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INTERVAL-EXPRESSED TREE-LIKE CONTINUA WITH THE FIXED POINT PROPERTY

M. M. MARSH

Abstract. Let $\mathcal{T}$ be the class of tree-like continua that admit representations as inverse limits on $[0,1]$ with surjective upper semi-continuous set-valued functions. We show (1) if $X \in \mathcal{T}$ with interval-valued bonding functions f_i, where there exists $m \geq 1$ such that, for each $i \geq m$, the graph of f_i^{-1} contains the graph of an interval-valued function, then X has the fixed point property, and (2) if $X \in \mathcal{T}$ with set-valued bonding functions f_i, where for each $i \geq 1$, f_i^{-1} is an interval-valued function, then X is a λ-dendroid. We also provide an example of an indecomposable, non-arclike continuum in $\mathcal{T}$ that has the fixed point property.

1. Introduction

A *compactum* is a compact metric space. A *continuum* is a connected compactum. A continuous function will be referred to as a *map* or *mapping*. A continuum X has the *fixed point property* (fpp) provided that whenever f is a self-mapping on X, there is a point $x \in X$ such that $f(x) = x$. If $\epsilon > 0$, a mapping $f\colon X \to Y$ is an ϵ-*mapping* if $\mathrm{diam}(f^{-1}(y)) < \epsilon$ for each $y \in Y$. A continuum X is *tree-like* (*arclike*) if for each $\epsilon > 0$, there exists an ϵ-mapping of X onto a tree (an arc).

Let $\mathcal{T}$ be the class of tree-like continua that admit representations as inverse limits on $[0,1]$ with surjective upper semi-continuous set-valued

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functions. We refer to continua in $\mathcal{T}$ as *interval-expressed tree-like continua*. This is a large class of continua that includes all arclike continua, as well as non-arclike continua that may contain any arclike continuum, or may themselves be either hereditarily decomposable or indecomposable. We provide an indecomposable example at the end of the paper. The subclass of $\mathcal{T}$ with interval-valued bonding functions has been characterized by properties of the bonding functions in [13]. In this paper, we characterize the subclass of $\mathcal{T}$ with set-valued bonding functions f_i, where each f_i^{-1} is interval-valued. In [15], the author began an investigation considering the following question.

Question 1. *Do all interval-expressed tree-like continua have the fpp?*

Several positive partial answers to Question 1 are given in [15]. We provide several more positive partial results in this paper.

Either a positive or negative answer to Question 1 would be of interest in light of D.P. Bellamy's example of a tree-like continuum without the fpp [1]. A negative answer would be particularly interesting, providing a "generalized" inverse limit representation of an example with simple factor spaces, namely $[0, 1]$. Perhaps such an example could shed some light toward an answer to the open question "Is there a planar tree-like continuum without the fpp?". We refer the reader to [15] for additional discussion and references related to tree-like continua without the fpp.

2. Basic definitions

Let X and Y be compacta. We refer to functions $f\colon X \to 2^Y$ as *set-valued functions* from X to Y and we write $f\colon X \to Y$ is a set-valued function. Note that throughout, we are assuming that, for $x \in X$, the value $f(x)$ of a set-valued function is a closed set. The *graph* of f, which we denote by $G(f)$, is the set in $X \times Y$ consisting of all points (x, y) with $y \in f(x)$.

A set-valued function $f\colon X \to Y$ is *upper semi-continuous at the point* $x \in X$ if for each open set V in Y containing the closed set $f(x)$, there is an open set U in X such that $x \in U$, and $f(p) \subset V$ for each $p \in U$. If $f\colon X \to Y$ is upper semi-continuous at each point of X, then f is said to be *upper semi-continuous*.

The set-valued function $f\colon X \to Y$ is *surjective* if for each $y \in Y$, there exists $x \in X$ such that $y \in f(x)$. If the set-valued function $f\colon X \to Y$ is surjective, we let $f^{-1}\colon Y \to X$ be the set-valued function such that $x \in f^{-1}(y)$ if and only if $y \in f(x)$. Clearly, $G(f^{-1})$ is homeomorphic to $G(f)$. A set-valued function $f\colon X \to Y$ is *continuum-valued* if for each $x \in X$, the set $f(x)$ is a subcontinuum of Y. If $x \in X$ and $f(x)$ is degenerate, we will sometimes treat $f(x)$ as a point of Y. For $f\colon X \to Y$

a set-valued function, and $A \subset X$, we let $f|_A$ be the set-valued function whose domain is A, and $f|_A(x) = f(x)$ for $x \in A$.

For $i \geq 1$, let X_i be a compactum, and let $f_i \colon X_{i+1} \to X_i$ be a surjective upper semi-continuous set-valued function. Throughout, we let $\{X_i, f_i\}_{i\geq 1}$ denote an inverse sequence, and its inverse limit is given by

$$\varprojlim\{X_i, f_i\} = \{\boldsymbol{x} = (x_1, x_2, \ldots) \in \prod_{i\geq 1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } i \geq 1\}.$$

For $n \in \mathbb{N}$, we define the set below.

$$G_{n+1} = G'(f_1, \ldots, f_n) = \{\boldsymbol{x} \in \prod_{i=1}^{n+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}.$$

We point out that some authors use the notation G'_n for the set we denote by G_{n+1}. We find it advantageous in this paper to have the subscript denote the number of coordinates of points in G_{n+1} rather than the number of bonding functions in the finite sequence. We refer to these sets as *partial graphs* in the inverse sequence. For consistency of notation, we let $G_1 = X_1$.

Let $X = \varprojlim\{X_i, f_i\}$ with surjective upper semi-continuous set-valued bonding functions. For $1 \leq i < j$, we denote the set-valued composition function $f_i \circ f_{i+1} \circ \ldots \circ f_j \colon X_{j+1} \to X_i$ by $f_{i,j+1}$. For convenience of notation, we let $f_{i,i}$ denote the identity function on X_i. For $n \geq 1$, we define $\pi_n \colon X \to G_n$ by $\pi_n(x_1, x_2, \ldots) = (x_1, \ldots, x_n)$. Assuming that $\mathrm{diam}(X_n) = 1$ for each $n \geq 1$, and taking the usual metric on $\prod_{i=1}^{\infty} X_i$, we note that for each $n \geq 1$, π_n is a $\frac{1}{2^n}$-mapping.

Let $\{X_i, g_i\}_{i\geq 1}$ be an inverse sequence with mappings for bonding functions. Then we refer to the inverse sequence and its inverse limit, respectively, as an *ordinary inverse sequence* and an *ordinary inverse limit*.

Let $\{X_i, f_i\}_{i\geq 1}$ be an inverse sequence with upper semi-continuous surjective set-valued bonding functions. For $i \geq 1$, define the mapping $\rho_i \colon G_{i+1} \to G_i$ by $\rho_i(x_1 \ldots, x_i, x_{i+1}) = (x_1 \ldots, x_i)$. Note that $\{G_i, \rho_i\}_{i\geq 1}$ is an ordinary inverse sequence. W.T. Ingram showed in [5, Corollary 4.2] that $\varprojlim\{X_i, f_i\}$ is homeomorphic to $\varprojlim\{G_i, \rho_i\}$. This representation of an inverse limit with set-valued functions as an ordinary inverse limit will be used in some of the proofs of our results.

Hereafter, we consider only inverse sequences on $[0,1]$ with set-valued functions. If $H \subset [0,1] \times [0,1]$, we let $H^{-1} = \{(y, x) \mid (x, y) \in H\}$. The notation $X \overset{T}{\approx} Y$ will indicate that X is homeomorphic to Y.

3. k-TAIL SEQUENCES WITH INTERVAL-VALUED INVERSE FUNCTIONS

An *interval* or a *subinterval of* $[0,1]$ is a possibly degenerate subcontinuum of $[0,1]$.

Definition 1. Let $\{[0,1], f_i\}_{i\geq 1}$ be an inverse sequence, where for each $i \geq 1$, $f_i\colon [0,1] \to [0,1]$ is a surjective upper semi-continuous set-valued function. Fix $k \geq 1$. We say that $\{J_i, \hat{f}_i\}_{i\geq k}$ is a *k-tail sequence with interval-valued inverse functions* in the inverse sequence $\{[0,1], f_i\}_{i\geq 1}$ if for each $i \geq k$,

(1) J_i is a subinterval of $[0,1]$,
(2) $G(\hat{f}_i) \subset G(f_i)$, and
(3) $\hat{f}_i^{-1}\colon J_i \to J_{i+1}$ is a surjective upper semi-continuous interval-valued function.

If $J_k = [0,1]$, we say that $\{J_i, \hat{f}_i\}_{i\geq k}$ is a *surjective k-tail sequence with interval-valued inverse functions*.

Definition 2. Let $\{J_i, \hat{f}_i\}_{i\geq k}$ be a k-tail sequence with interval-valued inverse functions in the inverse sequence $\{[0,1], f_i\}_{i\geq 1}$ as in Definition 1. Let $X = \varprojlim\{[0,1], f_i\}$. Let Y be the limit of the inverse sequence

$$f_{1,k}(J_k) \xleftarrow{\hat{f}_1} f_{2,k}(J_k) \xleftarrow{\hat{f}_2} \dots \xleftarrow{\hat{f}_{k-1}} J_k \xleftarrow{\hat{f}_k} J_{k+1} \xleftarrow{\hat{f}_{k+1}} \dots,$$

where for $1 \leq i \leq k-1$, $\hat{f}_i = f_i|_{f_{i+1,k}(J_k)}$. We call Y the *subcompactum of X generated by* $\{J_i, \hat{f}_i\}_{i\geq k}$.

Regarding Definition 2, we observe that if f_i is interval-valued for $1 \leq i < k$, then Y is a continuum. If $k = 1$, it follows from [4, Theorem 2.8] that Y is a continuum. If $k > 1$, it follows from [14, Corollary 3] that Y is a continuum.

It will be helpful to note that if $\{J_i, \hat{f}_i\}_{i\geq k}$ is a surjective k-tail sequence with interval-valued inverse functions, then the inverse sequence associated with the subcompactum Y of X generated by $\{J_i, \hat{f}_i\}_{i\geq k}$ is

$$[0,1] \xleftarrow{f_1} [0,1] \xleftarrow{f_2} \dots \xleftarrow{f_{k-1}} [0,1] \xleftarrow{\hat{f}_k} J_{k+1} \xleftarrow{\hat{f}_{k+1}} \dots,$$

We use the following observation in some of the proofs of our results.

Observation 1. *Let $\{J_i, f_i\}_{i\geq 1}$ be an inverse sequence on intervals with surjective upper semi-continuous set-valued functions, and let $\{G_i, \rho_i\}_{i\geq 1}$ be the ordinary inverse sequence on the partial graphs associated with $\{J_i, f_i\}_{i\geq 1}$ as described in the next-to-last paragraph before this section. Suppose that, for some $n \geq 1$, f_n^{-1} is interval-valued. Then $\rho_n\colon G_{n+1} \to G_n$ is a monotone mapping.*

Proof. It is straightforward to observe that for $(x_1, \ldots, x_n) \in G_n$,

$$\rho_n^{-1}(x_1, \ldots, x_n) = \{(x_1, \ldots, x_n)\} \times f_n^{-1}(x_n).$$

We see that ρ_n is monotone. □

A continuum X is *decomposable* if it is the union of two proper subcontinua. Otherwise, X is *indecomposable*. If each subcontinuum of X is decomposable, then X is *hereditarily decomposable*. A continuum is *hereditarily unicoherent* if the intersection of each pair of its subcontinua is connected. A continuum is a *dendrite* if it is locally connected and hereditarily unicoherent. A continuum is a λ-*dendroid* if it is hereditarily unicoherent and hereditarily decomposable. A mapping $f\colon X \to Y$ of compacta is *monotone* if for each $y \in f(X)$, $f^{-1}(y)$ is a continuum.

Theorem 1. *Let $X = \varprojlim\{[0,1], f_i\}$, where for each $i \geq 1$, f_i is a surjective upper semi-continuous set-valued function, and G_i is a λ-dendroid. Suppose that $\{J_i, \hat{f}_i\}_{i \geq k}$ is a k-tail sequence with interval-valued inverse functions, and for $1 \leq i < k$, f_i is interval-valued. Let Y be the subcontinuum of X generated by $\{J_i, \hat{f}_i\}_{i \geq k}$. Then X is a tree-like continuum, and Y is a λ-dendroid.*

Proof. As noted in the next-to-last paragraph before this section, X is homeomorphic to the ordinary inverse limit $\varprojlim\{G_n, \rho_n\}$. Since, for each $n \geq 1$, G_n is a λ-dendroid, it follows from [3] that, for each $n \geq 1$, G_n is tree-like. Since, for each $n \geq 1$, the projection mapping of $\varprojlim\{G_n, \rho_n\}$ onto G_n is a $\frac{1}{2^n}$-mapping, it follows that X is tree-like. Hence, Y is tree-like, and thus, hereditarily unicoherent. To see that Y is a λ-dendroid, we need to establish that Y is hereditarily decomposable.

For $n \geq 1$, let $\hat{G}_n$ be the n^{th} partial graph for the inverse sequence associated with Y. That is, we let $\hat{G}_1 = f_{1,k}(J_k)$, and for $n \geq 2$, we let $\hat{G}_n = G'(\hat{f}_1, \ldots, \hat{f}_{n-1})$. For each $n \geq 1$, $\hat{G}_n \subset G_n$, and by one of [4, Theorem 2.8] or [14, Corollary 3], $\hat{G}_n$ is a continuum. So, we have that each $\hat{G}_n$ is a λ-dendroid. As observed for X above, Y is homeomorphic to $\varprojlim\{\hat{G}_n, \hat{\rho}_n\}$, where for $n \geq 1$, $\hat{\rho}_n$ is the natural projection map that drops the $(n+1)^{\text{th}}$ coordinate. It follows from Observation 1 that $\hat{\rho}_n$ is a monotone mapping for $n \geq k$.

By a standard theorem in the theory of ordinary inverse limits (see [10, Theorem 2.1.38]), we have that Y is also homeomorphic to $\hat{Y} = \varprojlim\{\hat{G}_n, \hat{\rho}_n, n \geq k\}$. So, $\hat{Y}$ is a tree-like continuum. For $n \geq k$, let $p_n\colon \hat{Y} \to \hat{G}_n$ denote the projection mapping of the ordinary inverse limit $\hat{Y}$ onto the n^{th} factor space. Since for $n \geq k$, $\hat{\rho}_n$ is a monotone mapping, it follows from [10, Theorem 2.1.13] that p_n is a monotone mapping for $n \geq k$. Suppose $\hat{Y}$ contains an indecomposable subcontinuum H. By [11, 6.10], $\hat{p}_k|_H$ is monotone; and by [11, 8.2], $\hat{p}_k(H)$ is an indecomposable subcontinuum of $\hat{G}_k$. Since $\hat{G}_k$ is a λ-dendroid, we have a contradiction. Hence, $\hat{Y}$ is hereditarily decomposable, as is Y. We have that Y is a λ-dendroid. □

Corollary 1. *Let X be an interval-expressed tree-like continuum with interval-valued functions f_i. Suppose $\{J_i, \hat{f}_i\}_{i\geq k}$ is a k-tail sequence with interval-valued inverse functions, and let Y be the subcontinuum of X generated by $\{J_i, \hat{f}_i\}_{i\geq k}$. Then Y is a λ-dendroid.*

Proof. By [13, Corollaries 3 and 4], it follows that for each $n \geq 2$, G_n is a λ-dendroid. So, we may apply Theorem 1 to get that Y is a λ-dendroid. □

4. Main results

Theorem 2. *Let X be an interval-expressed tree-like continuum with interval-valued functions f_i. Suppose for each $n \geq 1$, there exist $k \geq n$, and a surjective k-tail sequence with interval-valued inverse functions $\{J_i, \hat{f}_i\}_{i\geq k}$. Then X has the fpp.*

Proof. Suppose $g\colon X \to X$ is a fixed-point-free mapping, and let $\epsilon > 0$ be chosen so that $d(x, g(x)) \geq \epsilon$ for all $x \in X$. Let $m \geq 1$ be chosen so that $\frac{1}{2^m} < \epsilon$. So, for $n \geq m$, π_n is an ϵ-mapping.

By hypothesis, there exist $k \geq m$, and a surjective k-tail sequence $\{J_i, \hat{f}_i\}_{i\geq k}$ with interval-valued inverse functions. Let Y be the subcontinuum of X generated by $\{J_i, \hat{f}_i\}_{i\geq k}$. By Corollary 1, Y is a λ-dendroid. Recall from the proof of Theorem 1, that by letting $\hat{Y} = \varprojlim\{\hat{G}_n, \hat{\rho}_n, n \geq k\}$, and letting $p_k\colon \hat{Y} \to \hat{G}_k$ denote the associated k^{th} projection mapping, we have that p_k is monotone. Since $\{J_i, \hat{f}_i\}_{i\geq k}$ is a surjective k-tail sequence with interval-valued inverse functions, we have that $J_k = [0, 1]$, and it follows from the definition of Y and its associated inverse sequence, noted in the second paragraph after Definition 2, that $\hat{G}_k = G_k$. We claim that p_k is topologically equivalent to $\pi_k|_Y$. That is, the diagram below commutes, where h is a homeomorphism and id is the identity function.

$$\begin{array}{ccc} \hat{Y} & \xrightarrow{p_k} & \hat{G}_k \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ Y & \xrightarrow{\pi_k|_Y} & G_k \end{array}$$

Define h by $h(((x_1,\ldots,x_k),(x_1,\ldots,x_{k+1}),\ldots)) = (x_1,x_2,\ldots)$. It is clear that h is a homeomorphism. We check that the diagram commutes. Let $y = ((x_1,\ldots,x_k),(x_1,\ldots,x_{k+1}),\ldots)$ be a point in $\hat{Y}$. Then $\pi_k h(y) = \pi_k(x_1,x_2,\ldots) = (x_1,\ldots,x_k)$, and $\mathrm{id}(p_k(y)) = (x_1,\ldots,x_k)$. Hence, p_k is topologically equivalent to $\pi_k|_Y$. Since p_k is monotone, it follows that $p_k^{-1}\colon \hat{G}_k \to \hat{Y}$ is continuum-valued. So, the function $\gamma_k = (\pi_k|_Y)^{-1}$ is also continuum-valued. We also note that, even though γ_k is a continuum-valued function, we have the equality $\pi_k\gamma_k = \mathrm{id}|_{G_k}$.

Define the set-valued function $r_k\colon X \to Y$ by $r_k = \gamma_k\pi_k$. Since π_k is a mapping and γ_k is continuum-valued, we observe that r_k is continuum-valued. Consider the continuum-valued function $r_k g|_Y\colon Y \to Y$. Since Y is a λ-dendroid, it follows from [12] that $r_k g|_Y$ has a fixed point. So, there is a point $x \in Y$ such that $x \in r_k g(x)$. So, $x \in \gamma_k\pi_k g(x)$, and $\pi_k(x) \in \pi_k\gamma_k\pi_k g(x)$. From the last sentence of the previous paragraph, we have that $\pi_k\gamma_k\pi_k g(x) = \pi_k g(x)$, which is a point of G_k. So, in fact, $\pi_k(x) = \pi_k g(x)$. Hence, $d(x, g(x)) < \epsilon$, which is a contradiction. □

Lemma 1. *Let $f\colon [0,1] \to [0,1]$ be a surjective upper semi-continuous interval-valued function such that $G(f)$ is hereditarily unicoherent. If there exists an upper semi-continuous interval-valued function $g\colon [0,1] \to [0,1]$ such that $G(g) \subset G(f^{-1})$, then for each subinterval J_1 of $[0,1]$, there exist a subinterval J_2 of $[0,1]$ and a surjective upper semi-continuous set-valued function $\hat{f}\colon J_2 \to J_1$ such that $G(\hat{f}) \subset G(f)$, and $\hat{f}^{-1}$ is interval-valued.*

Proof. Let c_2 and c_1 be, respectively, the projection maps from $[0,1]\times[0,1]$ onto the domain and range of f. Let $J_2' = c_2(G(g^{-1}))$. If J_1 is degenerate, say $J_1 = \{x_1\}$, we pick $x_2 \in J_2'$ such that $x_2 \in g(x_1)$, and we let $J_2 = \{x_2\}$. We define $\hat{f}\colon J_2 \to J_1$ by $\hat{f}(x_2) = \{x_1\}$. So, the graph of $\hat{f}$ is a point in $G(g^{-1}) \subset G(f)$. Clearly, $\hat{f}^{-1}$ is interval-valued.

Suppose J_1 is nondegenerate. Since $c_1|_{G(g^{-1})}\colon G(g^{-1}) \to [0,1]$ is weakly confluent, there is a subcontinuum H of $G(g^{-1})$ such that $c_1(H) = J_1$. Let $J_2 = c_2(H)$. We note that $J_2 \subset J_2'$. Define $\hat{f}\colon J_2 \to J_1$ to be the surjective set-valued function whose graph is H. Since $H \subset G(g^{-1})$, we get that $G(\hat{f}^{-1}) = H^{-1} \subset G(g) \subset G(f^{-1}) \overset{T}{\approx} G(f)$.

To see that $\hat{f}^{-1}$ is interval-valued, let $t \in J_1$. Now, $g(t)$ is either a point or a nondegenerate interval, and $\hat{f}^{-1}(t) = g(t) \cap H^{-1}$. Since $G(g)$ is hereditarily unicoherent, it follows that $\hat{f}^{-1}(t)$ is either a point or a nondegenerate interval. So, $\hat{f}^{-1}$ is interval-valued. Thus, $\hat{f}\colon J_2 \to J_1$ is the desired set-valued function. □

Theorem 3. *Let X be an interval-expressed tree-like continuum with interval-valued functions f_i. Suppose there exists an $m \geq 1$ such that for $i \geq m$, $G(f_i^{-1})$ contains the graph of an interval-valued function $g_i \colon [0,1] \to [0,1]$. Then X has the fpp.*

Proof. Let $n \geq 1$. Pick $k \geq \max\{n, m\}$. We construct a surjective k-tail sequence with interval-valued inverse functions. Let $J_k = [0,1]$. By Lemma 1, there exist an interval J_{k+1} and a surjective upper semi-continuous set-valued function $\hat{f}_k \colon J_{k+1} \to J_k$ such that $G(\hat{f}_k) \subset G(f_k)$, and $\hat{f}_k^{-1}$ is interval-valued. Again, by Lemma 1, there exist an interval J_{k+2} and a surjective upper semi-continuous interval-valued function $\hat{f}_{k+1} \colon J_{k+2} \to J_{k+1}$ such that $G(\hat{f}_{k+1}) \subset G(f_{k+1})$, and $\hat{f}_{k+1}^{-1}$ is interval-valued. Repeatedly applying Lemma 1, we construct the desired k-tail sequence $\{J_i, \hat{f}_i\}_{i \geq k}$. By Theorem 2, X has the fpp. □

Our third fixed point result is for certain tree-like continua in $\mathcal{T}$ whose bonding functions need not be interval-valued if their inverses are interval-valued. In fact, in this setting, we establish a characterization of the inverse limit space. We first need two definitions and a lemma.

A set-valued function $f \colon [0,1] \to [0,1]$ has a *flat spot* if there exist $s \in [0,1]$, and a nondegenerate interval $J \subset [0,1]$ such that $J \times \{s\} \subset G(f)$. We additionally say that *f has a flat spot at s*. Given an inverse sequence $\{[0,1], f_i\}_{i \geq 1}$ with set-valued functions, a *flat spot at x_i for f_i composes to a nondegenerate value of f_j* in the composition $f_j \circ f_{j+1} \circ \ldots \circ f_i$ if $f_j(x_i)$ is nondegenerate for $j = i-1$, or if there exists a point x_{j+1} in $f_{j+1,i}(x_i)$ such that $f_j(x_{j+1})$ is nondegenerate for $j < i-1$.

Lemma 2. *Let $\{[0,1], f_i\}_{i=1}^{n}$ be a finite inverse sequence, where for each $1 \leq i \leq n$, f_i is a surjective, upper semi-continuous, set-valued function, and f_i^{-1} is interval-valued. The following statements are equivalent.*

(1) *G_{n+1} is a λ-dendroid.*
(2) *for each $1 \leq i \leq n$, $\dim(G(f_i)) = 1$, and no flat spot of f_i composes to a nondegenerate value of f_j for $1 \leq j < i$.*

Proof. Consider the reverse sequence of inverse functions $f_n^{-1}, \ldots, f_1^{-1}$ as defined in [7, Section 5]. As noted there, we may consider $\{[0,1], f_i^{-1}\}_{i=n}^{1}$ to be a finite inverse sequence, although the indexing is reverse ordered. We have that the hypothesis of Corollary 4 in [13] is satisfied for the reverse sequence of inverse functions treated as an inverse sequence. So, $G'(f_n^{-1}, \ldots, f_1^{-1})$ being a λ-dendroid is equivalent to having for each $1 \leq i \leq n$, $\dim(G(f_i^{-1})) = 1$, and f_i^{-1} having no flat spot that composes to a non-degenerate value of f_j^{-1} for $1 \leq i < j \leq n$.

It was noted in [7, top of page 318] that G_{n+1} is homeomorphic to the partial graph $G'(f_n^{-1}, \dots, f_1^{-1})$ of the reverse sequence of inverse functions. We also point out that the statement "f_i has no flat spot that composes to a non-degenerate value of f_j for $1 \leq j < i$", is equivalent to the statement "f_i^{-1} has no flat spot that composes to a non-degenerate value of f_j^{-1} for $1 \leq i < j \leq n$. It follows that (1) is equivalent to (2). □

Theorem 4. *Let* $X = \varprojlim\{[0,1], f_i\}$*, where for each* $i \geq 1$*,* f_i *is a surjective upper semi-continuous set-valued function, and* f_i^{-1} *is an interval-valued function. The following statements are equivalent.*

1. X *is one dimensional.*
2. *For each* $n \geq 1$*,* G_n *is a* λ*-dendroid.*
3. X *is a* λ*-dendroid.*
4. *For each* $i \geq 1$*,* $\dim(G(f_i)) = 1$*, and no flat spot of* f_i *composes to a nondegenerate value of* f_j *for* $j < i$*.*

Proof. (1)⇒(2). By either [2, Theorem 2.3] or [6, Theorem 3.4], X has trivial shape. In [6], Ingram attributes Theorem 3.4 to V. Nall. It follows that X is tree-like. Recall that X is homeomorphic to the ordinary inverse limit $\varprojlim\{G_i, \rho_i\}$ defined in Section 2. As we have seen, for $n \geq 1$, both ρ_n and the projection mapping $p_n\colon \varprojlim\{G_i, \rho_i\} \to G_n$ are monotone mappings. Since X is tree-like, by [11, 7.22], each G_n is tree-like, and thus one dimensional and hereditarily unicoherent. Suppose for some $n \geq 1$, G_n contains an indecomposable subcontinuum H. By [11, 5.1], the composition mapping $\rho_1 \circ \dots \circ \rho_{n-1}\colon G_n \to [0,1]$ is monotone; and as we saw in the proof of Theorem 1, it follows from [11, 6.10 and 8.2] that $\rho_1 \circ \dots \circ \rho_{n-1}(H)$ is an indecomposable subcontinuum of $[0,1]$, which is a contradiction. So, for each $n \geq 1$, G_n is hereditarily decomposable. We have that for each $n \geq 1$, G_n is a λ-dendroid.

(2)⇒(3). Since $\{[0,1], f_i\}_{i\geq 1}$ is a 1-tail sequence with interval-valued inverse functions, and X is the subcontinuum generated by $\{[0,1], f_i\}_{i\geq 1}$, we have by Theorem 1 that X is a λ dendroid.

(3)⇒(1). This implication is obvious.

It follows from Lemma 2 that (2) and (4) are equivalent, and so the proof is complete. □

Corollary 2. *Let* X *be an interval-expressed tree-like continuum with functions* f_i*, where each* f_i^{-1} *is interval-valued. Then* X *is a* λ*-dendroid, and hence has the fpp.*

Proof. Since X is tree-like, X is one dimensional. By Theorem 4, X is a λ-dendroid. It follows from [12] that X has the fpp. □

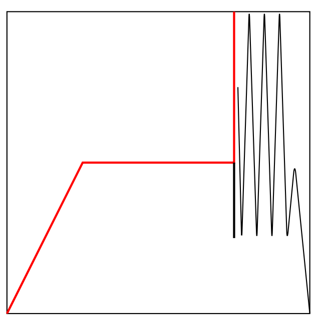
For interval-expressed tree-like continua, the following two corollaries follow from Theorem 4 and the results in the references cited.

Corollary 3. (J.P. Kelly [8, Theorem 4.9]) *Let X be an interval-expressed tree-like continuum with functions f_i, where for each $i \geq 1$, both f_i and f_i^{-1} are interval-valued. Then X is a dendrite.*

Corollary 4. (W.J. Charatonik and F.A. Mena [2, Theorem 2.1]) *Let X be an interval-expressed tree-like continuum with functions f_i, where for each $i \geq 1$, $G(f_i)$ is a dendrite, and f_i^{-1} is interval-valued. Then X is a dendrite.*

5. An example

We provide an example of an indecomposable, non-arclike, tree-like continuum that satisfies Theorem 3. The author is unaware of any other theorems that would show the example has the fpp.



The red arc is the graph of g^{-1}.

Figure 1. The graph of f.

Example. Let f be the surjective upper semi-continuous interval-valued function whose graph is shown in Figure 1. Let $X = \varprojlim\{[0,1], f\}$. Then X is an indecomposable, non-arclike, tree-like continuum having the fpp.

Proof. Note that the red arc in $G(f)$ is the graph of the inverse of an upper semi-continuous interval-valued function $g\colon [0,1] \to [0,1]$. It follows from [13, Corollary 3] that X is tree-like. By Theorem 3, X has the fpp. So, it remains to show that X is indecomposable, and not arclike. To show X is indecomposable involves the notions of indecomposable set-valued functions and inverse sequences that have the full projection property. For definitions and discussions of these terms, as well as theorems that relate them to indecomposability of an inverse limit with set-valued functions, see [4, Sections 3.5 and 3.6], [9, Sections 2 and 3], or [16, Chapter 5].

We will not need a formal definition of f to proceed. The picture of the graph of f in Figure 1 will be sufficient. One should note, nevertheless, we are assuming that $f(\frac{3}{4}) = [\frac{1}{4}, 1]$, and for $t \neq \frac{3}{4}$, $f(t)$ is a singleton. Also, $f([\frac{1}{4}, \frac{3}{4})) = \{\frac{1}{2}\}$, and the graph of $f|_{[\frac{3}{4},1]}$ contains a "topologist's $\sin\frac{1}{x}$-continuum". For $n \geq 1$, let G_n be the partial graph as defined in Section 2, and let $P_n = \{(p_1, \ldots, p_n) \in G_n \mid p_i \neq \frac{3}{4} \text{ for all } 1 \leq i \leq n\}$. As the proof proceeds, we will also need to consider, for $n \geq 2$, the set-valued function $F_{n-1}\colon [0,1] \to G_{n-1}$ where $G(F_{n-1}^{-1}) = G_n$, and the set $T_n = \{t \in [0,1] \mid F_{n-1}(t) \text{ is degenerate}\}$. Note that $0, 1 \in T_n$ for each $n \geq 2$. The remainder of the proof requires a few steps.

Step 1. For $n \geq 1$, P_n is dense in G_n. We use induction on n. Let $n = 1$. We have that $G_1 = [0,1]$, and $P_1 = [0,1] \setminus \{\frac{3}{4}\}$. Clearly, P_1 is dense in G_1.

Assume P_n is dense in G_n for some $n \geq 1$. Let $x = (x_1, \ldots, x_{n+1})$ be in G_{n+1}. Let $U = \prod_{i=1}^{n+1} U_i$, where for each $i \geq 1$, U_i is a connected open set in $[0,1]$ containing x_i, and $U_i \subset (\frac{1}{2}, 1]$ if $x_i \geq \frac{3}{4}$. We assume, without loss of generality, that $x \notin P_{n+1}$. Hence, there exists $1 \leq j \leq n+1$ where $x_j = \frac{3}{4}$. Viewing Figure 1, we see that for $j \leq i \leq n+1$, $x_i \geq \frac{3}{4}$.

Suppose $x_{n+1} = \frac{3}{4}$. Then $\frac{1}{4} \leq x_n \leq 1$. If $x_n < \frac{3}{4}$, then $0 \leq x_i \leq \frac{1}{2}$ for $1 \leq i \leq n-1$, and we have that $x_i \neq \frac{3}{4}$ for $1 \leq i \leq n$. It is clear from the graph of f that there is a sequence of points $\{(s_i, x_n)\}_{i \geq 1}$ in $G(f)$ converging to $(\frac{3}{4}, x_n)$ with $s_i > \frac{3}{4}$ for each $i \geq 1$. So, let $p_{n+1} = s_k$ for some $k \geq 1$ where $s_k \in U_{n+1}$. Then $(x_1, \ldots, x_n, p_{n+1}) \in P_{n+1} \cap U$ as desired. So, assume that $x_n \geq \frac{3}{4}$. By inductive assumption, we pick $(p_1, \ldots, p_n) \in P_n \cap \prod_{i=1}^{n} U_i$. As above, there is a sequence of points $\{(t_i, p_n)\}_{i \geq 1}$ in $G(f)$ converging to $(\frac{3}{4}, p_n)$ with $t_i > \frac{3}{4}$ for each $i \geq 1$. So, we pick $p_{n+1} = t_m$ where $t_m \in U_{n+1}$. Again, we have that $(p_1, \ldots, p_n, p_{n+1}) \in P_{n+1} \cap U$ as desired.

Suppose $x_{n+1} \neq \frac{3}{4}$. From the existence of the integer j above, we have that $\frac{3}{4} < x_{n+1} < 1$, and $\frac{3}{4} \leq x_n < 1$. We note that f is a locally one-to-one open mapping at x_{n+1} since $\frac{3}{4} \leq x_n < 1$. We pick open segments V_n and V_{n+1} such that $f(V_{n+1}) = V_n$, $x_n \in V_n \subset U_n$, and $x_{n+1} \in V_{n+1} \subset U_{n+1} \cap (\frac{3}{4}, 1)$. By inductive assumption, we let $(p_1, \ldots, p_n) \in P_n \cap (U_1 \times \ldots \times U_{n-1} \times V_n)$. Let $p_{n+1} \in V_{n+1}$ where $f(p_{n+1}) = p_n$. We have that $(p_1, \ldots, p_{n+1}) \in P_n \cap (U_1 \times \ldots \times U_{n-1} \times V_n \times V_{n+1}) \subset P_n \cap U$.

Step 2. For $n \geq 2$, $P_n \subset G(F_{n-1}|_{T_n}^{-1})$. Fix $n \geq 2$. If $F_{n-1}(t)$ is nondegenerate for some $t \in [0,1]$, then either $t = \frac{3}{4}$, or for some $1 \leq i \leq n-2$, $f^i(t) = \frac{3}{4}$, where f^i denotes i compositions of f; so, $t \notin \sigma_n(P_n)$, where σ_n is projection of G_n onto the n^{th} coordinate. That is, $[0,1] \setminus T_n \subset [0,1] \setminus \sigma_n(P_n)$. So, $\sigma_n(P_n) \subset T_n$, and it follows that $P_n \subset G(F_{n-1}|_{T_n}^{-1})$.

Step 3. For $n \geq 2$, G_n is irreducible between the points $p = (0, \ldots, 0, 0)$ and $q = (0, \ldots, 0, 1)$. We note that for $t \in T_n \backslash \{0, 1\}$, the point $(F_{n-1}(t), t)$ separates $G(F_{n-1}^{-1}) = G_n$ between the points p and q. Suppose K is a subcontinuum of G_n containing the points p and q. Then K must contain each point $(F_{n-1}(t), t)$ for $t \in T_n$. That is, $G(F_{n-1}|_{T_n}^{-1}) \subset K$. By Step 2, $P_n \subset K$. By Step 1, $\overline{P}_n = G_n$; so, we have that $K = G_n$. Hence, G_n is irreducible between the points p and q.

Step 4. The inverse sequence $\{[0, 1], f\}_{i \geq 1}$ has the full projection property. This follows from Step 3 and [9, Theorem 22].

Step 5. The tree-like continuum X is indecomposable. From Figure 1, we note that f is an indecomposable set-valued function as defined by Kelly and Meddaugh in [9, Definition 14, page 1723]. It follows from this, Step 4, and [9, Theorem 19] that X is indecomposable.

Step 6. To see that X is not arclike, let $L = \{(x_1, x_2, \frac{3}{4}, \frac{3}{4}, \ldots) \mid x_1 \in f|_{[\frac{1}{4}, 1]}(x_2)\}$. It is easy to see that $L \subset X$, $L \overset{T}{\approx} G(f|_{[\frac{1}{4}, 1]})$, and $G(f|_{[\frac{1}{4}, 1]})$ contains a simple triod. It follows that X is not arclike.

□

References

[1] D.P. Bellamy, *A tree-like continnum without the fixed-point property*, Houston J. Math. 6(1979), 1–14.

[2] W.J. Charatonik and F.A. Mena, *Local connectedness of inverse limits*, Topology and Its Appl. 265(2019), 1–12.

[3] H. Cook, *Tree-likeness of dendroids and λ-dendroids*, Fund. Math. 68(1970), #1, 19–22.

[4] W.T. Ingram, An Introduction to Inverse Limits with set-valued Functions, Springer Briefs in Mathematics, New York, 2012.

[5] W.T. Ingram, *Inverse limits of families of set-valued functions*, Bol. Soc. Mat. Mex. (3) 21(2015), #1, 53–70.

[6] W.T. Ingram, *Inverse limits with set-valued functions having graphs that are arcs*, preprint

[7] W.T. Ingram and M.M. Marsh, *Chainability of inverse limits on* $[0, 1]$ *with interval-valued functions*, Topology Proc. 56(2020), 305–320.

[8] J.P. Kelly, *Monotone and weakly confluent set-valued functions and their inverse limits*, Topology and Its Appl. 228(2017), 486–500.

[9] J.P. Kelly and J. Meddaugh, *Indecomposability in inverse limits with set-valued functions*, Topology and Its Appl. 160(2013), 1720–1731.

[10] S. Macías, Topics on Continua, Chapman and Hall/CRC, Boca Raton, FL, 2005.

[11] T. Maćkowiak, *Continuous mappings on continua*, Dissertationes Mathematicae CLVIII(1979), 1–90.

[12] R. Mańka, *Association and fixed points*, Fund. Math. 91(1976), 105–121.

[13] M.M. Marsh, *A characterization of tree-like inverse limits on* $[0, 1]$ *with interval-valued functions*, Topology Proc. 50(2017), 101–109.

[14] M.M. Marsh, *Connectedness of inverse limits with functions f_i where either f_i or f_i^{-1} is a union of continuum-valued functions*, Topology and Its Appl. 264(2019),473–488.

[15] M.M. Marsh, *Some fixed point theorems for tree-like continua*, submitted for publication

[16] S. Varagona, *Simple techniques for detecting decomposability or indecomposability of generalized inverse limits*, Thesis (Ph.D.)-Auburn Univresity. 2012. 80pp.

Department of Mathematics & Statistics, California State University, Sacramento, Sacramento,CA 95819-6051

E-mail address: mmarsh@csus.edu