Atriodic tree-like continua as inverse limits on $[0, 1]$ with interval-valued functions

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Abstract

For an inverse sequence on $[0, 1]$ with interval-valued functions, it is shown that chainability of the graphs of the bonding functions, and chainability of the partial graphs associated with the inverse sequence are necessary conditions for an atriodic tree-like inverse limit. An analogous result is proved for inverse sequences with functions f_i where each f_i^{-1} is interval-valued. It follows that non-chainable, atriodic tree-like continua cannot be realized as inverse limits on $[0, 1]$ with functions from either of these subclasses of set-valued functions.

Keywords: triod, inverse limit, atriodic, tree-like, chainable, interval-valued function
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1. Introduction

In the setting of inverse limits on $[0, 1]$ with interval-valued functions $\{f_i\}_{i\geq 1}$, or with functions $\{f_i\}_{i\geq 1}$ where each f_i^{-1} is interval-valued, we prove that chainability of the graphs $G(f_i)$, and chainability of the partial graphs G_1^{i+1} are necessary conditions to have an atriodic tree-like inverse limit. So, indeed, chainability of the graphs $G(f_i)$ and the partial graphs G_1^{i+1} are necessary conditions to have a chainable inverse limit. For such inverse limits, these results provide a partial answer to Problem 4, and an answer to Problem 6 of W.T. Ingram in [10]. In [18, Corollary 3] and [19, Theorem 4], the

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author characterizes tree-likeness of inverse limits in these settings. In [12, Corollaries 4.2 and 5.4], Ingram and the author provide sufficient conditions to have a chainable inverse limit in these settings. One of the conditions in [12], that each function f_i be C-set-valued, implies that each $G(f_i)$ is chainable. However, the two properties are not equivalent, and there are simple, well-known examples that illustrate the non-necessity of the C-set-valued assumption, see [6, Example 2.2] and [7, Example 5.1]. The results in this paper show how triods in the graphs $G(f_i)$ give rise to triods in the partial graphs associated with the inverse sequence, and these triods, in turn, produce triods in the inverse limit space. Unfortunately, triods can arise in the partial graphs and in the inverse limit even when each $G(f_i)$ is an arc, see [6, Example 2.13] and [10, Example 3]. Hence, in order to have a characterization of chainable inverse limits in this setting, we need a property to add to chainability of the graphs $G(f_i)$ that is a bit weaker than C-set-valued.

A *compactum* is a nonempty compact metric space. All spaces considered in this paper are compacta. A *continuum* is a connected compactum. A continuous function will be referred to as a *mapping*. A continuum X is *chainable* if for each $\epsilon > 0$, X admits a finite ϵ-chain of open sets covering X. A continuum X is *arclike,* (*tree-like*) if for each $\epsilon > 0$, X admits an ϵ-mapping onto $[0, 1]$ (a tree). It is well-known that for a continuum X, the following are equivalent.

(i) X is chainable.

(ii) X is arclike.

(iii) X is representable as an inverse limit of an inverse sequence on $[0, 1]$ with mappings for bonding functions.

See pages 3 and 4 at the end of Section 2 in [7] for specific definitions and discussion of these equivalences.

In the study of inverse limits on $[0, 1]$ with set-valued functions, the situation is quite different. In fact, the diversity of inverse limit spaces in this setting is extreme. Hence, attempts to understand what types of compacta or continua can be realized as such inverse limits have typically involved placing additional properties on the bonding functions. Some properties of the set-valued functions $\{f_i\}_{i\geq 1}$ and their graphs that have proven fruitful are the following. The first three are considered in references already mentioned. References are noted for the last three.

(1) Each f_i is interval-valued.

(2) Each f_i^{-1} is interval-valued.

(3) Each $G(f_i)$ is one-dimensional, and no flat spot of f_i composes to a nondegenerate value of f_j for $1 \leq j < i$.

(4) Each $G(f_i)$ is an arc, see [2, 3, 10].

(5) Each $f_i = f$ for a given f, see [13, 14, 15, 22, 23].

(6) Each f_i is a specific type of function (e.g., irreducible, sinusoid, Markov, C-set-valued, or N-type), see [1, 9, 12, 14, 24].

The properties in items (1) and (4) consider two subclasses of inverse limits on $[0, 1]$ with set-valued functions that may be considered "most like" ordinary inverse limits on $[0, 1]$ in the sense that the bonding functions f_i share a property that mappings have. In these two subclasses, perhaps there is enough structure to understand the nature of the inverse limits. The results in this paper show that atriodic tree-like continua in the two subclasses whose bonding functions satisfy items (1) or (2) must be chainable. Also, some progress is made toward a characterization of chainable inverse limits in terms of additional properties of the bonding functions in these two subclasses.

2. Basic definitions

Let X and Y be compacta. We refer to functions $f: X \to 2^Y$ as *set-valued functions* from X to Y and we write $f: X \to Y$ is a set-valued function. Note that throughout, we are assuming that, for $x \in X$, the value $f(x)$ of a set-valued function is a closed set. The *graph* of f, which we denote by $G(f)$, is the set in $X \times Y$ consisting of all points (x, y) with $y \in f(x)$.

A set-valued function $f: X \to Y$ is *upper semi-continuous at the point* $x \in X$ if for each open set V in Y containing the closed set $f(x)$, there is an open set U in X such that $x \in U$, and $f(p) \subset V$ for each $p \in U$. If $f: X \to Y$ is upper semi-continuous at each point of X, then f is said to be *upper semi-continuous*. Hereafter, all set-valued functions considered will be upper semi-continuous.

The set-valued function $f: X \to Y$ is *surjective* if for each $y \in Y$, there exists $x \in X$ such that $y \in f(x)$. If the set-valued function $f: X \to Y$ is surjective, we let $f^{-1}: Y \to X$ be the set-valued function such that $x \in f^{-1}(y)$

if and only if $y \in f(x)$. Clearly, $G(f^{-1})$ is homeomorphic to $G(f)$. A set-valued function $f\colon X \to Y$ is *continuum-valued* if for each $x \in X$, the set $f(x)$ is a subcontinuum of Y. If $x \in X$ and $f(x)$ is degenerate, we will sometimes treat $f(x)$ as a point of Y. For $f\colon X \to Y$ a set-valued function, and $A \subset X$, we let $f|_A$ be the set-valued function whose domain is A, and $f|_A(x) = f(x)$ for $x \in A$.

For $i \geq 1$, let X_i be a compactum, and let $f_i\colon X_{i+1} \to X_i$ be a surjective, set-valued function. Throughout, we let $\{X_i, f_i\}_{i\geq 1}$ denote an inverse sequence, and its inverse limit is given by

$$\varprojlim\{X_i, f_i\} = \{x = (x_1, x_2, \ldots) \in \prod_{i\geq 1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } i \geq 1\}.$$

For $n \in \mathbb{N}$, we define the set below.

$$G_1^{n+1} = G'(f_1, \ldots, f_n) = \{x \in \prod_{i=1}^{n+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}.$$

We point out that some authors use the notation G'_n for the set we denote by G_1^{n+1}. We refer to these sets as *partial graphs* in the inverse sequence. For consistency of notation, we let $G_1^1 = X_1$.

A set-valued function $f\colon X \to Y$ has a *flat spot at p* if $p \in Y$ and there exists a nondegenerate continuum $X' \subset X$ such that $X' \times \{p\} \subset G(f)$. Let $X = \varprojlim\{X_i, f_i\}$ with surjective, set-valued bonding functions. For $1 < i < j$, *a flat spot at x_j for f_j composes to a nondegenerate value of f_i* in the composition $f_i \circ f_{i+1} \circ \ldots \circ f_j$ if $f_i(x_j)$ is nondegenerate for $i = j-1$, or if there exists a point x_{i+1} in $f_{i+1} \circ \ldots \circ f_{j-1}(x_j)$ such that $f_i(x_{i+1})$ is nondegenerate for $i < j-1$.

For $n \geq 1$, we define the set-valued function $F_n\colon X_{n+1} \to G_1^n$, where $(x_1, \ldots, x_n)$ is in $F_n(t)$ if and only if $(x_1, x_2, \ldots, x_n, t)$ is in G_1^{n+1}. So, $G_1^{n+1} = G(F_n^{-1})$. V. Nall introduced this function in [21], and showed that F_n is upper semi-continuous. If f_i is continuum-valued for each $1 \leq i \leq n$, the author showed in [17] that F_n is continuum-valued.

For $n \geq 1$, we define $\hat{\pi}_n\colon X \to G_1^n$ by $\hat{\pi}_n(x_1, x_2, \ldots) = (x_1, \ldots, x_n)$. Assuming that $\mathrm{diam}(X_i) = 1$ for each $i \geq 1$, and taking the usual metric on $\prod_{i=1}^{\infty} X_i$, we note that for each $n \geq 1$, $\hat{\pi}_n$ is a $\frac{1}{2^n}$-mapping. The notation $X \overset{T}{\approx} Y$ will indicate that X is homeomorphic to Y.

A continuum X is *irreducible* if there exist points p and q in X such that no proper subcontinuum of X contains both p and q. In this case, we say that X is *irreducible between the points p and q*. A continuum X is *decomposable* if it is the union of two proper subcontinua. Otherwise, X is *indecomposable*. If each subcontinuum of X is decomposable, then X is *hereditarily decomposable*. A continuum is *hereditarily unicoherent* if the intersection of each pair of its subcontinua is connected. A continuum is a λ-*dendroid* if it is hereditarily unicoherent and hereditarily decomposable.

3. Triods in continuum folders

For convenience to the reader, we repeat a few definitions from [5]. Let X and Y be continua. A mapping $f\colon X \to Y$ is *monotone* if, for each $y \in Y$, $f^{-1}(y)$ is connected. Consider the class $\mathcal{M}$ of continua that admit a monotone mapping onto $[0,1]$. If $X \in \mathcal{M}$ and $\eta\colon X \to [0,1]$ is surjective and monotone, we refer to each $\eta^{-1}(t)$, for $t \in [0,1]$, as a *fiber* of X. This structure gives rise to an upper semi-continuous decomposition of X, where the fibers are elements of the decomposition.

We refer to a member X of $\mathcal{M}$ together with a monotone surjective map $\eta\colon X \to [0,1]$ as a *continuum folder*. So, a continuum folder is a pair (X, η), but we will typically refer to X as a continuum folder, with an assumed monotone map $\eta\colon X \to [0,1]$. Let $\mathcal{G}$ be a class of continua. If each fiber of X is either a point or belongs to $\mathcal{G}$, we call X a $\mathcal{G}$ *folder* or a *folder of continua from* $\mathcal{G}$.

For M a continuum, an $\{M\}$ folder, which we denote simply by M *folder*, is a continuum folder where each fiber is either a point or is homeomorphic to M.

Basic properties of continuum folders and arc folders, as well as examples, can be found in [5]. We note that if a continuum folder X is irreducible, then it is irreducible between points $p \in \eta^{-1}(0)$ and $q \in \eta^{-1}(1)$.

If X is a compactum, and $A \subset X$, we let $\mathrm{cl}(A)$ denote the closure of A in X. Two subsets A and B of X are *mutually separated* in X if $\mathrm{cl}(A) \cap B = \emptyset = \mathrm{cl}(B) \cap A$. A continuum T is a *triod* if there exists a subcontinuum K of T such that $T \setminus K$ is the union of three nonempty sets, each two of which are mutually separated in T. A unicoherent continuum T is a triod if there exist three subcontinua J_1, J_2, and J_3 of T such that $T = J_1 \cup J_2 \cup J_3$, $J_1 \cap J_2 \cap J_3 \neq \emptyset$, and for each $i \in \{1,2,3\}$, $J_i \setminus (J_j \cup J_k) \neq \emptyset$ for $\{i,j,k\} = \{1,2,3\}$, see [20, pages 208-209]. Since, throughout, we will be working in hereditarily unicoherent continua, we will say that $T = J_1 \cup J_2 \cup J_3$ is a triod, meaning that the conditions above are satisfied. A continuum is *atriodic* if it contains no triod.

An *interval* or a *subinterval* of $[0,1]$ is a possibly degenerate subcontinuum of $[0,1]$. Given an interval $[u,v]$, unless specified, we do not assume $u \leq v$. Let (M,η) be a folder of continua. We say that M contains a *one-sided triod* (at u) if there exists a nondegenerate interval $[u,v] \subset [0,1]$, and continua A, B, and C such that $A \cup B \subset \eta^{-1}(u)$, $\eta(C) = [u,v]$, and $T = A \cup B \cup C$ is a triod. We also define a one-sided triod (at v) to be one defined as above with A and B lying in $\eta^{-1}(v)$. We say that M contains a *two-sided triod* (at t) if there exist a nondegenerate interval $[u,v] \subset [0,1]$, $u < t < v$, and subcontinua A and B of M such that $\eta(A) = [u,t]$, $\eta(B) = [t,v]$, and $T = A \cup B \cup \eta^{-1}(t)$ is a triod. We say that M contains a *three-fibered triod* if there exist a nondegenerate interval $[u,v] \subset [0,1]$, $u < t < v$, and a subcontinuum K of M such that $\eta(K) = [u,v]$, and $(\eta^{-1}(u) \cup K) \cup (\eta^{-1}(t) \cup K) \cup (\eta^{-1}(v) \cup K)$ is a triod.

Lemma 1. *Suppose that (M,η) is a hereditarily unicoherent folder of continua. The following statements are equivalent.*

(1) M contains no two-sided triod.

(2) The continuum $\mathrm{cl}(\eta^{-1}(u,v))$ is irreducible between $\eta^{-1}(u)$ and $\eta^{-1}(v)$ for each nondegenerate interval $[u,v] \subset [0,1]$.

(3) The continuum $\mathrm{cl}(\eta^{-1}(0,1))$ is irreducible between $\eta^{-1}(0)$ and $\eta^{-1}(1)$.

Proof. (1) $\Rightarrow$ (2): We prove the contrapositive statement. Suppose $0 \leq u < v \leq 1$, N is a subcontinuum of M such that $\eta(N) = [u,v]$, and N is a proper subcontinuum of $\mathrm{cl}(\eta^{-1}(u,v))$. It follows that there exists $x \in \eta^{-1}(t) \setminus N$ for some $u < t < v$. Since M is hereditarily unicoherent, $N \cap \eta^{-1}(s)$ is a continuum for each $s \in [u,v]$. Let $A = \eta|_N^{-1}([u,t])$ and $B = \eta|_N^{-1}([t,v]$. It is straightforward to check that $A \cup B \cup \eta^{-1}(t)$ is a two-sided triod in M.

(2) $\Rightarrow$ (3): This implication is immediate.

(3) $\Rightarrow$ (1): We prove the contrapositive statement. Suppose M contains a two-sided triod T; say $T = A \cup B \cup \eta^{-1}(t)$, where $\eta(A) = [u,t]$, $\eta(B) = [t,v]$, and $u < t < v$. Let $A' = \mathrm{cl}(\eta^{-1}(0,u]) \cup A$ and $B' = B \cup \mathrm{cl}(\eta^{-1}[v,1))$. We have that $A' \cup B'$ is a proper subcontinuum of $\mathrm{cl}(\eta^{-1}(0,1))$ that meets $\eta^{-1}(0)$ and $\eta^{-1}(1)$. So, $\mathrm{cl}(\eta^{-1}(0,1))$ is not irreducible between $\eta^{-1}(0)$ and $\eta^{-1}(1)$. □

Theorem 1. *Suppose that (M, η) is a hereditarily unicoherent folder of atriodic continua that contains a triod. Then M contains either a one-sided triod or a two-sided triod.*

Proof. Suppose M contains no two-sided triod. Let T be a triod in M. Since M is hereditarily unicoherent, T is unicoherent. Let $T = J_1 \cup J_2 \cup J_3$ as described for unicoherent continua. Since the fibers of M are atriodic, $\eta(T) = [u, v]$ is a non-degenerate interval.

Suppose $\eta(J_i) \neq [u, v]$ for $i \in \{1, 2, 3\}$. Assume, without loss of generality, that $u \in \eta(J_1)$ and $v \in \eta(J_2)$. Since $J_1 \cap J_2 \neq \emptyset$, $\eta(J_1 \cup J_2) = [u, v]$. By Lemma 1, $\mathrm{cl}(\eta^{-1}(u, v)) \subset J_1 \cup J_2$. So, $\eta(J_3 \setminus (J_1 \cup J_2)) \subset \{u, v\}$. Suppose, without loss of generality, that $u \in \eta(J_3 \setminus (J_1 \cup J_2))$. Since $J_3 \cap J_2 \neq \emptyset$, $\eta(J_3 \cup J_2) = [u, v]$. By Lemma 1, $\mathrm{cl}(\eta^{-1}(u, v)) \subset J_3 \cup J_2$. By our supposition in this paragraph, $v \notin \eta(J_1 \cup J_3)$. So, it follows that $J_1 \setminus (J_2 \cup J_3) \subset \eta^{-1}(u)$, and $J_3 \setminus (J_1 \cup J_2) \subset \eta^{-1}(u)$. We let $L_1 = J_1 \cap \eta^{-1}(u)$, $L_2 = J_3 \cap \eta^{-1}(u)$, and $L_3 = \mathrm{cl}(\eta^{-1}(u, v])$. It is straightforward to check that $L_1 \cup L_2 \cup L_3$ is a one-sided triod.

So, we assume, without loss of generality, that $\eta(J_1) = [u, v]$. Once again by Lemma 1, $\mathrm{cl}(\eta^{-1}(u, v)) \subset J_1$, and $\eta(J_2 \setminus (J_1 \cup J_3)) \cup \eta(J_3 \setminus (J_1 \cup J_2)) \subset \{u, v\}$. Suppose that $u \in \eta(J_3 \setminus (J_1 \cup J_3))$ and $v \in \eta(J_2 \setminus (J_1 \cup J_2))$. Then $\eta(J_2 \cup J_3) = [u, v]$. By Lemma 1, $\eta(J_1 \setminus (J_2 \cup J_3)) \subset \{u, v\}$. Assume, without loss of generality, that $u \in \eta(J_1 \setminus (J_2 \cup J_3))$. As in the previous paragraph, letting $L_1 = J_1 \cap \eta^{-1}(u)$, $L_2 = J_3 \cap \eta^{-1}(u)$, and $L_3 = \mathrm{cl}(\eta^{-1}(u, v])$, we have that $L_1 \cup L_2 \cup L_3$ is a one-sided triod.

Lastly, assume that $\eta(J_2 \setminus (J_1 \cup J_3)) = \{u\} = \eta(J_3 \setminus (J_1 \cup J_2))$. We may form a one-sided triod as we did in the previous paragraph, letting $L_1 = J_2 \cap \eta^{-1}(u)$, $L_2 = J_3 \cap \eta^{-1}(u)$, and $L_3 = \mathrm{cl}(\eta^{-1}(u, v])$. □

Corollary 1. *Suppose that (M, η) is a hereditarily unicoherent folder of chainable continua such that $\mathrm{cl}(\eta^{-1}(0, 1))$ is irreducible between $\eta^{-1}(0)$ and $\eta^{-1}(1)$. If M contains no one-sided triod, then M is chainable.*

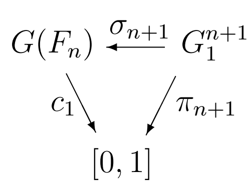
Proof. By Lemma 1, M contains no two-sided triod. So, by Theorem 1, M is atriodic. It follows from Proposition 6 in [5] that M is chainable. □

4. Continuum folders and inverse sequences on $[0, 1]$ with interval-valued functions

For each product $X \times Y$ of compacta X and Y, let $c_1 \colon X \times Y \to X$ and $c_2 \colon X \times Y \to Y$ denote coordinate projection.

If Y is a continuum, and $f\colon [0,1] \to Y$ is a continuum-valued function, the graph of f is a continuum [11, Theorem 4.1]. We note that for $t \in [0,1]$, $(c_1|_{G(f)})^{-1}(t) = \{t\} \times f(t)$, giving us that $c_1|_{G(f)}$ is a monotone mapping, and hence making $(G(f), c_1|_{G(f)})$ a continuum folder with fibers $\{t\} \times f(t)$. We call this continuum-folder structure the *natural continuum folder structure* on $G(f)$.

Suppose that $\{[0,1], f_i\}_{i \geq 1}$ is an inverse sequence, where for each $i \geq 1$, $f_i\colon [0,1] \to [0,1]$ is a surjective, interval-valued function. Recall from the next-to-last paragraph in Section 2, $F_n\colon [0,1] \to G_1^n$ is a surjective, continuum-valued function for each $n \geq 1$. So, from the comments above, for each $n \geq 1$, $(G(f_n), c_1|_{G(f_n)})$ is an arc-folder, and $(G(F_n), c_1|_{G(F_n)})$ is a continuum folder. For $n \geq 1$, let $\sigma_{n+1}\colon G_1^{n+1} \to G(F_n)$ be the homeomorphism given by $\sigma_{n+1}(x_1, \ldots, x_n, x_{n+1}) = (x_{n+1}, x_1, \ldots, x_n)$, and let $\pi_{n+1}\colon G_1^{n+1} \to [0,1]$ be projection onto the $(n+1)^{\text{th}}$ coordinate. We have, for each $n \geq 1$, the commuting diagram below. From the diagram, we see that (G_1^{n+1}, π_{n+1}) is also a continuum-folder. We note that $\sigma_{n+1}^{-1}(G(F_n)) = G(F_n^{-1})$. For subsets A of $G(F_n)$, we let $A^{-1} = \sigma_{n+1}^{-1}(A)$.



In a more general setting, a direct proof that π_{n+1} is monotone can be found in [9, Lemma 4.1]. We call this continuum-folder representation of G_1^{n+1} the *natural continuum folder structure* on G_1^{n+1}.

It will be an aid to clarity and notation to have the definitions of one-sided, two-sided, and three-fibered triods restated for the natural continuum folder structures on the graphs of continuum-valued functions $f\colon [0,1] \to Y$. It should be clear to the reader that these restated definitions are equivalent to those given for general continuum folders (M, η).

Let $f\colon [0,1] \to Y$ be a surjective, continuum-valued function. We say that $G(f)$ contains a *one-sided triod* (at r) if there exists a nondegenerate interval $[r,s]$, two subcontinua A and B of $\{r\} \times f(r)$, and a continuum-valued function $\hat{f}\colon [r,s] \to Y$ such that $G(\hat{f}) \subset G(f)$ and $T = A \cup B \cup G(\hat{f})$ is a triod. We also define a one-sided triod (at s) to be one defined as above with A and B lying in $\{s\} \times f(s)$. We say that $G(f)$ contains a *two-sided triod* (at s) if there exist a nondegenerate interval $[r, t]$, $r < s < t$, and a continuum-valued function $\hat{f}\colon [r, t] \to Y$ such that $G(\hat{f}) \subset G(f)$ and $T = G(\hat{f}|_{[r,s]}) \cup G(\hat{f}|_{[s,t]}) \cup (\{s\} \times f(s))$ is a triod. We say that $G(f)$ contains a *three-fibered triod* if there exist a nondegenerate interval $[r, t] \subset [0, 1]$, $r < s < t$, and a continuum-valued function $\hat{f}\colon [r, t] \to Y$ such that $G(\hat{f}) \subset G(f)$, and $((\{r\} \times f(r)) \cup G(\hat{f})) \cup ((\{s\} \times f(s)) \cup G(\hat{f})) \cup ((\{t\} \times f(t)) \cup G(\hat{f}))$ is a triod. We note that $G(\hat{f}|_{[r,s]}) \cup G(\hat{f}|_{[s,t]}) \cup (\{s\} \times f(s))$ is a two-sided triod (at s) contained in the three-fibered triod above. In fact, it is clear that each three-fibered triod may be realized as a two-sided triod.

Before proceeding to inverse limits, we provide a lemma about interval-valued functions between intervals. A mapping $g\colon X \to Y$ between continua is *weakly confluent* provided that whenever K is a subcontinuum of Y, there exists a subcontinuum H of X such that $g(H) = K$.

Lemma 2. *Let $f\colon I \to J$ be a surjective, interval-valued function between nondegenerate intervals I and J with* $\dim(G(f)) = 1$*. If $J' = [e_1, e_2]$ is a nondegenerate subinterval of J, then there exist a subinterval $I' = [r, s]$ of I, and a surjective, interval-valued function $\hat{f}\colon I' \to J'$ such that*

(1) $G(\hat{f}) \subset G(f)$,

(2) $G(\hat{f})$ is irreducible between $I \times \{e_1\}$ and $I \times \{e_2\}$,

(3) $G(\hat{f}) \cap ([r, s] \times \{e_1\}) = \{(r, e_1)\}$, $G(\hat{f}) \cap ([r, s] \times \{e_2\}) = \{(s, e_2)\}$, and

(4) if $G(\hat{f})$ contains no one-sided triod, then $G(\hat{f})$ is chainable.

Proof. We first note, by Proposition 19(6) in [5], that $G(f)$ is a λ-dendroid.

(1), (2), and (3). Since $c_2|_{G(f)}$ is weakly confluent, there exists a continuum $N' \subset G(f)$ such that $c_2(N') = [e_1, e_2]$. Let N be a subcontinuum of N' that is irreducible between $I \times \{e_1\}$ and $I \times \{e_2\}$. Let (r, e_1) and (s, e_2) be points of N. If $r = s$, then $[e_1, e_2] \subset f(r)$. Since $G(f)$ is hereditarily unicoherent, $\{r\} \times [e_1, e_2] \subset N$. Since N is irreducible between $I \times \{e_1\}$ and $I \times \{e_2\}$, $N = \{r\} \times [e_1, e_2]$. We let $\hat{f}$ be the function with degenerate domain $\{r\}$ defined by $\hat{f}(r) = [e_1, e_2]$. Items (1), (2), and (3) follow.

If $r \neq s$, we assume, without loss of generality, that $r < s$, and suppose $(u, e_1) \in N$ and $u \neq r$. One of u, r, and s separates the remaining two in I. Suppose, without loss of generality, that r separates u from s. Then

$N \cap G(f|_{[r,s]})$ is a proper subcontinuum of N meeting $I \times \{e_1\}$ and $I \times \{e_2\}$, which contradicts the irreducibility of N. We let $\hat{f}$ be the interval-valued function whose graph is N. Items (1), (2), and (3) are established.

(4). Let $h\colon [r,s] \to [0,1]$ be an increasing homeomorphism, and let $\eta = h \circ c_1|_{G(\hat{f})}$. It is clear that $(G(\hat{f}), \eta)$ is a continuum folder. Suppose $G(\hat{f})$ contains a two-sided triod; say $T = A \cup B \cup \eta^{-1}(t)$ is a two-sided triod in $G(\hat{f})$. Then clearly $A \cup B \cup \eta^{-1}(0) \cup \eta^{-1}(1)$ is a proper subcontinuum of $G(\hat{f})$ that contains (r, e_1) and (s, e_2), contradicting the irreducibility of $G(\hat{f})$ between (r, e_1) and (s, e_2). So, $G(\hat{f})$ contains no two-sided triod. By Lemma 1, the continuum $\mathrm{cl}(\eta^{-1}(0,1))$ is irreducible between $\eta^{-1}(0)$ and $\eta^{-1}(1)$. By hypothesis and Corollary 1, $G(\hat{f})$ is chainable. □

We say the function $\hat{f}$, given by Lemma 2, is *an irreducible portion of* f *between* $I \times \{e_1\}$ and $I \times \{e_2\}$. If $e_1 < e_2$ and $r < s$ in Lemma 2, we say that $\hat{f}$ *is increasing between* r and s. If $e_1 < e_2$ and $r > s$, we say that $\hat{f}$ *is decreasing between* r and s.

Let X, Y and Z be continua, and let $f : X \to Y$ and $g : Y \to Z$ be surjective, continuum-valued functions. Define the set-valued function $F : X \to G(g^{-1})$ by $F(x) = G((g|_{f(x)})^{-1})$. We refer to F as the function *induced* by f and g. We observe that $G(F^{-1}) = G'(g, f)$. Furthermore, since both f and g are continuum-valued, it follows that for each $x \in X$, $G(g|_{f(x)})$ is a continuum. Hence, F is continuum-valued.

Theorem 2. *Let* $\{[0,1], f_i\}_{i\geq 1}$ *be an inverse sequence, where for each* $i \geq 1$, $f_i\colon [0,1] \to [0,1]$ *is a surjective, interval-valued function,* $\dim(G(f_i)) = 1$, *and no flat spot of* f_i *composes to a non-degenerate value of* f_j *for* $1 \leq j < i$. *If for some* $n \geq 1$, $G(f_n)$ *contains a triod, then* $G(F_n)$*, with its natural continuum folder structure, contains either a two-sided triod or a one-sided triod. In particular, if* $G(f_n)$ *contains a two-sided triod, then* $G(F_n)$ *contains a two-sided triod, and if* $G(f_n)$ *contains a one-sided triod, then* $G(F_n)$ *contains a one-sided triod.*

Proof. By Theorem 1, $G(f_n)$, with its natural continuum folder structure, contains either a two-sided triod or a one-sided triod. From [18, Corollary 4], we have that $G(F_i) \overset{T}{\approx} G_1^{i+1}$ is a λ-dendroid for each $i \geq 1$.

Case 1. Suppose $G(f_n)$ contains a two-sided triod. We assume, without loss of generality, that the interval $[r,t]$, as in the definition in the third paragraph after the diagram, is the interval $[0,1]$. Then there exist $0 < s < 1$, and a continuum-valued function $\hat{f}_n\colon [0,1] \to [0,1]$ such that $G(\hat{f}_n) \subset G(f_n)$ and $T = G(\hat{f}_n|_{[0,s]}) \cup G(\hat{f}_n|_{[s,1]}) \cup (\{s\} \times f_n(s))$ is a triod. Let $(s,x) \in (\{s\} \times f_n(s)) \setminus G(\hat{f}_n)$. Let $(0,a) \in G(\hat{f}|_{[0,s]})$, and $(1,b) \in G(\hat{f}|_{[s,1]})$. Clearly, $(0,a)$ is not in $G(\hat{f}|_{[s,1]}) \cup (\{s\} \times f_n(s))$, and $(1,b)$ is not in $G(\hat{f}|_{[0,s]}) \cup (\{s\} \times f_n(s))$.

Let $\hat{F}_n\colon [0,1] \to G_1^n$ be the continuum-valued function induced by $\hat{f}_n$ and F_{n-1}. So, by definition, for $t \in [0,1]$, $\hat{F}_n(t) = G((F_{n-1}|_{\hat{f}_n(t)})^{-1})$. Also, it is clear that $G(\hat{F}_n) \subset G(F_n)$. Let $\hat{A} = G(\hat{F}_n|_{[0,s]})$, and $\hat{B} = G(\hat{F}_n|_{[s,1]})$. We claim that $\hat{A} \cup \hat{B} \cup (\{s\} \times F_n(s))$ is a two-sided triod in $G(F_n)$ with its natural continuum folder structure.

It is clear from the definitions, and from the hereditary unicoherence of $G(F_n)$ that each of $\hat{A}$, $\hat{B}$, and $\{s\} \times F_n(s)$ is a continuum, and that the three continua have a nonempty intersection.

Pick a point $\hat{a} \in \{0\} \times F_{n-1}(a) \times \{a\}$, $\hat{b} \in \{1\} \times F_{n-1}(b) \times \{b\}$, and $\hat{x} \in \{s\} \times F_{n-1}(x) \times \{x\}$. We note that $\hat{a} \in \hat{A} \setminus (\hat{B} \cup (\{s\} \times F_n(s)))$, $\hat{b} \in \hat{B} \setminus (\hat{A} \cup (\{s\} \times F_n(s)))$, and $\hat{x} \in (\{s\} \times F_n(s)) \setminus (\hat{A} \cup \hat{B})$. So, $\hat{A} \cup \hat{B} \cup (\{s\} \times F_n(s))$ is a two-sided triod as claimed.

Case 2. Suppose $G(f_n)$ contains a one-sided triod. We assume, without loss of generality, that the interval $[r,s]$, as in the definition, is the interval $[0,1]$, A and B are arcs in $\{0\} \times f_n(0)$, and $\hat{f}_n\colon [0,1] \to [0,1]$ is a continuum-valued function such that $G(\hat{f}_n) \subset G(f_n)$ and $T = A \cup B \cup G(\hat{f}_n)$ is a one-sided triod. Since $A \cup B \subset \{0\} \times f_n(0)$, we can pick points $(0,a) \in A \setminus (B \cup G(\hat{f}_n))$, $(0,b) \in B \setminus (A \cup G(\hat{f}_n))$, and $(1,x) \in G(\hat{f}_n)$. Clearly, $(1,x) \notin A \cup B$.

As in Case 1, let $\hat{F}_n\colon [0,1] \to G_1^n$ be the continuum-valued function induced by $\hat{f}_n$ and F_{n-1}. Since A and B are arcs in $\{0\} \times f_n(0)$, $c_2(A)$ and $c_2(B)$ are arcs in $f_n(0)$. Let $\hat{A} = \{0\} \times G((F_{n-1}|_{c_2(A)})^{-1})$, and $\hat{B} = \{0\} \times G((F_{n-1}|_{c_2(B)})^{-1})$. As in Case 1, we use the points $(0,a)$, $(0,b)$, and $(1,x)$ that were chosen above to pick points $\hat{a} \in \hat{A}$, $\hat{b} \in \hat{B}$, and $\hat{x}$ in $G(\hat{F}_n)$ in such a way that each of these points is not in the union of the remaining two of the three continua. It is clear from the construction that $\hat{A} \cup \hat{B} \cup G(\hat{F}_n)$ is a one-sided triod in the continuum folder $G(F_n)$. □

5. Main theorem for inverse limits with interval-valued functions

We make an observation before the main theorem of this section.

Observation 1. *Let $X = \varprojlim\{[0,1], f_i\}$, where f_i is a surjective, set-valued function for each $i \geq 1$. If for some $n \geq 1$ and some $t \in [0,1]$, $F_n(t)$ contains* *a triod, then X contains a triod.*

Proof. Let T be a triod in $F_n(t)$ for some $n \geq 1$ and $t \in [0,1]$. Let $p = (x_1, x_2, \ldots, x_n, t) \in F_n(t) \times \{t\}$, and let $x = (x_1, x_2, \ldots, x_n, t, x_{n+2}, \ldots)$ be a point of $\hat{\pi}_{n+1}^{-1}(p)$. Then $T \times \{t\} \times \{x_{n+2}\} \times \ldots$ is a triod in X. $\square$

Let $\{X_i, f_i\}_{i \geq 1}$ be an inverse sequence with surjective, set-valued bonding functions. For $i \geq 1$, let $\rho_i \colon G_1^{i+1} \to G_1^i$ be the mapping that drops the $(i+1)^{\text{th}}$ coordinate. We note that $\{G_1^{i+1}, \rho_i\}_{i \geq 1}$ is an ordinary inverse sequence; that is, an inverse sequence with mappings for bonding functions. Ingram showed in [8, Corollary 4.2] that $\varprojlim\{X_i, f_i\}$ is homeomorphic to $\varprojlim\{G_1^{i+1}, \rho_i\}$. In [7, Corollary 4.3], Ingram points out that $\varprojlim\{X_i, f_i\}$ is chainable if G_1^{n+1} is chainable for each $n \geq 1$.

Theorem 3. *Let $X = \varprojlim\{[0,1], f_i\}$, where for each $i \geq 1$, f_i is a surjective, interval-valued function. If X is atriodic and tree-like, then X is chainable, and for each $n \geq 1$, both $G(f_n)$ and $G(F_n) \overset{T}{\approx} G_1^{n+1}$ are chainable.*

Proof. By [18, Corollary 3], for each $i \geq 1$, $\dim(G(f_i)) = 1$, and no flat spot of f_i composes to a nondegenerate value of f_j for $1 \leq j < i$. As we noted in the proof of Theorem 2, each $G(F_n)$ is a λ-dendroid. Since X is tree-like, X is hereditarily unicoherent. So, as the proof proceeds, all non-empty intersections of continua either in X or in some $G(F_n)$ are continua. By [4, Theorem 11], an atriodic λ-dendroid is chainable; so, by Observation 1, for all $n \geq 1$ and $t \in [0,1]$, $F_n(t) \subset G_1^n$ is chainable.

We begin by considering three cases to show that, for each $n \geq 1$, $G(F_n)$ is atriodic, and hence, chainable.

Case 1. Suppose, for some $n \geq 1$, $G(F_n)$ contains a one-sided triod T_1. We will use T_1 to construct a triod in X.

Applying the definition of a one-sided triod (at r_1), we assume that $T_1 = A_1 \cup B_1 \cup G(\hat{F}_n)$, where $\hat{F}_n \colon [r_1, s_1] \to G_1^n$ is a continuum-valued function with $G(\hat{F}_n) \subset G(F_n)$, and $A_1 \cup B_1 \subset \{r_1\} \times F_n(r_1)$. We assume $r_1 < s_1$. We note that for each $t > r_1$, $A_1 \cup B_1 \cup G(\hat{F}_n|_{[r_1,t]})$ is a one-sided subtriod of T_1. Since T_1 is a triod in $G(F_n)$, there exist points $(r_1, a_1, \ldots, a_n) \in A_1 \setminus (B_1 \cup G(\hat{F}_n))$, $(r_1, b_1, \ldots, b_n) \in B_1 \setminus (A_1 \cup G(\hat{F}_n))$, and $(s_1, z_1, \ldots, z_n) \in (\{s_1\} \times F_n(s_1)) \setminus (A_1 \cup B_1)$.

Let $\hat{f}_{n+1}\colon [r_2, s_2] \to [r_1, s_1]$ be an irreducible portion of f_{n+1} between $[0,1] \times \{r_1\}$ and $[0,1] \times \{s_1\}$. Let $\hat{F}_{n+1}\colon [r_2, s_2] \to G_1^{n+1}$ be the continuum-valued function induced by $\hat{f}_{n+1}$ and $\hat{F}_n$. We observe that $\hat{f}_{n+1}(r_2) = \{r_1\}$. For otherwise, $\hat{f}_{n+1}(r_2) = [r_1, t]$ for some $t > r_1$, in which case

$$\{r_2\} \times \hat{F}_{n+1}(r_2) = \{r_2\} \times G((\hat{F}_n|_{\hat{f}_{n+1}(r_2)})^{-1}) = \{r_2\} \times G((\hat{F}_n|_{[r_1,t]})^{-1}).$$

From comments above, this gives us that $\{r_2\} \times (A_1^{-1} \cup B_1^{-1} \cup G(\hat{F}_n|_{[r_1,t]}^{-1}))$ is a one-sided triod contained in the fiber $\{r_2\} \times F_{n+1}(r_2)$ of $G(F_{n+1})$, which, by Observation 1, gives a us a contradiction. It also follows that $r_2 \neq s_2$.

We assume that $r_2 < s_2$, so that $\hat{f}_{n+1}$ is increasing between r_2 and s_2. It is possible, of course, that $r_2 > s_2$ and $\hat{f}_{n+1}$ is decreasing between r_2 and s_2. Either case will allow the proof to proceed. By Lemma 2(3), we note that $\hat{f}_{n+1}^{-1}(r_1) = \{r_2\}$.

Let $A_2 = \{r_2\} \times A_1^{-1}$, and note that $A_2 \subset \{r_2\} \times F_n(r_1) \times \{r_1\} \subset \{r_2\} \times F_{n+1}(r_2)$. Similarly, letting $B_2 = \{r_2\} \times B_1^{-1}$, we have that $B_2 \subset \{r_2\} \times F_{n+1}(r_2)$. It is easily seen that $A_2 \cup B_2 \cup G(\hat{F}_{n+1}|_{[r_2,s_2]})$ is a one-sided triod in $G(F_{n+1})$. Thus, we have an analogous situation to that in the first line of this case with n having increased by 1. So, we repeat the construction above using f_{n+2}. Let $\hat{f}_{n+2}\colon [r_3, s_3] \to [r_2, s_2]$ be an irreducible portion of f_{n+2} between $[0,1] \times \{r_2\}$ and $[0,1] \times \{s_2\}$. The interval $[r_3, s_3]$ is nondegenerate, $\hat{f}_{n+2}(r_3) = \{r_2\}$ and $\hat{f}_{n+2}^{-1}(r_2) = \{r_3\}$.

Continuing this process, we obtain an inverse sequence

$$G_1^n \xleftarrow{\hat{F}_n} [r_1, s_1] \xleftarrow{\hat{f}_{n+1}} [r_2, s_2] \xleftarrow{\hat{f}_{n+2}} \cdots,$$

whose limit $\hat{L}$ is a subcontinuum of X. Each bonding function in the sequence is surjective except $\hat{F}_n$. Since each bonding function is continuum-valued, it follows from [6, Theorem 2.7] that $\hat{L}$ is a continuum.

Let $\hat{A} = A_1^{-1} \times \{r_2\} \times \{r_3\} \times \ldots$, $\hat{B} = B_1^{-1} \times \{r_2\} \times \{r_3\} \times \ldots$, and $\hat{X} = \hat{A} \cup \hat{B} \cup \hat{L}$. We observe that $(a_1, \ldots, a_n, r_1, r_2, \ldots) \in \hat{A} \setminus (\hat{B} \cup \hat{L})$, $(b_1, \ldots, b_n, r_1, r_2, \ldots) \in \hat{B} \setminus (\hat{A} \cup \hat{L})$, and $(z_1, \ldots, z_n, s_1, s_2, \ldots) \in \hat{L} \setminus (\hat{A} \cup \hat{B})$. It follows that $\hat{X}$ is a triod in X, contradicting that X is atriodic.

Case 2. Suppose, for some $n \geq 1$, $G(F_n)$ contains a three-fibered triod T_1.

Let $T_1 = ((\{r_1\} \times F_n(r_1)) \cup K_1) \cup ((\{s_1\} \times F_n(s_1)) \cup K_1) \cup ((\{t_1\} \times F_n(t_1)) \cup K_1)$ be a three-fibered triod as in the definition. So, $[r_1, t_1]$ is a nondegenerate interval, $r_1 < s_1 < t_1$, and $\hat{F}_n\colon [r_1, t_1] \to G_1^n$ a continuum-valued function where $G(\hat{F}_n) = K_1 \subset G(F_n)$.

Let $\hat{f}_{n+1}\colon [r_2, t_2] \to [r_1, t_1]$ be an irreducible portion of f_{n+1} between $[0, 1]\times \{r_1\}$ and $[0, 1]\times\{t_1\}$. Pick a point $s_2 \in \hat{f}^{-1}_{n+1}(s_1)\cap[r_2, t_2]$. Suppose $r_2 = t_2$, so that $\hat{f}_{n+1}(r_2) = [r_1, t_1]$. Then, since r_1, s_1, and t_1 are three distinct points, it follows that the union of the three continua $(\{r_2\} \times F_n(r_1) \times \{r_1\}) \cup (\{r_2\} \times K_1^{-1})$, $(\{r_2\} \times F_n(s_1) \times \{s_1\}) \cup (\{r_2\} \times K_1^{-1})$, and $(\{r_2\} \times F_n(t_1) \times \{t_1\}) \cup (\{r_2\} \times K_1^{-1})$ is a triod in $F_{n+1}(r_2)$. By Observation 1, X contains a triod, which is a contradiction.

So, we assume, without loss of generality, that $r_2 < t_2$. It may be the case that $s_2 \in \{r_2, s_2\}$. From Lemma 2(3), we note that $\hat{f}^{-1}_{n+1}(r_1) = \{r_2\}$, and $\hat{f}^{-1}_{n+1}(t_1) = \{t_2\}$.

We repeat the construction for $n+2$. That is, we let $\hat{f}_{n+2}\colon [r_3, t_3] \to [r_2, t_2]$ be an irreducible portion of f_{n+2} between $[0, 1] \times \{r_2\}$ and $[0, 1] \times \{t_2\}$. Pick a point $s_2 \in \hat{f}^{-1}_{n+2}(s_1) \cap [r_3, t_3]$. If $r_3 = t_3$, in a manner analogous to the assumption above that $r_2 = t_2$, we see that this produces a triod in $F_{n+2}(r_3)$, which, by Observation 1, gives us a contradiction. So, $r_3 \neq t_3$. From Lemma 2(3), we note that $\hat{f}^{-1}_{n+1}(r_2) = \{r_3\}$, and $\hat{f}^{-1}_{n+1}(t_2) = \{t_3\}$.

For $i \geq 3$, we pick continuum-valued functions $\hat{f}_{n+i}\colon [r_{i+1}, t_{i+1}] \to [r_i, t_i]$ and points $s_{i+1} \in \hat{f}^{-1}_{n+i}(s_i) \cap [r_{i+1}, t_{i+1}]$ analogously as $\hat{f}_{n+1}$, s_2, $\hat{f}_{n+2}$, and s_3 were chosen. As in Case 1, we get an inverse sequence

$$G_1^n \xleftarrow{\hat{F}_n} [r_1, t_1] \xleftarrow{\hat{f}_{n+1}} [r_2, t_2] \xleftarrow{\hat{f}_{n+2}} \cdots,$$

whose inverse limit $\hat{K}$ is a subcontinuum of X.

We note that since r_1, s_1, and t_1 are three distinct points, the continua $L_1 = F_n(r_1) \times \{r_1\} \times \{r_2\} \times \ldots$, $L_2 = F_n(s_1) \times \{s_1\} \times \{s_2\} \times \ldots$, and $L_3 = F_n(t_1) \times \{t_1\} \times \{t_2\} \times \ldots$ are pairwise disjoint. Letting $\hat{X} = (L_1 \cup \hat{K}) \cup (L_2 \cup \hat{K}) \cup (L_3 \cup \hat{K})$, we see that $\hat{X}$ is a triod. This contradicts that X is atriodic.

Case 3. By cases (1) and (2), we may assume that for all $n \geq 1$, $G(F_n)$ contains neither a one-sided triod nor a three-fibered triod. Suppose for some $n \geq 1$, $G(F_n)$ contains a two-sided triod T_1.

Let $T_1 = G(\hat{F}_n|_{[r_1,s_1]}) \cup G(\hat{F}_n|_{[s_1,t_1]}) \cup (\{s_1\} \times F_n(s_1))$, where $r_1 < s_1 < t_1$, and $\hat{F}_n\colon [r_1, t_1] \to G_1^n$ is a continuum-valued function with $G(\hat{F}_n) \subset G(F_n)$. For convenience, we let $A_1 = G(\hat{F}_n|_{[r_1,s_1]})$, and $B_1 = G(\hat{F}_n|_{[s_1,t_1]})$. So, $T_1 = A_1 \cup B_1 \cup (\{s_1\} \times F_n(s_1))$.

Let $\hat{f}_{n+1}\colon [r_2, t_2] \to [r_1, t_1]$ be an irreducible portion of f_{n+1} between $[0, 1]\times \{r_1\}$ and $[0, 1] \times \{t_1\}$. Let $\hat{F}_{n+1}\colon [r_2, t_2] \to G_1^{n+1}$ be the continuum-valued function induced by $\hat{f}_{n+1}$ and $\hat{F}_n$.

We note that if $x \in \hat{f}_{n+1}^{-1}(s_1)$, then s_1 is not in the interior of $\hat{f}_{n+1}(x)$. For otherwise, $\hat{f}_{n+1}(x)$ is an interval that meets both $[r_1, s_1)$ and $(s_1, t_1]$, in which case, $F_{n+1}(x)$ contains a triod that is homeomorphic to a two-sided subtriod (at s_1) of T_1. By Observation 1, X contains a triod, which is a contradiction. It follows that $r_2 \neq t_2$, and we may assume, without loss of generality, that $r_2 < t_2$.

Suppose $\hat{f}_{n+1}^{-1}(s_1)$ contains three distinct points, say $x_2 < y_2 < z_2$ are in $\hat{f}_{n+1}^{-1}(s_1)$. It follows that

$$((\{x_2\} \times F_{n+1}(x_2)) \cup G(\hat{F}_{n+1}|_{[x_2,z_2]})) \cup ((\{y_2\} \times F_{n+1}(y_2)) \cup G(\hat{F}_{n+1}|_{[x_2,z_2]}))$$

$$\cup((\{z_2\} \times F_{n+1}(z_2)) \cup G(\hat{F}_{n+1}|_{[x_2,z_2]}))$$

is a three-fibered triod in $G(F_{n+1})$, which is a contradiction to our assumptions in this case.

Let N_1 be the component of $c_2^{-1}([r_1, s_1]) \cap G(\hat{f}_{n+1})$ that contains the point (r_2, r_1), and let N_2 be the component of $c_2^{-1}([s_1, t_1]) \cap G(\hat{f}_{n+1})$ that contains the point (t_2, t_1). Since $\hat{f}_{n+1}^{-1}(s_1)$ does not contain three distinct points, it follows that $N_1 \cap N_2$ is non-empty, and is, in fact, a singleton, say $N_1 \cap N_2 = \{(s_2, s_1)\}$. By Lemma 2(2), we have that $G(\hat{f}_{n+1}) = N_1 \cup N_2$.

Suppose $s_2 \in \{r_2, t_2\}$. Assume that $s_2 = r_2$. Then $[r_1, s_1] \subset \hat{f}_{n+1}(r_2)$, and by the fourth paragraph of this case, we have that $\hat{f}_{n+1}(r_2) = [r_1, s_1]$. Also, $\hat{f}_{n+1}(r_2) = N_1$. By definition of $\hat{F}_{n+1}$, we have that $\hat{F}_{n+1}(r_2) = G((\hat{F}_n|_{[r_1,s_1]})^{-1})$. We recall that $G(\hat{F}_n|_{[r_1,s_1]}) = A_1$, and meets the continuum $\{s_1\} \times F_n(s_1)$, but neither is a subset of the other. Furthermore, these two continua lie in $F_{n+1}(r_2)$. Also, for $x \in (r_2, t_2] = (s_2, t_2]$, $\{x\} \times \hat{f}_{n+1}(x) \subset N_2$; so $\hat{f}_{n+1}(x) \subset [s_1, t_1]$. It follows that $\mathrm{cl}(G(\hat{f}_{n+1}|_{(r_2,t_2]})) = N_2$, and for $x \in (r_2, t_2]$, $\hat{F}_{n+1}(x) = G((\hat{F}_n|_{\hat{f}_{n+1}(x)})^{-1}) \subset B_1^{-1}$.

This gives us that the three continua $\mathrm{cl}(G(\hat{F}_{n+1}|_{(r_2,t_2]}))$, $\{r_2\} \times A_1^{-1}$, and $\{r_2\} \times F_n(s_1) \times \{s_1\}$ meet in a subcontinuum of $\{r_2\} \times \hat{F}_n(s_1) \times \{s_1\}$. Clearly, the last two continua lie in $\{r_2\} \times F_{n+1}(r_2)$. Also, it is easily seen that each of these three continua contains a point not in the union of the other two. We have that $\mathrm{cl}(G(\hat{F}_{n+1}|_{(r_2,t_2]})) \cup (\{r_2\} \times A_1^{-1}) \cup (\{r_2\} \times F_n(s_1) \times \{s_1\})$ is a one-sided triod (at r_2) in $G(F_{n+1})$, which is a contradiction to our assumptions in this case.

Hence, we have that $r_2 < s_2 < t_2$. Recall that $G(\hat{f}_{n+1}) = N_1 \cup N_2$. We define $\hat{f}_{1,n+1}\colon [r_2, s_2] \to [r_1, s_1]$, and $\hat{f}_{2,n+1}\colon [s_2, t_2] \to [s_1, t_1]$, respectively, to be the surjective continuum-valued functions whose graphs are N_1 and N_2. Observe that $G(\hat{f}_{1,n+1}) \cup G(\hat{f}_{2,n+1}) = G(\hat{f}_{n+1})$. Let $\hat{F}_{1,n+1}$ be the continuum-valued function induced by $\hat{f}_{1,n+1}$ and $\hat{F}_n|_{[r_1,s_1]}$, and let $\hat{F}_{2,n+1}$ be the continuum-valued function induced by $\hat{f}_{2,n+1}$ and $\hat{F}_n|_{[s_1,t_1]}$.

Let $A_2 = G(\hat{F}_{1,n+1})$, and $B_2 = G(\hat{F}_{2,n+1})$. It is easily seen that $A_2 \cup B_2 \cup (\{s_2\} \times F_n(s_1) \times \{s_1\})$ is a two-sided triod in $G(F_{n+1})$. Thus, we have an analogous situation to that in the first line of this case with n having increased by 1.

We continue the construction with the functions f_{n+i} for $i \geq 1$, getting three distinct points r_{i+1}, s_{i+1}, and t_{i+1}, with s_{i+1} between r_{i+1} and t_{i+1}, and getting functions $\hat{f}_{1,n+i}\colon [r_{i+1}, s_{i+1}] \to [r_i, s_i]$, and $\hat{f}_{2,n+i}\colon [s_{i+1}, t_{i+1}] \to [s_i, t_i]$. This gives us inverse sequences

$$G_1^n \xleftarrow{\hat{F}_n} [r_1, s_1] \xleftarrow{\hat{f}_{1,n+1}} [r_2, s_2] \xleftarrow{\hat{f}_{1,n+2}} \cdots, \text{ and}$$

$$G_1^n \xleftarrow{\hat{F}_n} [s_1, t_1] \xleftarrow{\hat{f}_{2,n+1}} [s_2, t_2] \xleftarrow{\hat{f}_{2,n+2}} \cdots,$$

with respective limits $\hat{A}$ and $\hat{B}$. In a manner similar to the first two cases, we see that $\hat{A} \cup \hat{B} \cup (\hat{F}_n(s_1) \times \{s_1\} \times \{s_2\} \times \ldots)$ is a triod in X, which gives us a contradiction.

By Observation 1 and Theorem 1, we have established that, for each $n \geq 1$, $G(F_n) \overset{T}{\approx} G_1^{n+1}$ is atriodic. Therefore, since atriodic λ-dendriods are chainable, as mentioned at the outset, we have G_1^{n+1} is chainable for each $n \geq 1$. By Theorem 2, for each $n \geq 1$, $G(f_n)$ is atriodic, and hence chainable. It follows from [8, Corollary 4.2], mentioned in the paragraph immediately preceding Theorem 3, that X is chainable. □

Corollary 2 below follows from Theorem 3, and from [7, Corollary 4.3], mentioned in the paragraph preceding Theorem 3.

Corollary 2. *Let $X = \varprojlim\{[0,1], f_i\}$, where for each $i \geq 1$, f_i is a surjective, interval-valued function. The inverse limit X is chainable if and only if both $G(f_n)$ and $G(F_n) \overset{T}{\approx} G_1^{n+1}$ are chainable for each $n \geq 1$.*

Question 1. *In Theorem 3, can the assumption of interval-valued be replaced with set-valued?*

6. Inverse limits with functions whose inverses are interval-valued

Let $\{[0,1], f_i\}_{i\geq 1}$ be an inverse sequence, where for each $i \geq 1$, f_i is a surjective, set-valued function, and f_i^{-1} is interval-valued. In the proof of the main theorem of this section, for $n \geq 1$, we consider the *reverse sequence of inverse functions*

$$[0,1] \xleftarrow{f_n^{-1}} [0,1] \xleftarrow{f_{n-1}^{-1}} \ldots\ldots \xleftarrow{f_2^{-1}} [0,1] \xleftarrow{f_1^{-1}} [0,1],$$

associated with the finite inverse sequence

$$[0,1] \xleftarrow{f_1} [0,1] \xleftarrow{f_2} \ldots\ldots \xleftarrow{f_{n-1}} [0,1] \xleftarrow{f_n} [0,1],$$

Although the indexing does not match the usual definition, we may consider the reverse sequence of inverse functions to be a finite inverse sequence of interval-valued functions. It may be helpful to the reader to refer to [12, Section 5] where discussion and examples illustrate how we use these reverse sequences in the proof of Theorem 4. However, all that is needed is to notice that the partial graph

$$G'(f_n^{-1}, \ldots, f_1^{-1}) = \{(x_{n+1}, \ldots, x_1) \mid x_{i+1} \in f_i^{-1}(x_i) \text{ for } 1 \leq i \leq n\}$$

of the reverse sequence of inverse functions is homeomorphic to the partial graph G_1^{n+1} of the original inverse sequence. This is obvious from the definition.

Theorem 4. *Let $X = \varprojlim\{[0,1], f_i\}$, where for each $i \geq 1$, f_i is a surjective, set-valued function, and f_i^{-1} is interval-valued. If X is atriodic and tree-like, then X is chainable, and for each $n \geq 1$, both $G(f_n)$ and $G(F_n) \overset{T}{\approx} G_1^{n+1}$ are chainable.*

Proof. By [19, Theorem 4], X is a λ-dendroid. As we previously noted, atriodic λ-dendroids are chainable. So, X is chainable.

Expressing $X \overset{T}{\approx} \varprojlim\{G_1^i, \rho_i\}$, it was observed in [19, Observation 1] that each bonding map ρ_i is monotone. It follows from well-established results in the theory of ordinary inverse limits, that the projection mappings $p_n\colon X \to G_1^n$ are also monotone mappings. By [16, Table V], monotone mappings preserve chainability. So, for each $n \geq 1$, G_1^{n+1} is chainable.

Lastly, we show that $G(f_n^{-1}) \overset{T}{\approx} G(f_n)$ is chainable for each $n \geq 1$. Fix $n \geq 1$, and let M be the limit of the inverse sequence

$$[0,1] \overset{f_n^{-1}}{\leftarrow} [0,1] \overset{f_{n-1}^{-1}}{\leftarrow} \dots \overset{f_1^{-1}}{\leftarrow} [0,1] \overset{\mathrm{id}}{\leftarrow} [0,1] \overset{\mathrm{id}}{\leftarrow} [0,1] \overset{\mathrm{id}}{\leftarrow} \dots .$$

Clearly, M is homeomorphic to $G'(f_n^{-1}, \dots, f_1^{-1})$, which, as noted above, is homeomorphic to the chainable continuum G_1^{n+1}. Since each of the bonding functions in the inverse sequence for M is interval-valued, it follows from Corollary 2 that the graph of each bonding function is chainable. In particular, $G(f_n^{-1})$ is chainable. $\square$

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