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WEAK CHAINABILITY OF ARC FOLDERS

BY

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Abstract. Arc folders are continua that admit mappings onto an arc where the preimage of each point is either an arc or a point. We show that all arc folders are weakly chainable. Equivalently, they are continuous images of the pseudo-arc. We conclude that a continuum X that admits a mapping $f\colon X \to Y$ onto a locally connected continuum Y, where the preimage of each point is either an arc or a point, is weakly chainable.

Weakly chainable continua were introduced in the 1960s, and proven, by Fearnley [6] and Lelek [9], to form the class of continuous images of the pseudo-arc. The class of weakly chainable continua is a natural extension of the class of locally connected continua, and there is an analogy between these two classes. Each has a single continuum as a common model: the pseudo-arc for weakly chainable continua, and, by the Hahn–Mazurkiewicz Theorem (see [17, p. 126]), an arc for locally connected continua. Both classes are invariant with respect to mappings. Both are closed with respect to countable products, and with respect to finite unions. Locally connected continua have been of significant interest since the beginning of the 20th century. The interest in weakly chainable continua has been growing since the 1960s (see [2], [7], [11–13], [15], [18], [19], [21]).

The class of weakly chainable continua is large, containing all locally connected continua, all chainable continua, and various interesting tree-like continua. The property of being weakly chainable is nonlocal for continua, and is much less intuitive than the one of being locally connected. Proving the weak chainability of a continuum can be a challenge; few tools are available.

In this paper, we add one such tool by showing that if a continuum X admits a mapping onto a locally connected continuum, where the preimage of each point is either an arc or a point, then X is weakly chainable. A critical intermediate step in our argument is to show that all arc folders are weakly chainable.

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A *continuum* is a nonempty, compact, connected metric space. A *map* or *mapping* is a continuous function. A mapping $f\colon X \to Y$ is *monotone* if, for each $y \in Y$, $f^{-1}(y)$ is connected. Consider the class $\mathcal{M}$ of continua that admit a monotone mapping onto $[0,1]$. If $X \in \mathcal{M}$ and $\eta\colon X \to [0,1]$ is surjective and monotone, we refer to each $\eta^{-1}(t)$, for $t \in [0,1]$, as a *fiber* of X. This structure gives rise to an upper semicontinuous decomposition of X, where the fibers are elements of the decomposition. The next three definitions from [8] are repeated here for convenience to the reader.

DEFINITION 1. We refer to a member X of $\mathcal{M}$ together with a monotone surjective map $\eta\colon X \to [0,1]$ as a *continuum folder*. So, a continuum folder is a pair (X, η), but we will typically refer to X as a continuum folder, with an assumed monotone map $\eta\colon X \to [0,1]$. Let $\mathcal{G}$ be a class of continua. If each fiber of X is either a point or belongs to $\mathcal{G}$, we call X a $\mathcal{G}$ *folder* or a *folder of continua from* $\mathcal{G}$.

DEFINITION 2. For M a continuum, an $\{M\}$ folder, which we call simply M *folder*, is a continuum folder where each fiber is either a point or is homeomorphic to M.

DEFINITION 3. If the decomposition of a continuum folder X into its fibers is continuous, we call X a *continuous continuum folder*.

Basic properties of continuum folders and arc folders, as well as examples, can be found in [8]. Although the main results of this paper are related to arc folders, we begin with some definitions that apply to continuum folders. We first define some concepts that generalize the notion of a continuous continuum folder. For X a compact metric space, and $B \subset X$, we let $\mathrm{cl}(B)$ denote the closure of B in X.

DEFINITION 4. A continuum folder X is *strongly left-cohesive* at a fiber $\eta^{-1}(t)$ provided either $t = 0$, or there exists a sequence $\{t_n\} \subset [0,t)$ converging to t such that the fibers $\eta^{-1}(t_n)$ converge to $\eta^{-1}(t)$ in the sense of the Hausdorff distance. Similarly, X is *strongly right-cohesive* at $\eta^{-1}(t)$ if either $t = 1$, or there exists a sequence $\{t_n\} \subset (t,1]$ converging to t such that the fibers $\eta^{-1}(t_n)$ converge to $\eta^{-1}(t)$ in the sense of the Hausdorff distance. We say that X is *strongly cohesive* at $\eta^{-1}(t)$ if it is strongly left-cohesive and strongly right-cohesive at $\eta^{-1}(t)$. We say that X is a *strongly cohesive continuum folder* if it is strongly cohesive at each of its fibers.

PROPOSITION 1. *Each continuum folder is strongly cohesive at all but countably many fibers.*

Proof. Let X be a continuum folder. Let d_H denote the Hausdorff distance in X. Let A be the set of all $t \in [0,1]$ such that X is not strongly left-cohesive at $\eta^{-1}(t)$. Thus if $t \in A$, then, for some $\varepsilon_t > 0$, $d_H(\eta^{-1}(t), \eta^{-1}(s)) > \varepsilon_t$

for every $s \in [0,1]$ with $s < t$. Let $\varepsilon > 0$, and let $A_\varepsilon = \{t \in A \mid \varepsilon_t > \varepsilon\}$. If $s, t \in A_\varepsilon$ and $s \neq t$, then $d_H(\eta^{-1}(t), \eta^{-1}(s)) > \varepsilon$. Otherwise, either $\varepsilon_t < \varepsilon$ or $\varepsilon_s < \varepsilon$, which would violate the definition of A_ε. Since the collection $\mathcal{A}_\varepsilon = \{\eta^{-1}(t) \mid t \in A_\varepsilon\}$ is a subset of the compact space of all subcontinua of X, $\mathcal{A}_\varepsilon$ must be finite. We have $A = \bigcup_{n=1}^{\infty} A_{1/n}$. Thus A is countable. Similarly, the set B of all $t \in [0,1]$ such that X is not strongly right-cohesive at t is countable. Hence, $A \cup B$ is countable, and the conclusion follows. ∎

Definition 5. A continuum folder X is *left-cohesive* at a fiber $\eta^{-1}(t)$ provided either $t = 0$, or $\eta^{-1}(t) \subset \mathrm{cl}(\eta^{-1}([0,t)))$. Similarly, X is *right-cohesive* at $\eta^{-1}(t)$ if either $t = 1$, or $\eta^{-1}(t) \subset \mathrm{cl}(\eta^{-1}((t,1]))$. We say that X is *cohesive* at $\eta^{-1}(t)$ if it is left-cohesive and right-cohesive at $\eta^{-1}(t)$. We say that X is a *cohesive continuum folder* if it is cohesive at each of its fibers.

Clearly, continuity of a continuum folder implies strong cohesion, and strong cohesion implies cohesion. We note that the hairy arc is a strongly cohesive, 1-dimensional arc folder. So, [8, Proposition 19(7)] cannot be generalized to strongly cohesive arc folders. The discussion in [10, bottom of p. 262 and top of p. 263] makes it clear that the hairy arc has these properties. The definition of the hairy arc and a proof of its uniqueness can be found in [1].

A continuum X is *irreducible* if there exist points p and q in X such that no proper subcontinuum of X contains both p and q. In this case, we say that X is *irreducible between the points p and q*. We note that if a continuum folder X is irreducible, then it is irreducible between points $p \in \eta^{-1}(0)$ and $q \in \eta^{-1}(1)$. A subcontinuum X of a continuum Y is *terminal in Y* if each subcontinuum of Y that intersects both X and $Y \setminus X$ must contain X. A continuum T is a *triod* if there exist subcontinua L_1, L_2, L_3, and K of T such that $T = L_1 \cup L_2 \cup L_3$, K is a proper subcontinuum of L_i for each $i \in \{1,2,3\}$, and $K = L_1 \cap L_2 = L_1 \cap L_3 = L_2 \cap L_3$ (see [5, p. 36] or [16, p. 218]). Following this definition, we will say that $T = L_1 \cup L_2 \cup L_3$ is a triod with K a *core of T*. A continuum is *atriodic* if it contains no triod.

Proposition 2. *If X is an irreducible, cohesive continuum folder, then for all $0 \leq s < t \leq 1$, the subfolder $\eta^{-1}([s,t])$ of X is irreducible and cohesive.*

Proof. It is clear that $\eta^{-1}([s,t])$ is a cohesive folder. Suppose that F is a subcontinuum of $\eta^{-1}([s,t])$ such that $\eta(F) = [s,t]$. Then $\eta^{-1}((s,t)) \subset F$; for otherwise, the continuum $\eta^{-1}([0,s]) \cup F \cup \eta^{-1}([t,1])$ would be a proper subcontinuum of X containing both $\eta^{-1}(0)$ and $\eta^{-1}(1)$, contradicting the irreducibility of X. So, $\mathrm{cl}(\eta^{-1}((s,t))) \subset F$. Since X is cohesive, we see that $\eta^{-1}([s,t]) \subset F$. Hence, $F = \eta^{-1}([s,t])$. So, $\eta^{-1}([s,t])$ is an irreducible folder. ∎

Proposition 3. *A continuum folder X is irreducible and cohesive if and only if each fiber of X is terminal in X.*

Proof. Suppose (X,η) is irreducible and cohesive, $t \in [0,1]$, and K is a continuum in X such that $K \setminus \eta^{-1}(t) \neq \emptyset \neq K \cap \eta^{-1}(t)$. Then for some $r < t \leq s$ (or $r > t \geq s$) we have $[r,s] = \eta(K)$ (or $[s,r] = \eta(K)$).

We assume, without loss of generality, that $r < s$. By Proposition 2, we know that $\eta^{-1}([r,s])$ is irreducible and cohesive. So, $\eta^{-1}(t) \subset \eta^{-1}([r,s]) = K$, and hence $\eta^{-1}(t)$ is terminal in X.

Suppose all fibers of X are terminal. Let K be a continuum in X meeting both $\eta^{-1}(0)$ and $\eta^{-1}(1)$. Then K must meet all fibers, and by the terminality of $\eta^{-1}(t)$ for each $t \in [0,1]$, each $\eta^{-1}(t)$ is contained in K. Thus $K = X$, and hence X is irreducible.

Suppose for some $t \in (0,1]$ the folder X is not left-cohesive at $\eta^{-1}(t)$. Then the continuum $\eta^{-1}(t) \cap \mathrm{cl}(\eta^{-1}([0,t)))$ is a proper subset of $\eta^{-1}(t)$. The continuum $\mathrm{cl}(\eta^{-1}([0,t)))$ meets $\eta^{-1}(t)$ and its complement, but it does not contain $\eta^{-1}(t)$. Thus $\eta^{-1}(t)$ is not terminal in X, a contradiction. The case of a fiber that is not right-cohesive is similar. Hence, all fibers in X are both left- and right-cohesive, and X is a cohesive folder. ■

Proposition 4. *Let X be an irreducible, cohesive continuum folder. Then*

(1) *X is hereditarily irreducible if and only if all fibers of X are hereditarily irreducible,*
(2) *X is atriodic if and only if all fibers of X are atriodic, and*
(3) *X is chainable if and only if all fibers of X are chainable.*

Proof. We note that all "left-to-right" implications are obvious. So, we prove the "right-to-left" implications.

(1) Suppose each fiber of X is hereditarily irreducible. Let H be a subcontinuum of X. If H is a subset of some fiber of X, then H is irreducible. So, we assume that $\eta(H) = [s,t]$ is a nondegenerate interval. By Proposition 2, H equals $\eta^{-1}([s,t])$ and is irreducible.

(2) Suppose each fiber of X is atriodic. Suppose that X contains a triod $T = L_1 \cup L_2 \cup L_3$ with core K. Since each fiber of X is atriodic, we may assume, without loss of generality, that $\eta(L_1) = [s,t]$ is a nondegenerate interval. By Proposition 2, $L_1 = \eta^{-1}([s,t])$. Also by Proposition 2, it follows that $\eta(K) \neq [s,t]$. Assume that $s \notin \eta(K)$. Then both L_2 and L_3 meet $\eta^{-1}((t,1])$, which violates the irreducibility of X. So, X contains no triod.

(3) Suppose each fiber of X is chainable. Then each fiber of X is atriodic. By (2), X is atriodic. By [8, Proposition 6], X is chainable. ■

By [9, Corollary 3], a continuum is weakly chainable if it is the continuous image of a chainable continuum. We use a fiber replacement technique to prove that every arc folder is the continuous image of a chainable arc folder, and hence, is weakly chainable. To accomplish this, we need the following definitions, which we state in the general setting of continuum folders.

Definition 6. Let (X,η) and (Y,ρ) be folders of continua. A map f from X to Y defines a *morphism* $f:(X,\eta)\to(Y,\rho)$ if, for every $t\in[0,1]$, there is an $s\in[0,1]$ such that $f(\eta^{-1}(t))\subset\rho^{-1}(s)$. Each morphism $f:(X,\eta)\to(Y,\rho)$ between folders of continua induces a unique map $\hat{f}:[0,1]\to[0,1]$ where $\rho\circ f=\hat{f}\circ\eta$.

If f is one-to-one and $\hat{f}$ is an increasing embedding, we say that $f:(X,\eta)\to(Y,\rho)$ is a *morphic embedding*, or simply an *embedding*. If X is a subset of Y and the inclusion map is a morphic embedding, then (X,η) is called a *subfolder* of (Y,ρ). If f maps X onto Y homeomorphically and the induced map $\hat{f}$ is a homeomorphism, we say that f is a *folder isomorphism* between (X,η) and (Y,ρ).

Definition 7. For a folder of continua (X,η) and $t\in(0,1)$, we let $X_t(L)$ and $X_t(R)$ be, respectively, disjoint copies of $\mathrm{cl}(\eta^{-1}([0,t)))$ and $\mathrm{cl}(\eta^{-1}((t,1]))$. Let

$$F_t(L)=\mathrm{cl}(\eta^{-1}([0,t)))\cap\eta^{-1}(t)\quad\text{and}\quad F_t(R)=\mathrm{cl}(\eta^{-1}((t,1]))\cap\eta^{-1}(t).$$

By [8, Lemma 1], we see that $F_t(L)$ and $F_t(R)$ are continua. Also, the rightmost fiber of $X_t(L)$ and the leftmost fiber of $X_t(R)$ are copies of, respectively, $F_t(L)$ and $F_t(R)$. So, $X_t(L)$ and $X_t(R)$ are folders of continua, and there exist natural morphic embeddings $h_L\colon X_t(L)\to\eta^{-1}([0,t])$ and $h_R\colon X_t(R)\to\eta^{-1}([t,1])$.

If $t=0$, $X_0(R)$ and $F_0(R)$ are defined analogously, and we let $F_0(L)=\eta^{-1}(0)$. If $t=1$, then $X_1(L)$ and $F_1(L)$ are defined analogously, and we let $F_1(R)=\eta^{-1}(1)$.

Suppose also that (Y,ρ) is a folder of continua, where there exists a mapping $g_t\colon Y\to\eta^{-1}(t)$ such that

$$g_t|_{\rho^{-1}(0)}\colon\rho^{-1}(0)\to F_t(L),\quad g_t|_{\rho^{-1}(1/2)}\colon\rho^{-1}(1/2)\to\eta^{-1}(t),$$
$$g_t|_{\rho^{-1}(1)}\colon\rho^{-1}(1)\to F_t(R)\quad\text{are all homeomorphisms.}$$

We call Y an *admissible insertion for X at $\eta^{-1}(t)$*.

Under these conditions, we identify $\rho^{-1}(0)$ with $h_L^{-1}g_t(\rho^{-1}(0))$, the rightmost fiber of $X_t(L)$. Also, we identify $\rho^{-1}(1)$ with $h_R^{-1}g_t(\rho^{-1}(1))$, the leftmost fiber of $X_t(R)$. Thusly, we obtain $X_t(L)\cup Y\cup X_t(R)$ with the described identifications. We refer to this construction as *inserting a copy of Y at $\eta^{-1}(t)$ in X*, and we denote the resulting space by $X\cup_t Y$.

We observe that $X\cup_t Y$ is a folder of continua, where the quotient map $\hat{\eta}\colon X\cup_t Y\to[0,1]$ can be chosen in a natural, but not unique, way so that its fibers $\hat{\eta}^{-1}(t)$ are copies of fibers in either X or Y. Any such choice will be sufficient for our constructions. Also, there is a natural morphism $g\colon X \cup_t Y \to X$ given by $g = h_L \cup g_t \cup h_R$. It is clear from the definitions that g, restricted to each of $X_t(L)$ and $X_t(R)$, is a morphic embedding, and g collapses Y to $\eta^{-1}(t)$.

Definition 8. We call the morphism g as above a *Y-collapsing morphism.* As mentioned in the definition of morphisms, we have the following commuting diagram, which we call the *commuting diagram for a collapsing morphism*:

$$\begin{array}{ccc} X & \xleftarrow{\;g\;} & X \cup_t Y \\ \eta\big\downarrow & & \big\downarrow\hat{\eta} \\ [0,1] & \xleftarrow{\;\hat{g}\;} & [0,1] \end{array}$$

Note that $\hat{g}$ is a monotone map with exactly one nondegenerate fiber.

We now turn our attention to arc folders.

Definition 9. Let $S_0 = [0,1]$. We call S_0, together with the identity map, the *trivial arc folder.*

Below we define four versions of paired one-sided and/or two-sided topologist's sine curves, lying in $[0,1] \times [0,1]$, which we will use repeatedly as insertion folders into arc folders in an inverse limit construction in the proof of Theorem 1. The first projection map restricted to each of these folders will be the quotient map onto $[0,1]$.

Definition 10. Let

$$M_1 = (\{0\} \times [0,1]) \cup \left\{ \left(x, \frac{1}{2}\left(1 + \sin\frac{3\pi}{8x}\right)\right) \,\middle|\, 0 < x \le \frac{1}{4} \right\}.$$

Let M_2 be the reflection of M_1 through the line $x = 1/4$, and let M_3 be the reflection of $M_1 \cup M_2$ through the line $x = 1/2$. Let $S_1 = M_1 \cup M_2 \cup M_3$.

Let $N_1 = ([0,1/4] \times \{0\}) \cup M_2$, and let N_2 be the reflection of N_1 through the line $x = 1/2$. Let $S_2 = N_1 \cup N_2$, $S_3 = M_1 \cup M_2 \cup N_2$, and $S_4 = M_3 \cup N_1$.

We note that $\pi_1\colon S_i \to [0,1]$ is monotone for each $i \in \{1,2,3,4\}$, with only $\pi_1^{-1}(0)$, $\pi_1^{-1}(1/2)$ and $\pi_1^{-1}(1)$ as possible nondegenerate fibers. So, for each $i \in \{1,2,3,4\}$, S_i is an arc folder, and we observe that S_i is cohesive. Also, $\pi_2\colon S_i \to [0,1]$ is a mapping whose restrictions to the nondegenerate fibers $\pi_1^{-1}(0)$, $\pi_1^{-1}(1/2)$, and/or $\pi_1^{-1}(1)$ are homeomorphisms. Hence, for some $i \in \{0,1,2,3,4\}$, S_i is an admissible insertion for an arc folder at each of its fibers (see Observation 1 below). We refer to each of the folders S_i, $i \in \{0,1,2,3,4\}$, as a *standard insertion* for an arc folder.

Observation 1. *For each arc folder X, and each $t \in [0,1]$, there exists, for some $i \in \{0,1,2,3,4\}$, a standard insertion S_i at $\eta^{-1}(t)$ in X.*

Proof. Note which of the sets $F_t(L)$, $\eta^{-1}(t)$, and $F_t(R)$ are degenerate, and which are nondegenerate. Pick the appropriate S_i so that $\pi_1^{-1}(0)$, $\pi_1^{-1}(1/2)$, and $\pi_1^{-1}(1)$ are, respectively, copies of the first three sets. Let g'_t be a mapping whose domain is the union of the second group of three sets, and whose restriction to each is a homeomorphism onto the respective member of the first group of three sets. Now, by the Tietze extension theorem, we extend g'_t to a mapping $g_t \colon S_i \to \eta^{-1}(t)$. ∎

For an arc folder X, let $\mathrm{NC}(X) = \{t \in [0,1] \mid X \text{ is not cohesive at } \eta^{-1}(t)\}$. We make the following immediate observation about our standard insertions for arc folders.

Observation 2. *Let X be an arc folder, $t \in [0,1]$, and $g_t \colon S_i \to \eta^{-1}(t)$ be a standard insertion for some $i \in \{0,1,2,3,4\}$. Let $g \colon X \cup_t S_i \to X$ be an S_i-collapsing morphism. Then $\mathrm{NC}(X \cup_t S_i) \subset \hat{g}^{-1}(\mathrm{NC}(X) \setminus \{t\})$, where $\hat{g} \colon [0,1] \to [0,1]$ is the induced map in the commuting diagram for g.*

Theorem 1. *For each arc folder (X_0, η_0) there is an arc folder (X, η) and a surjective morphism $g : (X, \eta) \to (X_0, \eta_0)$ such that (X, η) is chainable and cohesive.*

Proof. Let (X_0, η_0) be an arc folder, and let $\{A_i\}_{i \geq 1}$ be a sequence of fibers in X_0 such that

(i) each fiber of noncohesion in X_0 is an element of $\{A_i\}_{i \geq 1}$, and
(ii) $\{\eta_0(A_i) \mid i \geq 1\}$ is dense in $[0,1]$.

Note that each fiber of noncohesion is nondegenerate.

We will construct an inverse sequence, starting with X_0, and inserting copies of some S_j, $j \in \{0,1,2,3,4\}$, at each A_i. The bonding mappings will be S_j-collapsing morphisms. The inverse limit space X will be the desired chainable continuum, and the first projection mapping of X onto X_0 will be the desired surjective morphism.

Beginning with A_1, we choose the appropriate S_j for insertion at A_1. Let $t = \eta_0(A_1)$. Assume, without loss of generality, that $F_t(L)$ is degenerate, and both $\eta_0^{-1}(t)$ and $F_t(R)$ are nondegenerate. Then we insert a copy of S_4 at A_1. That is, we let $X_1 = X_0 \cup_{\eta_0(A_1)} S_4$, and we let $g_0^1 \colon X_1 \to X_0$ be the S_4-collapsing morphism. Also, we let $\eta_1 = \hat{\eta}_0 \colon X_1 \to [0,1]$ be a monotone quotient mapping as described in the paragraph preceding Definition 8. Note that X_1 is an arc folder, and that copies of A_i for $i \geq 2$ remain fibers of X_1. Although $\eta_0(A_i)$ may not be equal to $\eta_1(A_i)$ for given $i \geq 2$, without loss of generality, we identify each A_i for $i \geq 2$ with its copy in X_1. So, the fibers of noncohesion in X_1 lie among the A_i for $i \geq 2$ (see Observation 2).

We repeat the procedure for $A_2 \subset X_1$, getting $X_2 = X_1 \cup_{\eta_1(A_2)} S_j$ for appropriate $j \in \{0,1,2,3,4\}$, with collapsing morphism $g_1^2 \colon X_2 \to X_1$ and quotient map $\eta_2 = \hat{\eta}_1 \colon X_2 \to [0,1]$.

Inductively, we have an inverse sequence ladder, with commuting diagrams and inverse limits X and $[0,1]$ as indicated below.

$$\begin{array}{ccccccccccc}
X_0 & \xleftarrow{g_0^1} & X_1 & \xleftarrow{g_1^2} & \cdots \longleftarrow & X_n & \xleftarrow{g_n^{n+1}} & X_{n+1} & \longleftarrow & \cdots \longleftarrow & X \\
\eta_0 \big\downarrow & & \eta_1 \big\downarrow & & & \eta_n \big\downarrow & & \eta_{n+1} \big\downarrow & & & \eta \big\downarrow \\
[0,1] & \xleftarrow{r_0^1} & [0,1] & \xleftarrow{r_1^2} & \cdots \longleftarrow & [0,1] & \xleftarrow{r_n^{n+1}} & [0,1] & \longleftarrow & \cdots \longleftarrow & [0,1]
\end{array}$$

The bonding mappings r_i^{i+1} are the induced mappings that were denoted $\hat{g}_i^{i+1}$ in Definition 6. Each bonding map g_i^{i+1} is an S_j-collapsing morphism for some $j \in \{0,1,2,3,4\}$. Each η_i is chosen as in the paragraph preceding Definition 8. Each r_i^{i+1} is a monotone mapping with exactly one nondegenerate fiber. It follows that $\varprojlim\{[0,1], r_i^{i+1}\} \stackrel{T}{\approx} [0,1]$ (see [3]). Also, since each η_i is monotone, the induced mapping η between the inverse limits is monotone (see [4, Corollary 11]). So, X is a folder of continua. We also note that for each $n \geq 0$, the fibers of noncohesion in X_n lie among the copies A_i, for $i \geq n+1$, in X_n (recall Observation 2). For each $n \geq 1$, we let g_n and r_n denote, respectively, the projection mapping from the inverse limits X and $[0,1]$ onto the factor spaces X_n and $[0,1]$.

Although it should be clear from the construction that X is an irreducible, cohesive, arc folder, we provide some justification for these claims.

X is an arc folder. We need to see that each fiber $\eta^{-1}(t)$ in X is either a point or an arc. Suppose x and y are points of X with $\eta(x) = \eta(y)$. Since η is induced, we have $\eta_n g_n(x) = r_n\eta(x) = r_n\eta(y) = \eta_n g_n(y)$ for all $n \geq 0$. That is, $g_n(x)$ and $g_n(y)$ are in the same fiber of X_n for all $n \geq 0$. So, x and y belong to a fiber of X that is an inverse limit of fibers from the X_n's with bonding maps being the g_n^{n+1}'s restricted to the corresponding fibers. From the construction, for each fiber F of X_0, either $(g_n^{n+1}|_F)^{-1}$ is a homeomorphism of single fibers for all $n \geq 0$, or there exists a k such that S_j, for some $j \in \{0,1,2,3,4\}$, is inserted for the copy of F in X_k. For such F, $(g_n^{n+1}|_{F'})^{-1}$ is a homeomorphism of single fibers for each new fiber F' in S_j and all $n \geq k$. It follows that each fiber of X is of one of these two types, which is either an arc or a point. So, X is an arc folder.

X is cohesive. Recall that for each $n \geq 0$, the fibers of noncohesion in X_n lie among the copies A_i, for $i \geq n+1$, in X_n. So, all fibers of noncohesion in X_0 have been replaced in X by copies of the cohesive arc folders S_j. Furthermore, in the metric topology of $\prod_{n\geq 1} X_n$, the "widths" of the copies of the S_j's that were inserted in the construction, tend to zero. Since each copy of some S_j contains a copy of the fiber where it was inserted, each fiber of X_0 that was not one of the A_i fibers remains a fiber of cohesion. Hence, X has no fibers of noncohesion.

X is irreducible. Note that, by our construction, X contains a set M of degenerate fibers such that M is dense in X. Therefore X is irreducible.

It follows from Proposition 4 and [8, Proposition 19(5)] that X is chainable. The first projection mapping $g_0\colon X \to X_0$ is a morphism onto X_0. ■

Corollary 1. *Each arc folder is weakly chainable.*

Definition 11. Let X, Y, and Z be continua, and let $f\colon X \to Z$ and $g\colon Y \to Z$ be surjective mappings. The *fibered product* of f and g is the set in $X \times Y$ given by $[f, g] = \{(x, y) \mid f(x) = g(y)\}$.

With regard to Definition 11, see [22, p. 70]. We point out that these sets were called double graphs in [14] and [20].

The next proposition gives a general method for constructing a continuum folder.

Proposition 5. *Let $f\colon X \to Y$ be a monotone mapping from a continuum X onto a locally connected continuum Y, and let $g\colon [0, 1] \to Y$ be a surjective mapping. Then the fibered product $[f, g]$ is a continuum folder. Moreover, if $\mathcal{G}$ is some class of continua, and the preimages $f^{-1}(y)$ for $y \in Y$ are either points or members of $\mathcal{G}$, then $[f, g]$ is a $\mathcal{G}$ folder.*

Proof. By definition, $[f, g] = \{(x, t) \in X \times [0, 1] \mid f(x) = g(t)\}$. Let $\eta = \pi_2|_{[f,g]}\colon [f, g] \to [0, 1]$. Note that, for $t \in [0, 1]$,

$$\eta^{-1}(t) = \pi_2^{-1}(t) \cap [f, g] = \{(x, t) \mid x \in f^{-1}(g(t))\},$$

which is a nonempty continuum since f is monotone. It follows that $[f, g]$ is a continuum folder.

Clearly, if additionally the point preimages under f are either points or members of $\mathcal{G}$, then $[f, g]$ is a $\mathcal{G}$ folder. ■

Corollary 2. *Let $f : X \to Y$ be a surjective mapping between continua X and Y such that Y is locally connected, and $f^{-1}(y)$ is an arc or a point for each $y \in Y$. Then X is weakly chainable.*

Proof. By the Hahn–Mazurkiewicz Theorem [17, 8.14, p. 126], there exists a continuous surjection $g\colon [0, 1] \to Y$. By Proposition 5, the fibered product $[f, g]$ is an arc folder. Consequently, $[f, g]$ is weakly chainable by Corollary 1. Since X is the continuous image of $[f, g]$ by the projection of $X \times [0, 1]$ onto X, the continuum X is weakly chainable. ■

The following example shows that local connectedness of Y cannot be replaced with chainability in Corollary 2.

Example 1. There exists a mapping $f\colon X \to Y$ such that Y is chainable, and for each $y \in Y$, $f^{-1}(y)$ is either a point or an arc, yet X is not weakly chainable.

Proof. Let T be the simple triod in the plane consisting of the points $\{(x,0) \mid |x| \leq 1\} \cup \{(0,y) \mid 0 \leq y \leq 1\}$. Let S be a simple spiral in $\mathbb{R}^2 \setminus T$ such that $S \cup T = \mathrm{cl}(S)$. So, $X = T \cup S$ is a continuum that is not weakly chainable (see [9, p. 281]). Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping such that $F(x,y) = (x,0)$ for points $(x,y) \in T$, and $F|_{\mathbb{R}^2 \setminus T}$ is a homeomorphism onto $\mathbb{R}^2 \setminus \{(x,0) \mid |x| \leq 1\}$. Let $f = F|_X$. The image of f is topologically a topologist's sine curve, which is chainable. Also, for each $y \in f(X)$, $f^{-1}(y)$ is either a point or an arc. ∎

A proof of the following conjecture would give a significant generalization of our main results.

CONJECTURE 1. *Each folder of chainable continua is weakly chainable.*

REFERENCES

[1] J. M. Aarts and L. G. Oversteegen, *The geometry of Julia sets*, Trans. Amer. Math. Soc. 338 (1993), 897–918.

[2] I. Banič, M. Črepnjak, M. Merhar, U. Milutinović, and T. Sovič, *An Anderson–Choquet-type theorem and a characterization of weakly chainable continua*, Mediterr. J. Math. 14 (2017), no. 2, art. 52, 14 pp.

[3] C. E. Capel, *Inverse limit spaces*, Duke Math. J. 21 (1954), 233–245.

[4] J. J. Charatonik and W. J. Charatonik, *On projections and limit mappings of inverse systems of compact spaces*, Topology Appl. 16 (1983), 1–9.

[5] H. Cook and A. Lelek, *Weakly confluent mappings and atriodic Suslinian curves*, Canad. J. Math. 30 (1978), 32–44.

[6] L. Fearnley, *Characterizations of the continuous images of the pseudo-arc*, Trans. Amer. Math. Soc. 111 (1964), 380–399.

[7] G. A. Feuerbacher, *Weakly chainable circle-like continua*, Fund. Math. 106 (1980), 1–12.

[8] C. L. Hagopian, M. M. Marsh, and J. R. Prajs, *Folders of continua*, Topology Proc. 55 (2020), 13–33.

[9] A. Lelek, *On weakly chainable continua*, Fund. Math. 51 (1962/63), 271–282.

[10] S. Macías, L. G. Oversteegen, and M. P. Widener, *Confluent images of the hairy arc*, Topology Appl. 105 (2000), 261–283.

[11] P. Minc, *A fixed point theorem for weakly chainable plane continua*, Trans. Amer. Math. Soc. 317 (1990), 303–312.

[12] P. Minc, *On weakly chainable inverse limits with simplicial bonding maps*, Proc. Amer. Math. Soc. 119 (1993), 281–289.

[13] P. Minc, *A weakly chainable tree-like continuum without the fixed point property*, Trans. Amer. Math. Soc. 351 (1999), 1109–1121.

[14] J. Mioduszewski, *On a quasi-ordering in the class of continuous mappings of a closed interval onto itself*, Colloq. Math. 9 (1962), 233–240.

[15] T. A. Moebes, *An atriodic tree-like continuum that is not weakly chainable*, Ph.D. thesis, Univ. of Houston, 1980.

[16] R. L. Moore, *Foundations of Point Set Theory*, Amer. Math. Soc. Colloq. Publ. 13, Amer. Math. Soc., Providence, RI, 1962.

[17] S. B. Nadler, Jr., *Continuum Theory. An Introduction*, Monogr. Textbooks Pure Appl. Math. 158, Dekker, New York, 1992.
[18] L. G. Oversteegen and E. D. Tymchatyn, *On the span of weakly-chainable continua*, Fund. Math. 119 (1983), 151–156.
[19] L. G. Oversteegen and E. D. Tymchatyn, *On span and weakly chainable continua*, Fund. Math. 122 (1984), 159–174.
[20] J. T. Rogers, Jr., *Pseudo-circles and universal circularly chainable continua*, Illinois J. Math. 14 (1970), 222–237.
[21] M. Sobolewski, *A weakly chainable uniquely arcwise connected continuum without the fixed point property*, Fund. Math. 228 (2015), 81–86.
[22] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

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