* f_i in the composition $f_i \circ f_{i+1} \circ \ldots \circ f_j$ if $f_i(x_j)$ is nondegenerate for $i = j - 1$, or if there exists a point x_{i+1} in $f_{i+1,j}(x_j)$ such that $f_i(x_{i+1})$ is nondegenerate for $i < j - 1$.

Let $\{X_i, f_i\}_{i\geq 1}$ be an inverse sequence with upper semi-continuous surjective set-valued functions. For $n \geq 1$, we define the set-valued function $F_n\colon X_{n+1} \to G_n$, where $(x_1, x_2, \ldots, x_n) \in F_n(t)$ if and only if $(x_1, x_2, \ldots, x_n, t)$ is in G_{n+1}. V. Nall introduced this function in [25], and showed that F_n is upper semi-continuous. If f_i is continuum-valued for each $1 \leq i \leq n$, the author showed in [17] that F_n is continuum-valued. We call F_n *the function induced by* $(f_1, \ldots, f_n)$. We also note that $G(F_n^{-1}) = G_{n+1}$.

Let $\{X_i, g_i\}_{i\geq 1}$ be an inverse sequence with mappings for bonding functions. Then we refer to the inverse sequence and its inverse limit, respectively, as an *ordinary inverse sequence* and an *ordinary inverse limit*. For ordinary inverse sequences and for $1 \leq i < j - 1$, we denote the composition bonding mapping from the j^{th} factor space to the i^{th} factor space by $g_i^j\colon X_j \to X_i$.

Let $X = \varprojlim\{X_i, f_i\}$ be an inverse sequence with upper semi-continuous surjective set-valued bonding functions. For $i \geq 1$, define the mapping $\rho_i\colon G_{i+1} \to G_i$ by $\rho_i(x_1 \ldots, x_i, x_{i+1}) = (x_1 \ldots, x_i)$. Note that $\{G_i, \rho_i\}_{i\geq 1}$

is an ordinary inverse sequence. W.T. Ingram showed in [10, Corollary 4.2] that $X \stackrel{T}{\approx} \varprojlim\{G_i, \rho_i\}$. This representation of an inverse limit with set-valued functions will be important in the proofs of our fixed point theorems.

3. Universal mappings

This section contains mostly introductory propositions about universal mappings, which can be found in the references cited. In some cases, our statement of the proposition is less general than the one cited, but is more suited to our use of it. Those propositions not cited either have easy proofs, or a proof is given. A mapping $f\colon X \to Y$ is *universal* if for each mapping $g\colon X \to Y$, there is a point $x \in X$ such that $f(x) = g(x)$.

Proposition 1. (Prop.1,[9]) *If $f\colon X \to Y$ is universal, then f is surjective.*

Proposition 2. (Prop.5,[9]) *Let $X_0 \subset X$, and let $f\colon X \to Y$ be a mapping. If $f|_{X_0}\colon X_0 \to Y$ is universal, then f is universal.*

Proposition 3. *If X has the fpp, and $f\colon X \to Y$ is a homeomorphism, then f is universal.*

Proposition 4. (Th.12.29,[24]) *Each mapping from a continuum onto a chainable continuum is universal.*

Proposition 5. (Cor.1,[9]) *Let $X = \varprojlim\{X_i, g_i\}$ be an inverse limit on absolute neighborhood retracts, where for each $1 \leq i < j$, the mapping g_i^j is universal. Then X has the fpp.*

The next proposition involves λ-dendroids, whose definition can be found in the second paragraph of the next section.

Proposition 6. *If $f\colon X \to Y$ is a mapping between λ-dendroids, and there exists a subcontinuum H of X such that $f|_H\colon H \to Y$ is surjective and monotone, then f is universal.*

Proof. First we show that $f|_H$ is universal. Let $g\colon H \to Y$ be a mapping. Since $f|_H$ is monotone, $f|_H^{-1}\colon Y \to H$ is an upper semi-continuous continuum-valued function. So, $g \circ f|_H^{-1}\colon Y \to Y$ is an upper semi-continuous continuum-valued function on a λ-dendroid. By [12, Corollary], $g \circ f|_H^{-1}$ has a fixed point. That is, there is a point $y \in Y$ such that $y \in g \circ f|_H^{-1}(y)$. Hence, there exists $x \in f|_H^{-1}(y)$ such that $g(x) = y$. So, $x \in H$ and $f(x) = y$. Therefore, $g(x) = f(x)$. We have that $f|_H$ is universal. By Proposition 2, f is universal. □

Proposition 7. *Let $f_1, f_2, \ldots, f_n$ be a finite sequence of mappings, where for each $i \geq 1$, $f_i\colon X_{i+1} \to X_i$ is a mapping between λ-dendroids. If for each $i \geq 1$, there is a subcontinuum H_{i+1} of X_{i+1} such that $f_i|_{H_{i+1}}\colon H_{i+1} \to X_i$ is surjective and monotone, then $f_1 \circ \ldots \circ f_n\colon X_{n+1} \to X_1$ is universal.*

Proof. A simple induction argument shows that there is a subcontinuum K_{n+1} of X_{n+1} such that $f_1 \circ \ldots \circ f_n|_{K_{n+1}}\colon K_{n+1} \to X_1$ is surjective and monotone. By Proposition 6, $f_1 \circ \ldots \circ f_n\colon X_{n+1} \to X_1$ is universal. □

4. Universal mappings on trees

This section is devoted to showing that certain mappings between trees are universal. Although the conditions satisfied by the mappings may seem rather specialized, they turn out out to be just what we need to prove our main fixed point theorem. The question of what conditions on a mapping of trees will make it a universal mapping has been studied before. The author established a set of sufficient conditions in [14], and C. Eberhart and J.B. Fugate generalized this result in [4] by establishing a set of conditions that characterize universal maps on trees. However, neither of these sets of conditions are well-suited to our setting, so we offer a new theorem in this area which could have general use.

A continuum is *decomposable* if it is the union of two proper subcontinua. A continuum X is *hereditarily decomposable* if each nondegenerate subcontinuum of X is decomposable. A continuum X is *hereditarily unicoherent* provided that whenever two subcontinua H and K of X have a non-empty intersection, it follows that $H \cap K$ is a continuum. A continuum X is a λ-*dendroid* if it is hereditarily decomposable and hereditarily unicoherent. A continuum X is a *dendroid* if it is arcwise connected and hereditarily unicoherent. So, a dendroid is uniquely arcwise connected.

Let X be a dendroid. The *order of a point x in X*, denoted $o_X(x)$, is the cardinality of the set of arc components of $X \setminus \{x\}$. We let the set of *endpoints* and *branchpoints* of X be defined, respectively, as $E(X) = \{x \mid o_X(x) = 1\}$ and $B(X) = \{x \mid o_X(x) \geq 3\}$. We let $V(X) = E(X) \cup B(X)$ be the set of *vertices* of X. If x and y are points of X, we let both $[x, y]$ amd $[y, x]$ denote the unique arc in X with endpoints x and y. A locally connected dendroid is a *dendrite*. A *tree* is a dendroid with finitely many branchpoints, each having finite order.

If X is a tree, we say that A is an *arc emanating from the point* x in X if A is an arc in X, and x is an endpoint of A. We say that two arcs A and B in X are *abutting arcs* if their intersection is an endpoint of each of A and B.

Definition 1. Let Z and T be trees, M be an arc in T, and $p: Z \to T$ be a surjective mapping. Suppose there exist an arc L in Z, pairwise disjoint subtrees $T_1, \ldots, T_m$ of T such that $T = M \cup (\cup_{i=1}^m T_i)$, pairwise disjoint subtrees $Z_1, \ldots, Z_n$ of Z such that $Z = L \cup (\cup_{i=1}^n Z_i)$, and p satisfies the following conditions.

(1) $p(L) = M$,

(2) for each $1 \leq i \leq n$, there exists $1 \leq j \leq m$ such that $p|_{Z_i}: Z_i \to T_j$ is a homeomorphism, and

(3) for each $1 \leq j \leq m$, $p^{-1}(T_j) \subset \cup_{i=1}^n Z_i$.

Then we say that p *folds* Z *onto* T *along the arc* M. If for each $1 \leq j \leq m$, there is a unique $1 \leq i \leq n$ satisfying property (2), then $m = n$, and we say that p *lays* Z *onto* T *along* M. We note that if $p: Z \to T$ is a homeomorphism, then for each arc M in T, p lays Z onto T along the arc M.

In Figure 1, we provide a schematic diagram of a mapping p that folds a tree Z with three branchpoints onto a simple triod T. The maximal horizontal arc in Z is L, the maximal horizontal arc in T is M, the light blue arcs in Z are Z_1, Z_2, and Z_3, and the red arc in T is T_1.

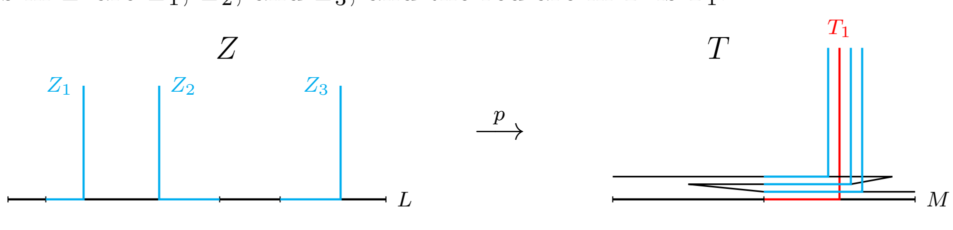

Figure 1. p folds Z onto T along M

Theorem 1. *Let Z and T be trees. If $p: Z \to T$ is a mapping that folds Z onto T along an arc in T, then p is universal.*

Proof. Let L, M, $T_1, \dots, T_m$, and $Z_1, \dots, Z_n$ be given as in Definition 1. If $T = M$, then by Proposition 4, p is universal. So, hereafter, we assume that M is a proper subset of T.

Let $g\colon Z \to T$ be a mapping. We show that p and g have a coincidence point, that is, a point $y \in Z$ such that $p(y) = g(y)$.

Since the T_i's are pairwise disjoint and $T = M \cup (\cup_{i=1}^m T_i)$, we note that M meets each T_i. For $1 \le i \le m$, let J_i be the possibly degenerate arc $M \cap T_i$. Let $T_{i1}, \dots, T_{ik_i}$ be the closures of the components of $T_i \setminus J_i$ for $1 \le i \le m$. Let $T_{ij} \cap J_i = \{w_{ij}\}$ for $1 \le i \le m$ and $1 \le j \le k_i$. Let $r\colon T \to M$ be the retraction such that $r(T_{ij}) = \{w_{ij}\}$ for $1 \le i \le m$ and $1 \le j \le k_i$.

Consider the mappings $p|_L\colon L \to M$ and $rg|_L\colon L \to M$. Since M is an arc, $p|_L$ is universal. So, there exists a point $z \in L$ such that $p(z) = rg(z)$. If $g(z) \in M$, then $p(z) = g(z)$ and the proof is complete. So, we assume that $g(z) \notin M$. We have that $g(z) \in T_{ij} \setminus \{w_{ij}\}$ for some $1 \le i \le m$ and $1 \le j \le k_i$, and $rg(z) = w_{ij} = p(z)$. By property (3) in Definition 1, $z \in Z_\ell$ for some $1 \le \ell \le n$. We assume that $\ell = 1$. Let K_j be the homeomorphic copy of T_{ij} in Z_1. So, $K_j \cap L = \{z\}$. We define the mapping $\hat{g}\colon K_j \to K_j$, where

$$\hat{g}(x) = \begin{cases} z & \text{if } g(x) \notin T_{ij} \\ (p|_{K_j})^{-1} g(x) & \text{if } g(x) \in T_{ij}. \end{cases}$$

It is straightforward to check that $\hat{g}$ is continuous.

Since K_j is a tree, there exists a point $y \in K_j$ such that $\hat{g}(y) = y$. If $g(y) \notin T_{ij}$, then $\hat{g}(y) = z$ and $y = z$. But this contradicts, from above, that $g(z) \in T_{ij}$. So, $g(y) \in T_{ij}$, and $y = \hat{g}(y) = (p|_{K_j})^{-1} g(y)$. Also, $p(y) = p((p|_{K_j})^{-1} g(y)) = g(y)$. Hence, p is universal. □

Corollary 1. *Let Z and T be trees. If there exists a subtree H of Z such that $p|_H\colon H \to T$ is a mapping that folds H onto T along an arc in T, then p is universal.*

Proof. Apply Proposition 2 and Theorem 1. □

5. Continuum-valued functions whose graphs are trees

We begin with two observations and some terminology about continuum-valued functions that have finitely many nondegenerate values, and whose graphs are trees. Let T be a tree, and let $g\colon [0,1] \to T$ be a surjective upper semicontinuous continuum-valued function whose graph is a tree. We also assume that the set $S = \{s \mid g(s) \text{ is nondegenerate}\}$ is finite.

An *interval* in $[0,1]$ is a subcontinuum of $[0,1]$. An *open segment* in $[0,1]$ is the interior of a nondegenerate interval in $[0,1]$. A sequence of sets $\{H_i\}_{i\geq 1}$ is *nested increasing* if for each $i \geq 1$, $H_i \subset H_{i+1}$.

Observation 1. *Let a, b be two adjacent points in $S \cup \{0,1\}$. If $\{J_i\}_{i\geq 1}$ is a nested increasing sequence of intervals in the open segment (a,b) such that $\cup_{i\geq 1} J_i = (a,b)$, then for each $i \geq 1$, $G(g|_{J_i})$ is an arc in $G(g)$. Furthermore, $\mathrm{cl}(\cup_{i\geq 1} G(g|_{J_i}))$ is an arc in $G(g)$ with endpoints of the form (a,x) and (b,y) for some $x \in g(a)$ and $y \in g(b)$.*

Proof. Since for $i \geq 1$, $J_i \subset (a,b)$, we have that $g|_{J_i}$ is single-valued, and therefore continuous. So, for $i \geq 1$, $G(g|_{J_i}) \overset{T}{\approx} J_i$. Hence, the first statement in the conclusion follows. Since $\{G(g|_{J_i})\}_{i\geq 1}$ is a nested increasing sequence of arcs in $G(g)$ and $G(g)$ is a tree, the second statement follows. □

For each pair a, b of adjacent points in $S \cup \{0,1\}$, we call the arc in $G(g)$ given by the furthermore statement in Observation 1, the *connecting arc* in $G(g)$ between $\{a\} \times g(a)$ and $\{b\} \times g(b)$, or simply a *connecting arc* in $G(g)$. If K is a connecting arc in $G(g)$, we also refer to K^{-1} as a *connecting arc* in $G(g^{-1})$.

Let $M_1, \ldots, M_{n+1}$ be the sequence of all connecting arcs in $G(g)$, ordered relative to the order on $[0,1]$. We construct an arc M in $G(g)$ that contains $\cup_{i=1}^{n+1} M_i$. If for each pair of abutting arcs $[a,b]$ and $[b,c]$ in $[0,1]$, where a, b, and c are consecutive members of $S \cup \{0,1\}$, we have that the associated connecting arcs, say M_i and M_{i+1}, are abutting arcs in $G(g)$, then $\cup_{i=1}^{n+1} M_i$ is the desired arc M. Suppose, on the other hand, for some $1 \leq i \leq n$, M_i and M_{i+1} are not abutting in $G(g)$. Then the "right" endpoint, say (b,x), of M_i is not the "left" endpoint, say (b,y), of M_{i+1}. In this case, since $g(b)$ is a tree, we let N_i be the unique arc in $\{b\} \times g(b)$ from (b,x) to (b,y). We let $M = (\cup_{i=1}^{n+1} M_i) \cup (\cup_{i=1}^{m} N_i)$, where the N_i's are chosen as needed for those M_i and M_{i+1} that are not abutting. We call M the *spanning arc in* $G(g)$. We note that the spanning arc M in $G(g)$ is unique, and contains all connecting arcs in $G(g)$. Additionally, M is a union of abutting arcs in $G(g)$, where for each $1 \leq i \leq n+1$, the projection of M_i to the domain of g is an interval between adjacent members of $S \cup \{0,1\}$, and for each $1 \leq i \leq m$, the projection of N_i to the domain of g is a singleton in $S \cup \{0,1\}$. We also refer to M^{-1} as *the spanning arc* in $G(g^{-1})$.

Now let $f\colon [0,1] \to [0,1]$ be a surjective upper semi-continuous interval-valued function whose graph is a tree. Let A be an arc in $G(f)$, and let (a,s)

and (b,t) be the endpoints of A. Assume, without loss of generality, that $a \leq b$.

Observation 2. *There is a continuum-valued function* $f_A\colon [a,b] \to [0,1]$ *such that* $G(f_A) \subset G(f|_{[a,b]})$ *and* $G(f_A) = A$.

Proof. If $a = b$, then $[s,t] \subset f(a)$, and $A = \{a\} \times [s,t]$. Define $f_A\colon \{a\} \to [0,1]$ by $f_A(a) = [s,t]$.

If $a < b$, let A' be the subarc of A that is irreducible between the possibly degenerate intervals $\{a\} \times f(a)$ and $\{b\} \times f(b)$, meeting each interval, respectively, at the points (a,u) and (b,v). It follows that $A = (\{a\} \times [u,s]) \cup A' \cup (\{b\} \times [v,t])$. We observe that the projection of A into the first coordinate is the interval $[a,b]$. Define $f_A\colon [a,b] \to [0,1]$ by $f_A(x) = A \cap f(x)$ for each $x \in [a,b]$. Thus, $G(f_A) = A$. □

In order to avoid cumbersome notation, throughout the remainder of this section we denote the function induced by a sequence of two functions with a capital letter and no subscript. So, for example, given surjective upper semi-continuous continuum-valued functions $f\colon [0,1] \to [0,1]$ and $g\colon [0,1] \to T$, we let $F : [0,1] \to G(g^{-1})$ denote the function induced by (g,f), and recalling the definition at the end of Section 2, we have that $(x,y) \in F(z)$ if and only if $(x,y,z) \in G'(g,f)$. Equivalently, $F(z) = \{(x,y) \in G(g^{-1}) \mid y \in f(z)\}$.

We let π_2, $\pi_{1,2}$, and $\pi_{2,3}$ be the natural projection mappings, relative to the order of the factors, from $T \times [0,1] \times [0,1]$ onto $[0,1]$, $T \times [0,1]$, and $[0,1] \times [0,1]$ respectively. Let $\tilde{\pi}_2$ be the projection of $G(f)$ onto $[0,1]$ that drops the first coordinates of points in $G(f)$.

If A is an arc in $G(f)$ such that $\tilde{\pi}_2(A) = [0,1]$, then we call A a *full arc* in $G(f)$.

We provide the two figures below as visual aids while reading the proofs of Lemmas 1 and 2. For the arc A discussed in Lemma 1, think of a "small" subarc of the red full arc in $G(f)$ that has the properties described.

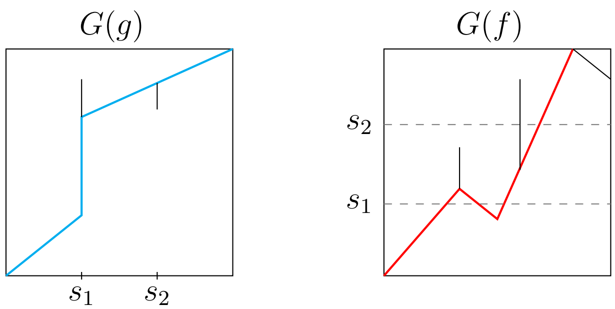

Figure 2. Examples of graphs for g and f

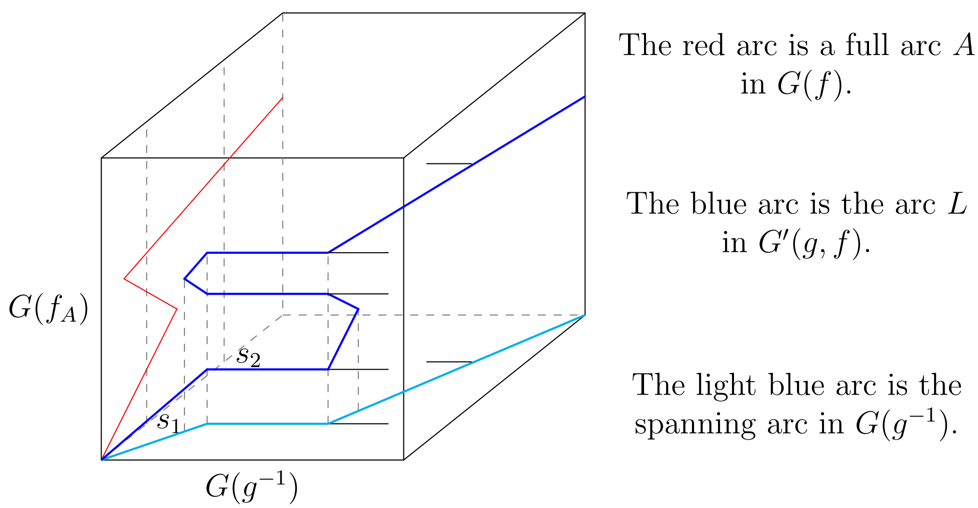

Figure 3. The partial graph $G'(g, f_A)$.

Lemma 1. *Suppose T is a tree, $f : [0,1] \to [0,1]$ and $g : [0,1] \to T$ are surjective upper semi-continuous continuum-valued functions whose graphs are trees. Let $S = \{s \mid g(s) \text{ is nondegenerate}\}$. Let A be an arc in $G(f)$ with endpoints (t, s) and (t', a) such that $A \cap (f^{-1}(s) \times \{s\}) = \{(t, s)\}$, and $\tilde{\pi}_2(A)$ is an arc $[s, b]$ where $[s, b] \cap S = \{s\}$. Let $f_A\colon [t', t] \to [0, 1]$ be the interval-valued function whose graph is A. Then there exist a unique arc L in $G'(g, f)$ and a point $x \in T$ such that the following statements hold.*

(1) $\pi_{2,3}(L) = A^{-1}$, $\pi_{2,3}|_L\colon L \to A^{-1}$ *is a homeomorphism, and* $L \cap (g(s) \times \{s\} \times [0, 1]) = \{(x, s, t)\}$.

(2) $\pi_{1,2}(L)$ *is the arc* $\mathrm{cl}(G(g|^{-1}_{(s,b]})) = \{(x,s)\} \cup G(g|^{-1}_{(s,b]})$*, which is a subset of a connecting arc in* $G(g^{-1})$*.*

(3) $L \cup (g(s) \times \{s\} \times \{t\}) = G'(g|_{[s,b]}, f_A) = G(F^{-1})$*, where* F *is the map induced by* $(g|_{[s,b]}, f_A)$*.*

(4) $G'(g|_{[s,b]}, f_A)$ *is a subtree of* $G'(g,f)$*, and* $\pi_{1,2}$ *lays* $G'(g|_{[s,b]}, f_A)$ *onto* $G(g|^{-1}_{[s,b]})$ *along the arc* $\mathrm{cl}(G(g|^{-1}_{(s,b]}))$*.*

Proof. Construction of L, (1) and (2). Let $\{A_i\}_{i\geq 1}$ be a nested increasing sequence of subarcs of A, where $\mathrm{cl}(\cup_{i\geq 1} A_i) = A$, and for each $i \geq 1$, A_i has endpoints (t_i, s_i) and (t', a) and $s_i \neq s$. Then, by the properties of A in the hypothesis, for $i \geq 1$, $\tilde{\pi}_2(A_i)$ is an interval in $(s,b]$, and $A = \{(t,s)\} \cup (\cup_{i\geq 1} A_i)$. So, for each $i \geq 1$, $A_i^{-1} \subset f^{-1}|_{(s,b]}$, and it follows from Lemma 3.4 in [11] that $\pi_{2,3}|^{-1}_{G'(g,f)}$ maps A_i^{-1} homeomorphically into $G'(g,f)$. For $i \geq 1$, let $L_i = \pi_{2,3}|^{-1}_{G'(g,f)}(A_i^{-1})$. Thus, $\{L_i\}_{i\geq 1}$ is the nested increasing sequence of arcs lying in $G'(g,f)$ such that for each $i \geq 1$, L_i is the unique arc in $G'(g,f)$ where $\pi_{2,3}(L_i) = A_i^{-1}$. Let $L = \mathrm{cl}(\cup_{i\geq 1} L_i)$. We need to show that L is an arc, afterwhich it will be clear, by the definition of the L_i's, that $\pi_{2,3}$ maps L homeomorphically onto A^{-1}. Toward this end, we look at the projections of L under $\pi_{2,3}$ and under $\pi_{1,2}$.

Since $\mathrm{cl}(\cup_{i\geq 1}\pi_{2,3}(L_i)) = \mathrm{cl}(\cup_{i\geq 1} A_i^{-1}) = \{(s,t)\} \cup (\cup_{i\geq 1} A_i^{-1})$, it follows that $L \setminus (\cup_{i\geq 1} L_i) \subset T \times \{(s,t)\}$. For $i \geq 1$, since the second coordinates of points in L_i are equal to the second coordinates of their images under $\pi_{2,3}$ in A_i^{-1}, we have, from the second sentence of the first paragraph, that $\pi_2(L_i) \subset (s,b]$. So, by Observation 1, for each $i \geq 1$, $\pi_{1,2}(L_i)$ is a subarc of $G(g|^{-1}_{(s,b]})$, and we have that $\{\pi_{1,2}(L_i)\}_{i\geq 1}$ is a nested increasing sequence of arcs whose union is $G(g|^{-1}_{(s,b]})$. Since, by Observation 1, $\mathrm{cl}(G(g|_{(s,b]}))$ is an arc with an endpoint (s,x) for some $x \in g(s)$, it follows that $L \setminus \cup_{i\geq 1} L_i \subset \{(x,s)\} \times [0,1]$. We now have that $L \setminus \cup_{i\geq 1} L_i \subset (\{(x,s)\} \times [0,1]) \cap (T \times \{(s,t)\}) = \{(x,s,t)\}$. Hence, L is an arc that meets $g(s) \times \{s\} \times [0,1]$ in exactly one point, namely (x,s,t). Properties (1) and (2) are established.

(3) and (4). Let $f_A\colon [t',t] \to [0,1]$ be the continuum-valued function whose graph is A, and let F be the function induced by $(g|_{[s,b]}, f_A)$. We note that $G(g|^{-1}_{[s,b]})$ is the union of the arc $G(g|^{-1}_{(s,b]}) \cup \{(x,s)\}$ and the tree $g(s) \times \{s\}$. By definition of the arc L, and properties established above, it is now easy to see that $G(F^{-1}) = G'(g|_{[s,b]}, f_A) = L \cup (g(s) \times \{s\} \times \{t\})$ with L meeting $g(s) \times \{s\} \times \{t\}$ at the point (x,s,t). So, $G(F^{-1})$ is the union of an arc and a

tree, and F has exactly one nondegenerate value, namely $F(t) = g(s) \times \{s\}$. Also, from the properties established above, it is easy to see that $\pi_{1,2}$ lays $G'(g|_{[s,b]}, f_A)$ onto $G(g|_{[s,b]}^{-1})$ along the arc $\mathrm{cl}(G(g|_{(s,b]}^{-1}))$. □

We refer to the subtree $G'(g|_{[s,b]}, f_A)$ of $G'(g, f)$ in Lemma 1 as the *subtree of $G'(g, f)$ generated by the arc A* in $G(f)$. In Lemma 2, we use a full arc in $G(f)$ to construct an arc L and a subtree of $G'(g, f)$ containing L.

Lemma 2. *Suppose T is a tree, $f : [0,1] \to [0,1]$ and $g : [0,1] \to T$ are surjective upper semi-continuous continuum-valued functions, and $G(f)$ and $G(g)$ are trees. Suppose that $S = \{s \mid g(s) \text{ is nondegenerate}\}$ is a finite set, $f^{-1}(y)$ has finitely many components for each $y \in [0,1]$, and if $f^{-1}(y)$ contains an interval, then $y \notin S$. Let A be a full arc in $G(f)$, let f_A be the interval-valued function with $G(f_A) = A$, and let F be the function induced by (g, f_A). Then there exists an arc L in $G'(g, f)$ such that L is a union of abutting subarcs, and the following statements hold.*

(1) $\pi_{2,3}|_L \colon L \to A^{-1}$ is monotone, and $(\pi_{2,3}|_L)^{-1}$ maps the endpoints of A^{-1} to the endpoints of L.

(2) $\pi_{1,2}$ maps L onto the spanning arc M^{-1} in $G(g^{-1})$, mapping each of the abutting subarcs of L either into $g(s) \times \{s\}$ for some $s \in S$ or into a connecting arc in $G(g^{-1})$.

(3) For $s \in S$, A^{-1} meets $\{s\} \times [0,1]$ in finitely many points of the form $(s, t_i(s))$ for $1 \le i \le k_s$, and the set $\{t_i(s) \mid s \in S \text{ and } 1 \le i \le k_s\}$ is the set of points where F has its nondegenerate values.

(4) $G'(g, f_A) = G(F^{-1})$ is a subtree of $G'(g, f)$, and $\pi_{1,2}$ folds $G'(g, f_A)$ onto $G(g^{-1})$ along M^{-1}.

Proof. Before we provide details of the proofs of the four rather technical properties, we offer an intuitive description of how our construction will proceed.

We begin by viewing $G(g)$ as the union of its spanning arc M and trees that are attached to M in places where g has nondegenerate values $g(s)$. These trees $\{s\} \times g(s)$ meet M in either a point or an arc as described in the definition of the spanning arc. Sitting above $G(g^{-1})$ in $G(g^{-1}) \times [0,1]$ is $G'(g, f)$. We construct a tree in $G'(g, f)$ that is a sort of folded roof above $G(g^{-1})$. To do this, we start with a full arc A in $G(f)$ that produces an arc L in $G'(g,f)$ so that $\pi_{1,2}(L) = M^{-1}$. To L we attach either single or multiple copies of the trees $g(s) \times \{s\}$. We may think of this as happening as follows. As we move from one endpoint of L to the other, we watch our second coordinates. For those points (x,u,t) in L where $u \notin S$, we add nothing to L, and continue moving along L. When we encounter a point (x,s,t) where $s \in S$ (these are the points t where the induced function F has nondegenerate values), we add the tree $g(s) \times \{s\} \times \{t\}$, which clearly homeomorphically projects by $\pi_{1,2}$ onto $g(s) \times \{s\}$. Furthermore, these added trees are attached to L at either points or arcs in exactly the same manner as those trees below them are attached to M^{-1}. Thus, $\pi_{1,2}$ collapses this folded roof down onto $G(g^{-1})$. Or, according to Definition 1, $\pi_{1,2}$ folds this "roof" above $G(g^{-1})$ onto $G(g^{-1})$.

We begin the proof by noting that since $G(g^{-1})$ is a tree, for each adjacent pair of points s and s' in $S \cup \{0,1\}$, the arc $\mathrm{cl}(G(g|^{-1}_{(s,s')}))$ is a connecting arc in $G(g^{-1})$.

Construction of L, (1) and (2). Let $S = \{s_1, \ldots, s_m\}$. Since $f^{-1}(y)$ has finitely many components for each $y \in [0,1]$, and $f^{-1}(s_i)$ is a finite set for each $1 \le i \le m$, we can pick an ordered sequence $A_1, A_2, \ldots, A_n$ of abutting subarcs of A such that $A = A_1 \cup \ldots \cup A_n$, and for each $1 \le i \le n$, there exists $j \in \{1, \ldots, m\}$ such that $A_i \cap (f^{-1}(s_j) \times \{s_j\})$ is an endpoint of A_i, and $\tilde{\pi}_2(A_i) \cap S = \{s_j\}$.

Thus, for the sequence of arcs $A_1^{-1}, \ldots, A_n^{-1}$, we can pick the sequence of arcs $L_1, \ldots, L_n$ in $G'(g,f)$ given by Lemma 1. The union of the arcs $L_1, \ldots, L_n$ may or may not be the arc L that we desire, but we construct the desired arc L with the arcs $L_1, \ldots, L_n$ as our starting point.

For each $1 \le j \le n$, the pair A_j^{-1}, A_{j+1}^{-1} has an endpoint (c, t_j) in common, where either $c \notin S$, or $c = s_k$ for some $1 \le k \le m$. We begin with $j = 1$, and demonstrate how we can construct a sequence of abutting arcs in $G'(g,f)$ that has the desired properties. We note first that one endpoint of A_1^{-1} is an endpoint of A^{-1}, so it follows from Lemma 1(1), that $(\pi_{2,3}|_L)^{-1}$ maps this endpoint to an endpoint of L_1, which will be an endpoint of L after our construction is complete.

Suppose $A_1^{-1} \cap A_2^{-1} = \{(c_1, t_1)\}$ where $c_1 \notin S$. So, the remaining endpoints of A_1^{-1} and A_2^{-1} are, respectively, of the form (s_i, t) and (s_j, t'), where either $i = j$ or $|i - j| = 1$. In either case, by Lemma 1, we see that L_1 and L_2 are abutting arcs in $G'(g,f)$ such that $\pi_{2,3}|_{L_1 \cup L_2} \colon L_1 \cup L_2 \to A_1^{-1} \cup A_2^{-1}$ is a homeomorphism, and $L_1 \cup L_2$ meets $(g(s_i) \times \{s_i\} \times [0,1])$ in exactly two points if $i = j$, and meets each of $(g(s_i) \times \{s_i\} \times [0,1])$ and $(g(s_j) \times \{s_j\} \times [0,1])$ in exactly one point otherwise. So, by Lemma 1(1), $(\pi_{2,3}|_{L_1})^{-1}$ takes the endpoint (s_i, t) of A_1^{-1} to an endpoint of L_1. Also, it is clear from the statement immediately before the proof of (1) and (2), that $\pi_{1,2}$ maps each of L_1 and L_2 into a connecting arc in $G(g^{-1})$ that is contained in M^{-1}.

Suppose $A_1^{-1} \cap A_2^{-1} = \{(s_k, t_1)\}$ for some $1 \leq k \leq m$. By Lemma 1, for each $i \in \{1, 2\}$, L_i is an arc in $G'(g, f)$ such that $\pi_{2,3}|_{L_i}\colon L_i \to A_i^{-1}$ is a homeomorphism, and L_i meets $g(s_k) \times \{s_k\} \times [0,1]$ in exactly one point (x_i, s_k, t_i). So, again the endpoint of A_1^{-1} that is not (s_k, t_1) has as its preimage under $(\pi_{2,3}|_{L_1})^{-1}$ an endpoint of L_1. Suppose that either $s_k \in \{0, 1\}$ or both $A_1 \setminus \{(t_1, s_k)\}$ and $A_2 \setminus \{(t_1, s_k)\}$ project under $\tilde{\pi}_2$ into the same component of $[0,1] \setminus \{s_k\}$. Then the argument from the previous paragraph applies to give the desired properties of L_1 and L_2. So, we assume that $A_1 \setminus \{(t_1, s_k)\}$ and $A_2 \setminus \{(t_1, s_k)\}$ project under $\tilde{\pi}_2$ into different components of $[0,1] \setminus \{s_k\}$. If $x_1 = x_2$, then L_1 and L_2 have the endpoint (x_1, s_k, t_1) in common. So, L_1 and L_2 are abutting arcs, and $\pi_{2,3}|_{L_1 \cup L_2}\colon L_1 \cup L_2 \to A_1^{-1} \cup A_2^{-1}$ is a homeomorphism. Also, it is clear that $\pi_{1,2}$ maps L_1 and L_2 into adjacent connecting arcs in $G(g^{-1})$.

If $x_1 \neq x_2$, let N_1 be the unique arc in $g(s_k) \times \{s_k\}$ from (x_1, s_k) to (x_2, s_k). Let $J_1 = N_1 \times \{t_1\}$ and note that J_1 is an arc with endpoints (x_1, s_k, t_1) and (x_2, s_k, t_1). Furthermore, $\pi_{1,2}$ maps J_1 homeomorphically onto N_1. We have that L_1, J_1, L_2 are abutting arcs in $G'(g, f)$, and since $\pi_{2,3}(J_1) = \{(s_k, t_1)\}$, it follows that $\pi_{2,3}$ is a monotone mapping of $L_1 \cup J_1 \cup L_2$ onto $A_1^{-1} \cup A_2^{-1}$. That $\pi_{1,2}$ maps each of these abutting arcs into $M^{-1} \subset G(g^{-1})$ is clear.

Now, moving one-by-one, from A_i^{-1} to A_{i+1}^{-1} for each $2 \leq i \leq n$, we piece together the arcs L_i, and J_j as necessary, getting an arc L in $G'(g, f)$ that is the union of the abutting arcs constructed. By the construction of L, we easily see that $\pi_{1,2}$ maps L onto the spanning arc M^{-1} in $G(g^{-1})$ in the manner desired.

(3). For each $s \in S$, A^{-1} meets $\{s\} \times [0,1]$ at exactly those points where some A_i^{-1} has exactly one endpoint in $\{s\} \times [0,1]$. Also, by the construction, we see that the induced function F of (g, f_A) has its nondegenerate values at precisely those points $t_i(s)$, where $s \in S$ and $1 \leq i \leq k_s$. That is, $F(t_i(s)) = g(s) \times \{s\}$ for each $s \in S$ and $1 \leq i \leq k_s$.

(4). This property follows from the construction of L, the properties of the abutting arcs L_i established by Lemma 1, the last sentence in the proof of (1) and (2), and the proof of (3). □

We call $G(F^{-1})$ or $G'(g, f_A)$, in Lemma 2, *the subtree of* $G'(g, f)$ *generated by the full arc* A in $G(f)$.

Remark. In order to keep notational complexity to a minimum, and because we need a full arc in the proof of Theorem 2, we used a full arc A in Lemma 1 to generate the arc L in $G'(g, f)$. We point out that the same construction could be carried out for an arbitrary arc A in $G(f)$, where $\tilde{\pi}_2(A)$ would be a subinterval of $[0, 1]$, and the induced map F would be induced by $(g|_{\tilde{\pi}_2(A)}, f_A)$ rather than (g, f_A). Properties (1) through (4) would follow subject to this modification.

Lemma 3. *We assume the same hypotheses as in Lemma 2, and additionally we let* $h: [0, 1] \to [0, 1]$ *be a surjective upper semi-continuous continuum-valued function such that* $G(h)$ *is a tree,* $h^{-1}(y)$ *has finitely many components for each* $y \in [0, 1]$*, and if* $h^{-1}(y)$ *contains an interval, then* $F(y)$ *is degenerate, where* F *is the induced map from Lemma 2. Then there exists a subtree* H *of* $G'(g, f, h)$ *such that the projection mapping* $\pi_{1,2,3}: T \times [0, 1] \times [0, 1] \times [0, 1] \to T \times [0, 1] \times [0, 1]$ *folds* H *onto* $G'(g, f_A)$ *along* L*, and* $\pi_{1,2} \circ \pi_{1,2,3}$ *folds* H *onto* $G(g^{-1})$ *along* M^{-1}*.*

Proof. We assume that we have the arc L and the subtree of $G'(g, f)$ generated by the full arc A as given by Lemma 2. So, properties (1) through (4) hold. Let $[a, b]$ be the domain of the function f_A. So, $[a, b]$ is also the domain of the function $F: [a, b] \to G(g^{-1})$. Let $S' = \{t \mid F(t) \text{ is nondegenerate}\}$. We choose an arc B in $G(h)$ such that B is irreducible between $[0, 1] \times \{a\}$ and $[0, 1] \times \{b\}$. Let h_B be the continuum-valued function whose graph is B. Now, we treat h, B, h_B, F, and S', respectively, as we treated f, A, f_A, g, and S in (1) through (4) above. Doing so, we construct an arc L' in $G'(F, h) \subset G'(g, f, h)$ where projection into the first two coordinates relative to $G'(F, h)$, equivalently the projection $\pi_{1,2,3}$ into the first three coordinates relative to $G'(g, f, h)$, maps L' onto L, and we attach copies of the trees $g(s) \times \{s\} \times \{t_i(s)\}$ for $s \in S$ and $1 \leq i \leq k_s$ at precisely the points $(x, s, t_i(s), u)$ where $t_i(s) \in h_B(u)$ and $s \in f_A(t_i(s))$. These are the values $u \in [0, 1]$ where the function $\hat{F}$ induced by (F, h) has its nondegenerate values. We note that $G(\hat{F}^{-1}) = G'(F, h_B) = G'(g, f_A, f_B) \subset G'(g, f, h)$. It follows as in the proof of parts (1) through (4), that $\pi_{1,2,3}$ folds $G'(F, h_B)$ onto $G'(g, f_A)$ along L, and in turn, $\pi_{1,2}$ folds $G'(g, f_A)$ onto $G(g^{-1})$ along M^{-1}. So, the composition mapping $\pi_{1,2} \circ \pi_{1,2,3}$ folds $G'(F, h_B)$ onto $G(g^{-1})$ along M^{-1}. □

Lemma 4. *Suppose T is a tree, $f : [0,1] \to [0,1]$ and $g : [0,1] \to T$ are surjective upper semi-continuous continuum-valued functions, and $G(f)$ and $G(g)$ are trees. Suppose that the sets $S_1 = \{s \mid f(s) \text{ is nondegenerate}\}$ and $S_2 = \{s \mid g(s) \text{ is nondegenerate}\}$ are finite, $f^{-1}(y)$ has finitely many components for each $y \in [0,1]$, and if $f^{-1}(y)$ contains an interval, then $y \notin S$. Then $G'(g,f)$ is a tree, and the function F induced by (g,f) has a finite set of values at which F is nondegenerate.*

Proof. We let $G = G'(g,f)$, and we note that $G = G(F^{-1})$.

First we show that G is arcwise connected. Let (a_1,s_1,t_1) and (a_2,s_2,t_2) be points of G. We assume, without loss of generality, that $s_1 \in S_2$ and $s_2 \notin S_2$. Since $G(f)$ is a tree, there exists a unique arc A in $G(f)$ with endpoints (t_1,s_1) and (t_2,s_2). By Lemma 2 and the Remark that follows it, there exists an arc L in G such that $\pi_{2,3}|_L\colon L \to A^{-1}$ is monotone. Furthermore, by Lemma 2(1), we have that (a_2,s_2,t_2) is one endpoint of L, and the other endpoint of L is of the form (x_1,s_1,t_1) for some $x_1 \in g(s_1)$. If $x_1 = a_1$, then we have the desired arc. Otherwise, since $g(s_1) \times \{s_1\} \times \{t_1\}$ is a tree, we can pick a unique arc L_1 in $g(s_1) \times \{s_1\} \times \{t_1\}$ joining the points (x_1,s_1,t_1) and (a_1,s_1,t_1). Now, L and L_1 are abutting arcs. So, $L \cup L_1$ is an arc in G with endpoints (a_1,s_1,t_1) and (a_2,s_2,t_2). So, G is arcwise connected.

It follows from Corollary 4 in [18] that G is a λ-dendroid. So, G is hereditarily unicoherent, and thus, G is a dendroid. It remains to show that $B(G)$ is finite, and that $o_G(p)$ is finite for $p \in B(G)$.

It follows from Lemma 3.1 in [11] that F has degenerate values at points not in the finite set $S_1 \cup f^{-1}(S_2)$. So, the set of branchpoints of G is a subset of the sets $F(t) \times \{t\}$, where $t \in S_1 \cup f^{-1}(S_2)$. Also, for $t \in S_1 \cup f^{-1}(S_2)$, the set $F(t) \times \{t\}$ has finitely many branchpoints relative to itself since $F(t)$ is a subtree of the tree $G(g^{-1})$. It follows that the set of branchpoints of G is finite.

Lastly, we show that the order of each branchpoint of G is finite. Suppose (x,s,t) is a branchpoint of G. If $s \notin S_2$, then there exists an interval $J \subset [0,1] \setminus S_2$ such that $s \in J$. By Lemma 3.4 in [11], $\{(z,y,u) \in G \mid y \in J\}$ is homeomorphic to $G(f^{-1}|_J)$. Since $G(f)$ is a tree, it follows that the order of (x,s,t) in G is finite. Suppose $s \in S_2$. If there exists a neighborhood U of (x,s,t) in G such that $U \subset g(s) \times \{x\} \times \{t\}$, then the order of (x,s,t) in G is finite since $g(s)$ is a tree.

If there is no such neighborhood U, then there are two types of "short" arcs that may emanate from (x,s,t). Those that lie in $g(s) \times \{x\} \times \{t\}$, for which we have just observed there are finitely many, and those whose second projection is an arc $[s,b]$ with $s \neq b$, and $[s,b] \cap S = \{s\}$. Let I be an arc of the second type. So, $\pi_{2,3}(I)$ is an arc A emanating from the point (s,t) in $G(f^{-1})$. By Lemma 1(1), I is the unique arc in G such that $\pi_{2,3}(I) = A$. Since $G(f^{-1})$ is a tree, it follows that there are finitely many such arcs I. Hence, the order of (x,s,t) in G is finite. We have that G is a tree. $\square$

6. Main fixed point results

Theorem 2. *Let $X = \varprojlim\{[0,1], f_i\}$ be an inverse limit with surjective upper semi-continuous interval-valued bonding functions. Suppose that for each $i \geq 1$, $G(f_i)$ is a tree, the set $S_i = \{s \mid f_i(s) \text{ is nondegenerate}\}$ is finite, $f_i^{-1}(t)$ has finitely many components for each $t \in [0,1]$, and for $1 \leq j < i$ no flat spot of f_i composes to a nondegenerate value of f_j. Then for $1 \leq n < m$, $\rho_n^m\colon G_m \to G_n$ is a universal mapping between trees. In particular, if $2 \leq n < m$, there is a subtree H_m of G_m such that $\rho_n^m|_{H_m}\colon H_m \to G_n$ folds H_m onto G_n along the spanning arc in $G_n = G(F_{n-1}^{-1})$.*

Proof. Recall from the last paragraph in Section 2, that X is homeomorphic to the ordinary inverse limit $\varprojlim\{G_i, \rho_i^{i+1}\}$. There are three steps in the proof.

First, we show by induction that for each $n \geq 2$, $G_n = G(F_{n-1})$ is a tree, and $F_{n-1}\colon [0,1] \to G_{n-1}$ has its nondegenerate values on a finite set. Then we observe that $\rho_1^m\colon G_m \to G_1$ is universal for all $m \geq 2$, and for $n \geq 1$, we use Lemma 2 to construct a subtree H_{n+1} of G_{n+1} so that $\rho_n|_{H_{n+1}}\colon H_{n+1} \to G_n$ folds H_{n+1} onto G_n along the spanning arc in G_n. Lastly, we show how to construct a subtree H_m of G_m, for $m > n+1$, so that $\rho_n^m|_{H_m}\colon H_m \to G_n$ folds H_m onto G_n along the spanning arc in G_n.

Step 1. Let $n = 2$. We have that $F_1 = f_1$. It follows from our hypothesis that $G_2 = G(f_1^{-1})$ is a tree, and F_1 has nondegenerate values for finitely many $t \in [0,1]$.

Assume that for some $n \geq 2$, G_n is a tree, and $F_{n-1}\colon [0,1] \to G_{n-1}$ has its nondegenerate values on a finite set. Letting f_n and F_{n-1}, respectively, play the roles of f and g in Lemma 4, we see that $G_{n+1} = G'(F_{n-1}, f_n)$ is a tree, and that the function F_n induced by (F_{n-1}, f_n) has nondegenerate values for finitely many $t \in [0,1]$.

Step 2. Let $n = 1$ and $m \geq 2$. Since $G_1 = [0,1]$, $\rho_1^m\colon G_m \to [0,1]$ is universal by Proposition 4.

Fix $n \geq 2$, and suppose $m = n+1$. We let f_n and F_{n-1}, respectively, play the roles of f and g in Lemma 2. We see that the appropriate conditions are satisfied, and we note that $G_{n+1} = G'(f_n, F_{n-1})$, and $\rho_n = \pi_{1,2}|_{G_{n+1}}$. Let M be the spanning arc in $G_n = G(F_{n-1}^{-1})$, and let A be an arc in $G(f_n)$ that is irreducible between $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$. So, A is a minimal full arc in $G(f_n)$. By Lemma 2, we let H_{n+1} be the tree in G_{n+1} generated by the full arc A. We recall that H_{n+1} contains an arc L where $\rho_n(L) = M^{-1}$, and H_{n+1} is the inverse graph of the function F induced by (f_{n_A}, F_{n-1}). Furthermore, F has its nondegenerate values on the set $\{t_i(s) \mid s \in S \text{ and } 1 \leq i \leq k_s\}$. By Lemma 2(4), ρ_n folds H_{n+1} onto $G(F_{n-1}^{-1}) = G_n$ along the spanning arc M^{-1} in G_n. It follows from Corollary 1 that ρ_n is universal.

Step 3. Let $2 \leq n < n+1 < m$. To show that there is a subtree H_m of G_m such that $\rho_n^m|_{H_m}\colon H_m \to G_n$ folds H_m onto G_n along the spanning arc M^{-1} in G_n, we proceed, using Lemma 3, from the already constructed subtree H_{n+1} of G_{n+1} with ρ_n folding H_{n+1} onto G_n along M^{-1}, to construct a subtree H_{n+2} of G_{n+2}, where ρ_n^{n+2} folds H_{n+2} onto G_n. It will be clear from the process how to proceed until we reach m.

So, f_{n+1} plays the role of h in Lemma 3, and we are now working in $G_{n+2} = G'(F_{n-1}, f_n, f_{n+1})$. Let B be an arc in $G(f_{n+1})$ chosen as in the proof of Lemma 3, and let $\hat{F}$ be the function induced by (F, f_n). Observe that $\rho_{n+1} = \pi_{1,2,3}$ from Lemma 3. Also, from the proof of Lemma 3, we generate an arc L' from B in G_{n+2}, where $\rho_{n+1}(L') = L$, and copies of the trees $g(s) \times \{s\} \times \{t_i(s)\}$, for $s \in S$ and $1 \leq i \leq k_s$, are attached to L' at those points $(x, s, t_i(s), u)$ where $t_i(s) \in f_{n+1_B}(u)$ and $s \in f_{n_A}(t_i(s))$. This is the set of points where $\hat{F}$ has its nondegenerate values. It follows from Lemma 3 that ρ_{n+1} folds $H_{n+2} = G'(F, f_{n+1_B}) = G(\hat{F}^{-1})$ onto H_{n+1} along L, and $\rho_n \circ \rho_{n+1}$ folds H_{n+2} onto G_n along M^{-1}. So, by Corollary 1, ρ_n^{n+2} is universal.

We continue, making analogous backward steps in the inverse sequence until we reach m. The result follows. □

Corollary 2. *Let $X = \varprojlim\{[0,1], f_i\}$ be an inverse limit with surjective upper semi-continuous interval-valued bonding functions. Suppose that for each $i \geq 1$, $G(f_i)$ is a tree, the set $S_i = \{s \mid f_i(s) \text{ is nondegenerate}\}$ is finite, $f_i^{-1}(t)$ has finitely many components for each $t \in [0,1]$, and for $1 \leq j < i$ no flat spot of f_i composes to a nondegenerate value of f_j. Then X is a tree-like continuum with the fpp.*

Proof. Apply Theorem 2 and Proposition 5. □

Corollary 3. *Suppose X is a tree-like continuum that admits a representation where $X = \varprojlim\{[0,1], f_i\}$ with surjective upper semi-continuous interval-valued bonding functions such that for each $i \geq 1$, $G(f_i)$ is a tree, the set $S_i = \{s \mid f_i(s) \text{ is nondegenerate}\}$ is finite, and $f_i^{-1}(t)$ has finitely many components for each $t \in [0,1]$. Then X has the fpp.*

Proof. The result follows from Corollary 3 in [18] and Corollary 2 above. $\square$

Question 1. *Suppose X is a tree-like continuum that admits a representation where $X = \varprojlim\{[0,1], f_i\}$ with surjective upper semi-continuous interval-valued bonding functions such that for each $i \geq 1$, $G(f_i)$ is a tree. Does X have the fpp.*

Question 2. *Suppose X is a tree-like continuum that admits a representation where $X = \varprojlim\{[0,1], f_i\}$ with surjective upper semi-continuous set-valued bonding functions such that for each $i \geq 1$, the partial graph G_i is a dendrite. Does X have the fpp.*

We give a partial answer to Question 2 in the next section.

7. Another fixed point theorem for inverse limits with set-valued bonding functions

Lemma 5. *Let $X = \varprojlim\{X_i, f_i\}$, where for each $i \geq 1$, X_i is a compactum, and $f_i\colon X_{i+1} \to X_i$ is a surjective upper semicontinuous set-valued function. If for each $i \geq 1$, $G(f_i^{-1})$ contains the graph of a continuum-valued function $g_i\colon X_i \to X_{i+1}$, then for each $i \geq 1$, $G(\rho_i^{-1})$ contains the graph of a continuum-valued function $\alpha_i\colon G_i \to G_{i+1}$.*

Proof. Fix $i \geq 1$. For $(x_1, \ldots, x_i) \in G_i$, let $\alpha_i(x_1, \ldots, x_i) = \{(x_1, \ldots, x_i)\} \times g_i(x_i)$. Clearly, $G(\alpha_i) \subset G(\rho_i^{-1})$ and α_i is continuum-valued. $\square$

Under the same assumptions as in Lemma 5, we observe the following.

Observation 3. *For each $i \geq 1$, there exists a subcompactum H_{i+1} of G_{i+1} such that $\rho_i|_{H_{i+1}}\colon H_{i+1} \to G_i$ is a monotone mapping.*

Proof. Fix $i \geq 1$. Let $H_{i+1} = \alpha_i(G_i)$, where α_i is the continuum-valued function given in Lemma 5. For $(x_1, \ldots, x_i) \in G_i$, $(\rho_i|_{H_{i+1}})^{-1}(x_1, \ldots, x_i) = \alpha_i(x_1, \ldots, x_i)$, which is a continuum. So, $\rho_i|_{H_{i+1}}$ is a monotone mapping. $\square$

Observation 4. *If for each $i \geq 1$, G_i is a continuum, then each H_{i+1} is a subcontinuum of G_{i+1}.*

In the setting of inverse limits on $[0,1]$ with set-valued functions whose partial graphs are dendrites, Theorem 3 generalizes Corollary 2.4 in [16].

Theorem 3. *Let $X = \varprojlim\{[0,1], f_i\}$, where for each $i \geq 1$, $f_i\colon [0,1] \to [0,1]$ is a surjective upper semicontinuous set-valued function. Suppose that, for each $i \geq 1$, G_{i+1} is a dendrite, and $G(f_i^{-1})$ contains the graph of an interval-valued function $g_i\colon [0,1] \to [0,1]$. Then X is a tree-like continuum with the fpp.*

Proof. By Observations 3 and 4, for each $i \geq 1$, there exists a subcontinuum H_{i+1} of G_{i+1} such that $\rho_i|_{H_{i+1}}\colon H_{i+1} \to G_i$ is a monotone mapping. For $1 \leq i \leq j-1$, by Proposition 7, the mapping $\rho_i^j\colon G_j \to G_i$ is universal. It follows from Proposition 5 that X has the fpp. Since $X \overset{T}{\approx} \varprojlim\{G_i, \rho_i^{i+1}\}$, it follows that X is tree-like. □

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