Continuum Fixed Point Theory

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- A mapping $f : X \to Y$ is an *ϵ-map* if for each $y \in Y$, the diameter of $f^{-1}(y)$ is less than ϵ.
- A continuum X has the *fixed point property (fpp)* if each mapping $f : X \to X$ has a fixed point.

Representations of continua

Let $\mathcal{P}$ be the class of finite connected polyhedra. Let $\mathcal{G} \subseteq \mathcal{P}$.

- A continuum X is *$\mathcal{G}$-like* if for each $\epsilon > 0$, there exist a member Y of $\mathcal{G}$ and a surjective ϵ-map $g \colon X \to Y$.

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Both inverse limit descriptions and open ϵ-coverings of continua have frequently been used to obtain fixed point results.

We will primarily be interested in continua that are *arc-like, tree-like*, or *$\mathcal{G}$-like* for a class of polyhedra with the fpp. We discuss the fpp as related to continua in these classes. Arc-like continua have also been called *snake-like continua* and *chainable continua*.

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Theorem (O.H. Hamilton, 1951)

Arc-like continua have the fpp.

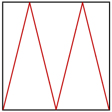
Proof.

Suppose that X is an arc-like continuum and that $f : X \to X$ is a fixed-point-free mapping. Let $\epsilon > 0$ be chosen so that $d(x, f(x)) \geq \epsilon$ for all $x \in X$. Let $g : X \to [0, 1]$ be an ϵ-map. Each map of a continuum onto $[0, 1]$ is universal; so g is universal. Hence, there exists a point $x \in X$ such that $g(x) = g(f(x))$. But then $d(x, f(x)) < \epsilon$, which is a contradiction. □

A continuum X is an *arc-continuum* if each proper subcontinuum of X is an arc. A continuum is *indecomposable* if it cannot be written as the union of two proper subcontinua. Otherwise, it is *decomposable*.

We let $\mathcal{K}$ be the class of *n-adic Knaster continua*. That is, $X \in \mathcal{K}$ if there is an $n \geq 2$ such that X is an inverse limit on $[0, 1]$ with a single open bonding map whose graph has n "ups or downs".

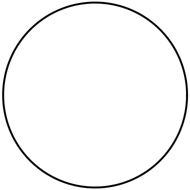
Members of $\mathcal{K}$ are indecomposable, arc-like, arc-continua.

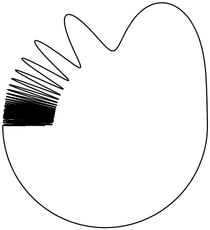


Circle-like continua

If X is circle-like and all bonding maps are inessential (homotopic to a constant map), then X is arc-like and will have the fpp.

If X is circle-like and all bonding maps are essential, X may or may not have the fpp.





Theorem (J. Mioduszewski and M. Rochowski, 1962)

Each inverse limit on polyhedra with universal bonding maps has the fpp.

Theorem (W. Holsztynski, 1967)

Same result for inverse limits of ANRs.

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Same result for inverse limits of ANRs.

So, one approach to get fixed point theorems on polyhedra-like continua would be to determine necessary conditions for a map between particular polyhedra to be universal. To use this approach, the polyhedra must have the fpp.

I will refer to this approach as the *MRH-method*.

Theorem (R.L. Russo (1979), C.L. Hagopian (2003))

If X is a tree-like continuum and K is a locally connected continuum that contains an n-cell ($n \geq 2$), then X is K-like.

As we will see, this means that there will be $\mathcal{G}$-like continua without the fpp, whenever $\mathcal{G}$ contains polyhedra of dimension ≥ 2, even if the polyhedra in $\mathcal{G}$ have the fpp. Nevertheless, additional conditions on the continua can result in fixed point theorems.

Disk-like continua

A mapping $f\colon X \to Y$ is *confluent (weakly confluent)* if for each subcontinuum K of Y, each (there exists a) component H of $f^{-1}(K)$ is mapped onto K by f.

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Theorem (S. Nadler, 1980)

If X is a disk-like continuum with weakly confluent bonding maps, then X has the fixed point property.

.... uses the MRH-method.

Question

If X is n-cell-like, for $n \geq 3$, with weakly confluent bonding maps, does X have the fixed point property?

Even-dimensional projective space-like continua

Theorem (J. Segal and T. Watanabe, 1992)

If X is an inverse limit on an even-dimensional complex projective space with essential bonding maps, then X has the fpp.

Theorem (Marsh, 2010)

If X is an inverse limit on an even-dimensional real (or quaternionic) projective space and the bonding maps have non-zero degree, then X has the fpp.

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Corollary (partial answer to a question of Bellamy)

Each projective plane-like continuum with essential bonding maps that lift to essential bonding maps on the 2-sphere has the fpp?

The question for projective plane-like continua whose bonding maps are essential, but have inessential lifts to the 2-sphere is of interest. Lifts of representatives of this homotopy class of maps on the projective plane have similarities to the squaring map on the unit circle; they also produce inverse limits with the fpp.

Cellular continua

A subcontinuum X of Euclidean n-space is *cellular* if X is the intersection of a decreasing sequence of topological n-cells. Cellular continua are important in decomposition theory of manifolds, but this characteristic of some continua has not proven to be particularly useful in continuum fixed point theory.

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For example, each non-separating planar continuum is an intersection of topological disks, but the well-known question below remains unanswered.

Question (W.L. Ayres (1930), Scottish Book (1935))

Does each non-separating planar continuum have the fpp?

Cellular continua

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Question (W.L. Ayres (1930), Scottish Book (1935))

Does each non-separating planar continuum have the fpp?

Additionally, there are several well-known examples of cellular (with 3-cells) continua without the fpp. (K. Borsuk, R.H. Bing, S. Kinoshita, R. Knill, A. Illanes)

Tree-like continua

Theorem (O.H. Hamilton, 1938)

If X is a tree-like continuum and contains no indecomposable subcontinuum, then X has the fpp for homeomorphisms.

Theorem (K. Borsuk, 1954)

Arc-wise connected tree-like continua have the fpp.

Bing called the following question one of the most interesting unsolved problems in geometric topology.

Question (R.H. Bing, 1969)

Do all tree-like continua have the fpp?

Since 1-dimensional, non-separating planar continua are tree-like (Bing), and planar tree-like continua are non-separating, this question is related to the non-separating planar question.

Theorem (J.H. Case and R.E. Chamberlin, 1960)

If a 1-dimensional continuum admits no essential mapping into a graph, then it is tree-like.

In 1975, J. Krasinkiewicz improved this result by replacing "graph" with "the figure eight".

A *dendroid* is an arc-wise connected, hereditarily unicoherent continuum. A *dendrite* is a locally connected dendroid. A *λ-dendroid* is a hereditarily decomposable, hereditarily unicoherent continuum.

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Theorem (H. Cook, 1970)

All dendroids, λ-dendroids, and hereditarily equivalent continua are tree-like.

Theorem (R. Manka, 1976)

λ-dendroids have the fpp.

Corollary (R. Manka, 1976)

Each tree-like continuum without the fpp must contain an indecomposable subcontinuum.

Lefschetz theory applied to non-ANRs

- *quasi-complexes*, defined by S. Lefschetz (1942)
- *weak semi-complexes*, defined by R.B. Thompson (1967)
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These definitions are related to covers and nerves of covers on the spaces. They ensure enough structure on the spaces so that the Lefschetz theorem applies. That is, if a mapping $f : X \to X$ on a space X admitting such a structure has a non-zero Lefschetz number, then f has a fixed point. All of these definitions (although different in general) coincide on the class of tree-like continua. In 1959, Chamberlin showed that not all tree-like continua are quasi-complexes.

Theorem (R.B. Thompson, 1967)

If a tree-like continuum admits a weak semi-complex structure, then it has the fpp.

In 1975, Borsuk defined conditions on a mapping, called *nearly extendable*, so that the Lefschetz theorem holds for these mappings. His result was later generalized by J. Dugundji (1977), by G. Gauthier (1980), and by J. Girolo (1988). Girolo's theory defined Q-simplicial maps, and defined a space X as Q-simplicial if the identity map on X is Q-simplicial. His theory unified and generalized many of the theories above, which allow a Lefschetz theorem.

Theorem (J.B. Fugate and L. Mohler, 1977)

If there is a tree-like continuum without the fpp, then there is an indecomposable tree-like continuum that admits a fixed-point-free homeomorphism.

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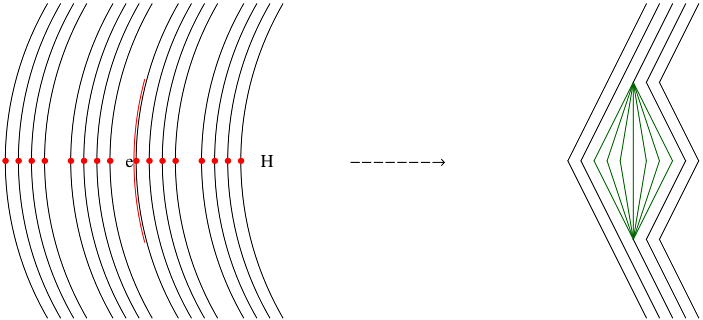
Example (D. Bellamy, 1979)

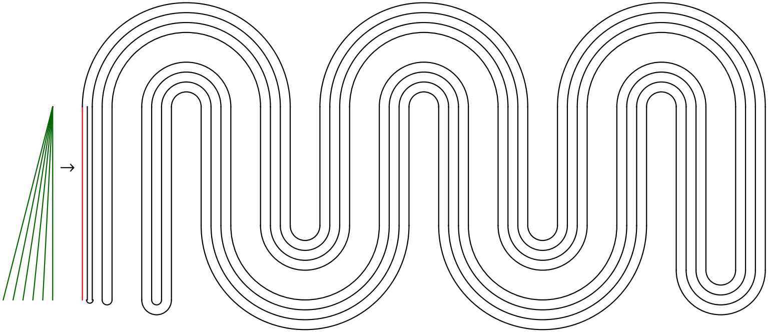
A tree-like continuum without the fpp.

Bellamy noted that by applying the Fugate-Mohler technique to his example, one gets an indecomposable tree-like arc-continuum admitting a fixed-point-free homeomorphism.

Bellamy's pioneering example



Modified 6-adic solenoid



Modified 6-adic Knaster continuum

Some properties of Bellamy's example are

- there are endpoints of period 2 and of period 3 under the fixed-point-free map,
- no points map "into" the period 2 orbit
- the fixed-point-free map is not a deformation
- not embeddable in the plane

- Do fan-like (or star-like) continua have the fpp? (asked by L.G. Oversteegen and J.T. Rogers, Jr.)

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Some more questions ....

Let f be a map on a tree-like continuum X.

- Must f have a periodic point? No, P. Minc
- If f has orbits of "sufficiently small" diameter, must f have a fixed point? No, P. Minc
- If f is a deformation, does f have a fixed point? Yes, C.L. Hagopian

Oversteegen-Rogers examples

In 1980, Oversteegen and Rogers, gave an inverse limit description of a tree-like arc-continuum admitting an "induced" fixed-point-free map. In this example, using the geometric nature of its description, they show that complements of open sets are embeddable in the plane. To obtain a somewhat simple description of the bonding maps, they have rather complicated factor spaces, in which the number of branchpoints, simple closed curves, and "Hawaiian earrings" increases in the inverse sequence.

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In 1982, Oversteegen and Rogers, defined two more examples using inverse limits. These examples have similar properties to the previous example, but additionally the fixed-point-free map is a homeomorphism, and the factor spaces are trees.

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In 1982, Oversteegen and Rogers, defined two more examples using inverse limits. These examples have similar properties to the previous example, but additionally the fixed-point-free map is a homeomorphism, and the factor spaces are trees.
Again, the number of branchpoints in the factor spaces increases in the inverse sequence. The nature of the "symmetry" in the factor spaces and the "folding" of the bonding maps is reminiscent of Bellamy's example.

Minc examples

For each positive integer j, Minc alters an n-adic Knaster continuum, where $n = 2(4^1 - 1)(4^2 - 1)\cdots(4^j - 1)$, by replacing an arc containing the endpoint with a fan over a 0-dimensional set. The resulting indecomposable tree-like continuum B_j admits a map with no periodic points of period less than or equal j. The continuum B_1 is quite similar to Bellamy's original example as it is a modified 6-adic Knaster continuum.

Applying the Fugate-Mohler technique, Minc defines tree-like arc-continua $\tilde{B}_j$ that admit homeomorphisms with analogous properties.

Minc uses the sequence $\{B_j\}$ and the sequence $\{\tilde{B}_j\}$ to construct a number of remarkable examples. Namely,

- (1992) A tree-like continuum X such that for each $\epsilon > 0$ and each $j \geq 1$, there exists a map $f\colon X \to X$ such that f has no periodic points of period $\leq j$, and the diameters of all trajectories (forward and backward orbits) of points are less than ϵ.

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- (1996) A periodic-point-free homeomorphism on a tree-like continuum. (a particularly sophisticated and technical construction)
- (1999) A map of a tree-like continuum with no invariant indecomposable subcontinuum.
- (1999) A weakly chainable tree-like continuum without the fpp. And a hereditarily indecomposable tree-like continuum without the fpp.

Some more positive results on tree-like continua

In conversation with Hagopian, Bing asked several fixed-point questions related to inverse limits of n-ods where the bonding maps “fix“ vertices, or “fix” some of the edges.
One of Bing's questions is answered below.

- Let F be a fan with one isolated edge L. If X is an inverse limit on F with bonding maps leaving edges other than L invariant, then X has the fpp. (Marsh, 1984)

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- Inverse limits on n-ods with edges $L_1, L_2, \ldots, L_n$, where the bonding maps satisfy $f(L_i) \subset \cup_{j=1}^{i} L_j$, then X has the fpp. (Marsh, 1986)

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- Let T be a tree. If X is an inverse limit on T with bonding maps satisfying a condition that is of a similar nature as the fan result, then X has the fpp. (Marsh, 1989)

Theorem (C.A. Eberhart and J.B. Fugate, 1981)

Inverse limits on trees with weakly arc-preserving bonding maps have the fpp.

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Theorem (Marsh, 1992)

If X is an inverse limit on trees with u-mappings for bonding maps, then X has the fpp.

....generalizes the Eberhart-Fugate result by allowing the bonding maps to have a “restricted” amount of folding across branchpoints.
.... uses the MRH-method.

In 1998, Eberhart and Fugate characterized universal maps between trees.

The following two theorems follow from the previous Marsh/Eberhart-Fugate theorems.

Theorem

If X is an inverse limit on a single tree with weakly confluent bonding maps, then X has the fpp.

Theorem

If X is an inverse limit on trees with confluent bonding maps, then X has the fpp.

The second result also follows from a result of H. Schirmer in 1967, and was again established independently by J.J. Charatonik and J. Prajs in 2005.

Theorem (Fugate and McLean, 1981)

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Theorem (Oversteegen and Tymchatyn, 1981)

Every point-wise periodic homeomorphism on an atriodic tree-like continuum has a fixed point.

Theorem (Hagopian, 1998)

Arc-component-preserving maps on tree-like continua have fixed points.

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Composant-preserving maps of indecomposable k-junctioned tree-like continua have fixed points.

Example (Hagopian, Marsh, and Prajs, 2012)

There exists a fixed-point-free homeomorphism on an indecomposable tree-like continuum that leaves all composants invariant.

Joint work with Hagopian

Let X and Y be tree-like continua with the fpp.

Question

If $X \cap Y$ has the fpp, does $X \cup Y$ have the fpp?

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No, in general. Yes, if $X \cap Y$ is a dendrite.

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Example

There exists a countably infinite wedge of tree-like continua $\{X_n\}$, each having the fpp, such that $X = \cup_{n \geq 1} X_n$ is tree-like and does not have the fpp.

A few open questions

1. Do simple triod-like continua (arc-continua) have the fpp?
2. Do weakly chainable arc-continua have the fpp?
3. Must each map of a tree-like arc-continuum have a periodic point?
4. Must each pointwise periodic homeomorphism on a tree-like continuum have a fixed point?
5. Must every composant-preserving map of an indecomposable (tree-like) planar continuum have a fixed point?
6. Do homogeneous tree-like continua have the fpp?