REPRESENTATION THEORY OF THE SYMMETRIC GROUP
2021 ANTC SEMINAR SERIES NOTES

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These notes are simply a record of what was covered in the six-part seminar on the *Representation Theory of the Symmetric Group* in Spring 2021 at California State University, Sacramento. The seminar was meant to be casual, example driven, perhaps fun, and accessible to anyone having completed a first course in linear algebra. Many articles, books, and sets of lecture notes exist on this topic; the development of this seminar benefited in particular from a book by James [Jam78] and seminar notes by Chan [Cha11] and Wildon [Wil14].

First Meeting Notes 02/12/2021

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1. Permutations and Sym(n)

Definition 1.1. A **permutation** of a set X (think $X = \{1, \dots, 7\}$ or $X = \{a, b, c\}$) is a function $\sigma : X \to X$ that is both one-to-one and onto. We might also say σ **permutes** X, σ **acts on** X, or σ **rearranges** X.

Let us now look at an example for the set $X = \{a, b, c\}$. Define $\sigma(a) = b, \sigma(b) = c, \sigma(c) = a$ and $\alpha(a) = b, \alpha(b) = c, \alpha(c) = c$. Note that σ is a permutation while α is not.

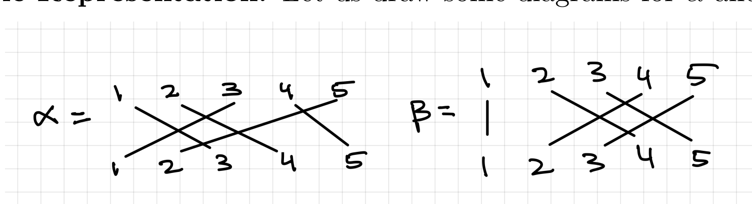
Definition 1.2. The collection of all permutations of a set X is denoted $\mathrm{Sym}(X)$ or S_X where $\mathrm{Sym}(X)$ is called the **symmetric group** on X.

This gives rise to the fact that if $\alpha, \beta \in \mathrm{Sym}(X)$ then $\alpha \circ \beta \in \mathrm{Sym}(X)$ and $\alpha^{-1} \in \mathrm{Sym}(X)$. Also note that the identity function, denoted 1, is also in $\mathrm{Sym}(X)$.

2. Representing Permutations

Consider $\alpha, \beta \in \mathrm{Sym}(5)$ where $\mathrm{Sym}(5)$ is the symmetric group on $\{1, 2, 3, 4, 5\}$ defined by $\alpha(1) = 3, \alpha(2) = 4, \alpha(3) = 1, \alpha(4) = 5, \alpha(5) = 2$ and $\beta(1) = 1, \beta(2) = 4, \beta(3) = 5, \beta(4) = 2, \beta(5) = 3$.

2.1. **Diagrammatic Representation.** Let us draw some diagrams for α and β.



And now let us draw the diagram for $\alpha \circ \beta$.

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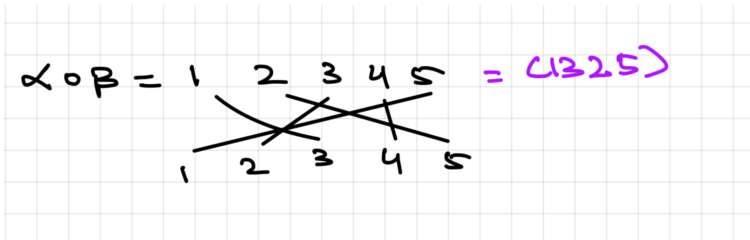
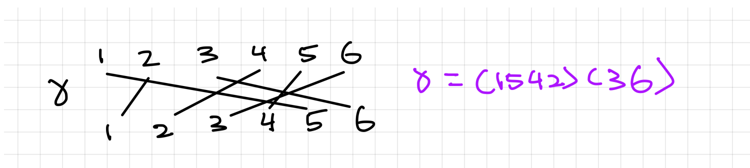


2.2. **Cycle Notation.** Again let us define $\alpha(1) = 3, \alpha(2) = 4, \alpha(3) = 1, \alpha(4) = 5, \alpha(5) = 2$ and $\beta(1) = 1, \beta(2) = 4, \beta(3) = 5, \beta(4) = 2, \beta(5) = 3$. Now let's follow elements in definition of α, so $\alpha : 1 \to 3 \to 1$, which written in cycle notation is (13) and $\alpha : 2 \to 4 \to 5 \to 2$, again written in cycle notation is (245) yields $\alpha = (13)(245)$. Note that each individual cycle is read from left to right. Similarly $\beta : 2 \to 4 \to 2$ and $\beta : 3 \to 5 \to 3$, so $\beta = (24)(35)$.

Let's now look at an example and find the cycle notation for γ:



Lastly, let's return to α and β from before and compose them using cycle notation to find

$$\alpha\beta = (13)(245)(24)(35) = (1325).$$

Note that in cycle notation we follow each element through the cycles from right to left (but within the cycles we read from left to right). For example following the element 1 we see that $1 \leftarrow 1$ by (35), $1 \leftarrow 1$ by (24), $1 \leftarrow 1$ by (245), and $3 \leftarrow 1$ by (13). Thus, $\alpha\beta(1) = 3$. Similarly we see that $5 \leftarrow 3$ by (35), $5 \leftarrow 5$ by (24), $2 \leftarrow 5$ by (245), and $2 \leftarrow 2$ by (13), so $\alpha\beta(3) = 2$.

Second Meeting Notes 02/26/2021

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3. Representations

Let's keep looking at $\alpha = (13)(245), \beta = (24)(35) \in \mathrm{Sym}(5)$. This time we'll reresent them with 5×5 matrices, so consider the following vectors:

$$\overline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \overline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \overline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \overline{e_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \overline{e_5} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note that α acts on $\{1, 2, 3, 4, 5\}$ via $\alpha(1) = 3$, $\alpha(2) = 4$, $\alpha(3) = 1$, $\alpha(4) = 5$, $\alpha(5) = 2$. This gives rise to an "obvious" action on $\{\overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{e_4}, \overline{e_5}\}$ via $\alpha(\overline{e}_1) = \overline{e}_3$, $\alpha(\overline{e}_2) = \overline{e_4}$, $\alpha(\overline{e}_3) = \overline{e}_1$, $\alpha(\overline{e_4}) = \overline{e_5}$, $\alpha(\overline{e_5}) = \overline{e}_2$. In other words, $\alpha(\overline{e_i}) = \overline{e_{\alpha(i)}}$. Now, let's try to find a matrix M_α that acts the same on $\{\overline{e}_1, \dots, \overline{e_5}\}$ as α does. That is we want

$$\begin{aligned} M_\alpha \cdot \overline{e}_1 &= \overline{e}_3 \\ M_\alpha \cdot \overline{e}_2 &= \overline{e_4} \\ M_\alpha \cdot \overline{e}_3 &= \overline{e}_1 \\ M_\alpha \cdot \overline{e_4} &= \overline{e_5} \\ M_\alpha \cdot \overline{e_5} &= \overline{e}_2 \end{aligned}$$

Thus we find

$$M_\alpha \cdot \overline{e}_1 = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathrm{Col}_1(M_\alpha) = \overline{e}_3.$$

Similarly,

$$M_\alpha \cdot \overline{e}_2 = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathrm{Col}_2(M_\alpha) = \overline{e_4}$$

Continuing on, we find that

$$M_\alpha = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Similarly, for $\beta = (24)(35)$, we find that

$$M_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

This gives a function: $\rho : \mathrm{Sym}(n) \to \mathrm{GL}_n$ (where GL_n is the set of invertible $n \times n$ matrices) defined by $\rho(\alpha) = M_\alpha$ where M_α is the matrix such that $\mathrm{Col}_i(M_\alpha) = \overline{e_{\alpha(i)}}$.

Definition 3.1. The function $\rho : \mathrm{Sym}(n) \to \mathrm{GL}_n$ defined above is called the **natural (linear) representation** of $\mathrm{Sym}(n)$.

Let us now look at an example where we write down all permutations in Sym(3) using cycle notation and the natural representation.

Example 3.2. Using cycle notation we have

$$\mathrm{Sym}(3) = \{1, (12), (23), (13), (123), (132)\}.$$

And using the natural representation, we have

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{(123)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad M_{(132)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{(23)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad M_{(13)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Example 3.3. Recall that for $\alpha = (13)(245)$ and $\beta = (24)(35)$ from before we found that

$$M_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad M_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Computing the product $M_\alpha M_\beta$, we find that

$$M_\alpha M_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} = M_{(1325)} = M_{\alpha\beta}.$$

Fact 3.4. For all $\alpha, \beta \in \mathrm{Sym}(n)$,

(1) $M_1 = I_n$
(2) $M_\alpha \cdot M_\beta = M_{\alpha\beta}$
(3) $(M_\alpha)^{-1} = M_{\alpha^{-1}}$

Also, $M_\alpha \cdot \overline{e_i} \iff \alpha(i) = j$.

Modeling the properties above, we arrive at the definition of a representation.

Definition 3.5. A function $\rho : \mathrm{Sym}(n) \to \mathrm{GL}_m$ is called a **(linear) representation** of $\mathrm{Sym}(n)$ if

(1) $\rho(1) = I_m$
(2) $\rho(\alpha\beta) = \rho(\alpha)\rho(\beta)$
(3) $\rho(\alpha^{-1}) = \rho(\alpha)^{-1}$

The number m is called the **degree** or **dimension** of the representation. It can be checked that the second condition implies the first and third, so only the second is needed.

3.1. **Main Questions (take 1).**

Questions.

I. Are there representations of $\mathrm{Sym}(n)$ other than the natural one? If so what are they?
II. Are there representations of $\mathrm{Sym}(n)$ of degree m with $m < n$? If so how small of m is possible?

Third Meeting Notes 03/05/2021

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4. Modules

Let's start by recalling the definition of a representation and the special case of the natural representation.

Definition (See Definition 3.5)**.** A function $\rho : \mathrm{Sym}(n) \to \mathrm{GL}_m(F)$ is called a **(linear) representation** of $\mathrm{Sym}(n)$ if $\forall \alpha, \beta \in \mathrm{Sym}(n), \rho(\alpha\beta) = \rho(\alpha)\rho(\beta)$. We call m the **degree** or **dimension** of the representation.

This definition of a representation only gives one of the three axioms from Definition 3.5, but it can be shown that this one axiom implies the other two.

Example (See Definition 3.1)**.** The **natural representation** $\rho : \mathrm{Sym}(n) \to \mathrm{GL}_m(\mathbb{Z})$ where $\rho(\alpha) = M_\alpha$ is a representation of degree n.

Example 4.1. The following is a 2-dimensional representation of $\mathrm{Sym}(3)$:

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (123) \mapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \quad (132) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$(12) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (23) \mapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \quad (13) \mapsto \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

To verify it's a representation there are various things to check. For example, if $\alpha = (123)$ and $\beta = (12)$, then we need to verify that

$$\rho(\alpha)\rho(\beta) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} = \rho((13)) = \rho(\alpha\beta).$$

Also, there is the obvious question: where did this representation come from?

Let $\rho : \mathrm{Sym}(n) \to \mathrm{GL}_m(\mathbb{R})$ be any representation. Then $\alpha \mapsto \rho(\alpha)$ where $\rho(\alpha)$ is a matrix, and it may be helpful to just think of $\alpha \to M_\alpha$.

Matrices can be understood by how they multiply vectors. Let us recall some properties. If $A, B \in \mathrm{Mat}_{m\times m}$ and $\overline{v}, \overline{w} \in \mathbb{R}^m$ and $c \in \mathbb{R}$, then

- $A(\overline{v} + \overline{w}) = A\overline{v} + A\overline{w}$;
- $AC(\overline{v})) = C(A\overline{v})$;
- $A(B \cdot \overline{v}) = (AB) \cdot \overline{v}$.

Thus,

- $\rho(\alpha)(\overline{v} + \overline{w}) = \rho(\alpha)\overline{v} + \rho(\alpha)\overline{w}$;
- $\rho(\alpha)(c \cdot \overline{v}) = c\rho(\alpha)\overline{v}$;
- $\rho(\alpha)(\rho(\beta)\overline{v}) = (\rho(\alpha)\rho(\beta))\overline{v} = \rho(\alpha\beta)\overline{v}$.

Definition 4.2. A $\mathrm{Sym}(n)$**-module** is a vector space V together with a multiplication $\alpha \cdot \overline{v}$ defined for all $\alpha \in \mathrm{Sym}(n)$ and all $\overline{v} \in V$ such that $\alpha \cdot \overline{v} \in V$ and

(1) $\alpha \cdot (\overline{v} + \overline{w}) = \alpha \cdot \overline{v} + \alpha \cdot \overline{w}$
(2) $\alpha \cdot (c\overline{v}) = c(\alpha \cdot \overline{v})$
(3) $\alpha \cdot (\beta \cdot \overline{v}) = \alpha\beta \cdot \overline{v}$
(4) $1 \cdot \overline{v} = \overline{v}$

Remark 4.3. A $\mathrm{Sym}(n)$-module as has addition, scalar multiplication, and $\mathrm{Sym}(n)$-multiplication, and when working with a $\mathrm{Sym}(n)$-module, we may think of each $\alpha \in \mathrm{Sym}(n)$ as a matrix.

Fact 4.4. $\mathrm{Sym}(n)$-representations correspond to $\mathrm{Sym}(n)$-modules.

To see why this is true, first consider a representation $\rho : \mathrm{Sym}(n) \to \mathrm{GL}_m(F)$. Then we produce a module $V = F^m$ where we define $\alpha \cdot \overline{v} = \rho(\alpha)\overline{v}$. Conversely, we can also produce a representation from a module, as we'll see in the next example.

Definition 4.5. We define the **natural permutation module** perm_F^n to be the $\mathrm{Sym}(n)$-module corresponding to the natural representation $\rho : \mathrm{Sym}(n) \to \mathrm{GL}_n(F)$.

Example 4.6. Lets look at $\mathrm{perm}_{\mathbb{R}}^3$ as a $\mathrm{Sym}(3)$-module. We want to understand the multiplication $\alpha \cdot \overline{v}$. Let $\overline{v} \in \mathrm{perm}_{\mathbb{R}}^3$. Then

$$\overline{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} = a\overline{e}_1 + b\overline{e}_2 + c\overline{e}_3.$$

Consider $\alpha = (132)$. Remember from before, we determined the natural representation of $\mathrm{Sym}(3)$ and found that $M_\alpha\overline{e}_1 = \overline{e}_3$, $M_\alpha\overline{e}_2 = \overline{e}_1$, and $M_\alpha\overline{e}_3 = \overline{e}_2$. So using our module notation, this means that $\alpha \cdot \overline{e}_1 = \overline{e}_3$, $\alpha \cdot \overline{e}_2 = \overline{e}_1$, and $\alpha \cdot \overline{e}_3 = \overline{e}_2$. Thus,

$$\begin{aligned} \alpha \cdot \overline{v} &= \alpha(a\overline{e}_1 + b\overline{e}_2 + c\overline{e}_3) \\ &= a\alpha \cdot \overline{e}_1 + b\alpha \cdot \overline{e}_2 + c\alpha \cdot \overline{e}_3 \\ &= a\overline{e}_3 + b\overline{e}_1 + c\overline{e}_2 \\ &= b\overline{e}_1 + c\overline{e}_2 + a\overline{e}_3, \end{aligned}$$

so

$$(132) \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ c \\ a \end{bmatrix}.$$

Notice how α acts on $\overline{v}$ by permuting the rows of $\overline{v}$. Similarly,

$$\begin{aligned}(12) \cdot v = (12) \cdot (a\overline{e}_1 + b\overline{e}_2 + c\overline{e}_3 &= \\ &= a\overline{e}_2 + b\overline{e}_1 + c\overline{e}_3 \\ &= b\overline{e}_1 + a\overline{e}_2 + c\overline{e}_3,\end{aligned}$$

so

$$(12) \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ a \\ c \end{bmatrix}.$$

Fourth Meeting Notes (Talk by Barry Chin) 04/02/2021

Remark 4.7. If $\overline{v}$ is in $\mathrm{perm}^3_{\mathbb{R}}$, then we may write it in terms of the standard basis for $\mathbb{R}^3$ as

$$\overline{v} = a\overline{e}_1 + b\overline{e}_2 + c\overline{e}_3,$$

and as we saw above,

$$\alpha \cdot \overline{v} = \alpha(a\overline{e}_1 + b\overline{e}_2 + c\overline{e}_3) = a(\alpha \cdot \overline{e}_1) + b(\alpha \cdot \overline{e}_2) + c(\alpha \cdot \overline{e}_3).$$

This illustrates that to understand the multiplication $\alpha \cdot \overline{v}$ for any $\mathrm{Sym}(n)$-module, we only need to understand how α acts on the basis of the module. In other words, if $\overline{b}_1, \ldots, \overline{b}_d$ is a basis for a $\mathrm{Sym}(n)$-module, then as soon as we know $\alpha \cdot \overline{b}_1, \ldots, \alpha \cdot \overline{b}_d$, we are able to determine $\alpha \cdot \overline{v}$ for *all* $\overline{v}$ in the module.

4.1. **Submodules.**

Definition 4.8. Let V be a $\mathrm{Sym}(n)$-module, and let $W \subseteq V$. We call W a **submodule** of V if it is a subspace of V that is closed under $\mathrm{Sym}(n)$-multiplication, i.e. if for all $\alpha \in \mathrm{Sym}(n)$ and all $\overline{w} \in W$, we have $\alpha \cdot \overline{w} \in W$. This means that W is a $\mathrm{Sym}(n)$-module with respect to the same operations used for V.

Example 4.9 (submodules of $\mathrm{perm}^3_{\mathbb{R}}$)**.** Let $V = \mathrm{perm}^3_{\mathbb{R}}$. Let's look for submodules of V.

(1) Let's first consider

$$W_1 = \mathrm{span}(\overline{e}_1, \overline{e}_2) = \{a\overline{e}_1 + b\overline{e}_2 \mid a, b \in \mathbb{R}\} = \left\{ \left[\begin{smallmatrix} a \\ b \\ 0 \end{smallmatrix}\right] \mid a, b \in \mathbb{R} \right\}.$$

Let $w \in W_1$, so $w = a\overline{e}_1 + b\overline{e}_2 = \left[\begin{smallmatrix} a \\ b \\ 0 \end{smallmatrix}\right]$ for some $a, b \in \mathbb{R}$. To check if W_1 is a submodule, we need to check if $\alpha \cdot w \in W_1$ for *all* $\alpha \in \mathrm{Sym}(3)$. Observe that

- $(12) \cdot w = (12) \cdot (a\overline{e}_1 + b\overline{e}_2) = a\overline{e}_2 + b\overline{e}_1 = \left[\begin{smallmatrix} b \\ a \\ 0 \end{smallmatrix}\right] \in W_1$;
- $(123) \cdot w = (123) \cdot (a\overline{e}_1 + b\overline{e}_2) = a\overline{e}_2 + b\overline{e}_3 = \left[\begin{smallmatrix} 0 \\ a \\ b \end{smallmatrix}\right] \notin W_1$.

Since $(123) \cdot w \notin W_1$, W_1 is *not* a submodule.

(2) Next, let's consider

$$W_2 = \mathrm{span}(\overline{e}_1 + \overline{e}_2 + \overline{e}_3) = \{a\overline{e}_1 + a\overline{e}_2 + a\overline{e}_3 \mid a \in \mathbb{R}\} = \left\{ \left[\begin{smallmatrix} a \\ a \\ a \end{smallmatrix}\right] \mid a \in \mathbb{R} \right\}.$$

Let $w \in W_2$, so $w = a\overline{e}_1 + a\overline{e}_2 + a\overline{e}_3$ for some $a \in \mathbb{R}$. This time observe that

- $(123) \cdot w = (123) \cdot (a\overline{e}_1 + a\overline{e}_2 + a\overline{e}_3) = a\overline{e}_2 + a\overline{e}_3 + a\overline{e}_1 = w \in W_2$.

In fact, $\alpha \cdot w = w$ for *all* $\alpha \in \mathrm{Sym}(3)$. This shows that W_2 is a submodule. The multiplication $\alpha \cdot w$ is very boring, so this submodule is called the **trivial module**, denoted $\mathrm{triv}^3_{\mathbb{R}}$.

(3) Finally we consider

$$W_3 = \{a\overline{e}_1 + b\overline{e}_2 + c\overline{e}_3 \mid a+b+c=0\} = \left\{ \left[\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right] \mid a+b+c=0 \right\}.$$

Let $w \in W_3$, and write $w = a\overline{e}_1 + b\overline{e}_2 + c\overline{e}_3$ where $a+b+c=0$. Observe that

- $(123) \cdot w = (123) \cdot (a\overline{e}_1 + b\overline{e}_2 + c\overline{e}_3) = a\overline{e}_2 + b\overline{e}_3 + c\overline{e}_1 = \left[\begin{smallmatrix} c \\ a \\ b \end{smallmatrix}\right]$,

so as the sum of the coefficients of $(123) \cdot w$ is still zero (i.e. $c+a+b=0$), we see that $(123) \cdot w \in W_3$. And it is not hard to verify that $\alpha \cdot w \in W_3$ for *all* $\alpha \in \mathrm{Sym}(3)$, so W_3 is also a submodule. This submodule is called the **standard module**, denoted $\mathrm{std}^3_{\mathbb{R}}$.

We may define std^n_F and triv^n_F (for any n and F) in an analogous way to $\mathrm{std}^3_{\mathbb{R}}$ and $\mathrm{triv}^3_{\mathbb{R}}$ above, and it's easy to show that both are submodules of $\mathrm{perm}^n_{\mathbb{F}}$.

Definition 4.10. The **standard** and **trivial** modules are the submodules of perm^n_F defined by

$$\mathrm{std}^n_F = \{a_1\overline{e}_1 + \cdots + a_n\overline{e}_n \mid a_1 + \cdots + a_n = 0\};$$
$$\mathrm{triv}^n_F = \{a\overline{e}_1 + \cdots + a\overline{e}_n\} = \mathrm{span}(\overline{e}_1 + \cdots + \overline{e}_n).$$

We know that $\dim(\mathrm{perm}^n_F) = n$ and $\dim(\mathrm{triv}^n_F) = 1$, but what about $\dim(\mathrm{std}^n_F)$? At this point, we only know that $\dim(\mathrm{std}^n_F) < n$. And what might be a basis for std^n_F?

Example 4.11. Let's find a basis for $\mathrm{std}^3_{\mathbb{R}}$. Consider the vectors $\overline{f}_1$ and $\overline{f}_2$ defined as

$$\overline{f}_1 = \overline{e}_1 - \overline{e}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \overline{f}_2 = \overline{e}_2 - \overline{e}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Note that $\overline{f}_1, \overline{f}_2 \in \mathrm{std}^3_{\mathbb{R}}$. If we can show $\overline{f}_1$ and $\overline{f}_2$ are linearly independent, then we'll know they form a basis since $\dim(\mathrm{std}^3_{\mathbb{R}}) \leq 2$. Suppose $a\overline{f}_1 + b\overline{f}_2 = 0$. We want to show $a, b = 0$. Observe that

$$\begin{aligned} a\overline{f}_1 + b\overline{f}_2 = 0 &\iff a(\overline{e}_1 - \overline{e}_3) + b(\overline{e}_2 - \overline{e}_3) = 0 \\ &\iff a\overline{e}_1 - a\overline{e}_3 + b\overline{e}_2 + b\overline{e}_3 = 0 \\ &\iff a\overline{e}_1 + b\overline{e}_2 + (-a-b)\overline{e}_3 = 0. \end{aligned}$$

Since $\overline{e}_1, \overline{e}_2, \overline{e}_3$ are linearly independent, we conclude that $a = b = 0$. Hence $\overline{f}_1$ and $\overline{f}_2$ are linearly independent so form a basis for $\mathrm{std}^3_{\mathbb{R}}$.

This example generalizes to give the following fact.

Fact 4.12. A basis for the standard module std^n_F is $\{\overline{f}_1, \ldots, \overline{f}_{n-1}\}$ where $\overline{f}_i = \overline{e}_i - \overline{e}_n$. In particular, $\dim(\mathrm{std}^n_F) = n-1$.

Remark 4.13. There are many other "nice" bases for std^n_F; here's one:

$$\{\overline{e}_1 - \overline{e}_2, \overline{e}_2 - \overline{e}_3 \ldots, \overline{e}_{n-1} - \overline{e}_n\}.$$

Remember that every $\mathrm{Sym}(n)$-module corresponds to a representation of $\mathrm{Sym}(n)$. In Example 3.2, we saw the representation of $\mathrm{Sym}(3)$ corresponding to the natural module. Also, in Example 4.1, we saw the representation of $\mathrm{Sym}(3)$ corresponding to the standard module, but we didn't know it then. Let's try to see how to build the representation from Example 4.1.

Example 4.14. Let $\mathcal{B} = \{\overline{f}_1, \overline{f}_2\}$ be the basis for $\mathrm{std}^3_{\mathbb{R}}$ described above. With respect to $\mathcal{B}$, $\overline{f}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\overline{f}_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Now, observe that

- $(12) \cdot \overline{f}_1 = \overline{f}_1 = (12) \cdot (\overline{e}_1 - \overline{e}_3) = \overline{e}_2 - \overline{e}_3 = \overline{f}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$

- $(12) \cdot \overline{f}_2 = (12) \cdot (\overline{e}_2 - \overline{e}_3) = \overline{e}_1 - \overline{e}_3 = \overline{f}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus, the matrix for (12) with respect to the basis $\mathcal{B}$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Also, we find that

- $(123)\cdot\overline{f}_1 = (123)\cdot(\overline{e}_1-\overline{e}_3) = \overline{e}_2-\overline{e}_1 = \overline{e}_2-\overline{e}_3+\overline{e}_3-\overline{e}_1 = (\overline{e}_2-\overline{e}_3)-(\overline{e}_1-\overline{e}_3) = \overline{f}_2-\overline{f}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$;
- $(123) \cdot \overline{f}_2 = (123) \cdot (\overline{e}_2 - \overline{e}_3) = \overline{e}_3 - \overline{e}_1 = -\overline{f}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Thus, the matrix for (123) is $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$. The remaining elements of Sym(3) can be built from (12) and (123), which can be used to find the remaining matrices for this representation. For example,

$$(23) = (12)(123) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

Computing the remaining matrices, we find the exact representation given in Example 4.1.

4.2. **Main Questions (take 2).** We are trying to find representations of Sym(n), which we saw is equivalent to trying to find Sym(n)-modules. Here's the catalog of so far:

- $\mathrm{perm}_F^n = \mathrm{span}(\overline{e}_1, \dots, \overline{e}_n)$, which is n-dimensional;
- $\mathrm{std}_F^n = \mathrm{span}(\overline{e}_1 - \overline{e}_n, \dots, \overline{e}_{n-1} - \overline{e}_n)$, which is $n-1$-dimensional;
- $\mathrm{triv}_F^n = \mathrm{span}(\overline{e}_1 + \cdots + \overline{e}_n)$, which is 1-dimensional.

Questions.

I. Are there other submodules of perm_F^n? Are there submodules of std_F^n, other than std_F^n and $\{\overline{0}\}$?

II. Are there other modules (that may not be submodules of perm_F^n)?

4.3. **Irreducibility.** In our search for modules, we've been looking specifically at submodules of perm_F^n. There is the question of if there are any submodules we haven't yet found and the related question of if std_F^n has any "interesting" submodules at all (other that itself and the zero module $\{\overline{0}\}$, consisting of just the zero vector). The answer to both questions turn out to be (usually) no!

Definition 4.15. Let V be a Sym(n)-module. We say V is **irreducible** if the only submodules of V are V (itself) and $\{\overline{0}\}$ (the zero module).

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Remark 4.16. Irreducible modules have no "interesting" submodules—think of them as primes. They are the "building blocks" of all modules, so in our quest to find modules, we should focus on finding the irreducible ones.

The next proposition shows that the answer to the second part of Question **I.** above is "no" in the case when our scalars are from $F = \mathbb{R}$. In fact, the same argument shows that the answer is "no" anytime the so-called characteristic of F is *not* a divisor of n. However, when the characteristic of F does divide n (e.g. when $F = \mathbb{Z}/p\mathbb{Z}$ for p a prime divisor of n), then $\mathrm{triv}_F^n \leq \mathrm{std}_{\mathbb{R}}^n$, so $\mathrm{std}_{\mathbb{R}}^n$ is *not* irreducible in this case.

Proposition 4.17. The module $\mathrm{std}_{\mathbb{R}}^n$ is irreducible

Proof. Let $V = \mathrm{perm}_{\mathbb{R}}^n$, $S = \mathrm{std}_{\mathbb{R}}^n$, and $T = \mathrm{triv}_{\mathbb{R}}^n$. As before, we set $\overline{f}_i = \overline{e}_i - \overline{e}_n$. Recall that

$$S = \{\overline{v} \in V |\text{ sum of coefficients is 0 }\} = \mathrm{span}(\overline{f}_1, \dots, \overline{f}_{n-1})$$
$$T = \{\overline{v} \in V |\text{ all coefficients are equal }\} = \mathrm{span}(\overline{e}_1 + \cdots + \overline{e}_n).$$

Claim 1. $S \cap T = \{\bar{0}\}$.

Proof of Claim. Let $\overline{v} \in T$. This implies $\overline{v} = a\overline{e}_1 + \cdots + a\overline{e}_n$ for some $a \in \mathbb{R}$. Thus, if $\overline{v}$ is also in S, then $n \cdot a = 0$, which implies $a = 0$. (Here is where we are using that the characteristic of R does not divide n.) And, if $a = 0$, then $\overline{v} = \bar{0}$, so we see that $\overline{v} \in T \cap S$ implies $\overline{v} = \bar{0}$. $\Diamond$

Now let W be a submodule of S. We aim to show $W = S$ or $W = \{\bar{0}\}$. Set $B_1 = \mathrm{span}(\overline{f}_1)$. The next claim highlights an important property of B_1.

Claim 2. $\overline{v} \in B_1 \iff (1n) \cdot \overline{v} = -\overline{v}$.

Proof of Claim. Suppose $\overline{v} \in B_1$. Then $\overline{v} = c(\overline{e}_1 - \overline{e}_n)$ for some $c \in \mathbb{R}$, so

$$(1n) \cdot \overline{v} = (1n) \cdot (c\overline{e}_1 - c\overline{e}_n) = c\overline{e}_n - c\overline{e}_1 = -\overline{v}.$$

Conversely, assume $(1n) \cdot \overline{v} = -\overline{v}$ for an arbitrary $\overline{v} \in S$. Writing, $\overline{v} = a_1\overline{e}_1 + \cdots + a_n\overline{e}_n$, we have,

$$-(a_1\overline{e}_1 + a_2\overline{e}_2 + \cdots + a_n\overline{e}_n) = (1n) \cdot (a_1\overline{e}_1 + a_2\overline{e}_2 + \cdots + a_n\overline{e}_n) = a_1\overline{e}_n + a_2\overline{e}_2 + \cdots + a_n\overline{e}_1,$$

so comparing coefficients, we find that $-a_1 = a_n$ and $-a_i = a_i$ for all $1 < i < n$. This shows $\overline{v} = a_1\overline{e}_1 - a_1\overline{e}_n \in B_1$. $\Diamond$

We now consider two cases: $B_1 \subseteq W$ and $B_1 \not\subseteq W$.

Claim 3. If $B_1 \subseteq W$, then $W = S$.

Proof of Claim. As W is a submodule, we have $\alpha \cdot \overline{w} \in W$ for all $\alpha \in \mathrm{Sym}(n)$ and all $\overline{w} \in W$. Thus,

$$\begin{aligned} B_1 \subseteq W &\implies \overline{f}_1 \in W \\ &\implies \overline{f}_1, (12) \cdot f_1, (13) \cdot f_2, \ldots, (1n-1) \cdot f_1 \in W \\ &\implies \overline{f}_1, \overline{f}_2, \overline{f}_3, \ldots, \overline{f}_{n-1} \subseteq W \\ &\implies \mathrm{span}(\overline{f}_1, \overline{f}_2, \overline{f}_3, \ldots, \overline{f}_{n-1}) \subseteq W. \end{aligned}$$

Since $\mathrm{span}(\overline{f}_1, \overline{f}_2, \overline{f}_3, \ldots, \overline{f}_{n-1}) = S$, we find that $S \subseteq W \subseteq S$, so $W = S$. $\Diamond$

Claim 4. If $B_1 \not\subseteq W$, then $W = \{\bar{0}\}$.

Proof of Claim. Since B_1 is 1-dimensional, $B_1 \not\subseteq W$ implies $B_1 \cap W = \{\bar{0}\}$. We show that this implies that $(1n) \cdot \overline{w} = \overline{w}$ for all $\overline{w} \in W$. Let $\overline{u} = \overline{w} - (1n) \cdot \overline{w}$. We aim to show $\overline{u} = \bar{0}$. Observe,

$$(1n) \cdot \overline{u} = (1n) \cdot \overline{w} - (1n)(1n) \cdot \overline{w} = (1n) \cdot \overline{w} - \overline{w} = -\overline{u},$$

which shows $\overline{u} \in B_1$. Since W is a submodule, $\overline{u} \in W$, so as $B_1 \cap W = \{\bar{0}\}$, $\overline{u} = \bar{0}$. This shows that $(1n) \cdot \overline{w} = \overline{w}$ for all $\overline{w} \in W$.

Define $B_i = \mathrm{span}(\overline{f}_i)$. The argument in the previous claim adapts to show that if any $B_i \subseteq W$, then $W = S$. Thus, in the case we are considering, $B_i \not\subseteq W$ for all $1 \leq i \leq n-1$. Moreover, the argument we just gave then shows $(in) \cdot \overline{w} = \overline{w}$ for all $\overline{w} \in W$. Writing $\overline{w} = a_1\overline{e}_1 + \cdots + a_n\overline{e}_n$, and using that $(in) \cdot \overline{w} = \overline{w}$ for all $1 \leq i \leq n-1$, we find that $\overline{w} = a\overline{e}_1 + \cdots + a\overline{e}_n \in T$. Thus $W \subseteq T$. Since W is a submodule of S, we have $W \subseteq S \cap T$, which is equal to $\{\bar{0}\}$ by the first claim. $\Diamond$

$\square$

5. The irreducible modules of Sym(n)

We now know that to find other modules we should look outside of perm_F^n. Let's start by revisiting how we built perm_F^n.

Step 1: $\mathrm{Sym}(n)$ permutes $\{1, 2, \ldots, n\}$

Step 2: Consider a basis for F^n labeled $\overline{e}_1, \ldots, \overline{e}_n$. Then $\alpha \cdot \overline{e}_i = \overline{e}_{\alpha(i)}$.

Step 3: $\mathrm{perm}_F^n = \mathrm{span}(\overline{e}_1, \ldots, \overline{e}_n)$ with this $\mathrm{Sym}(n)$-multiplication.

So one idea would be to revisit Step 1 and find other sets that $\mathrm{Sym}(n)$ permutes. For example, we could consider the set of ordered pairs: $\{(i, j) \mid 1 \le i, j \le n\}$ with a "coordinatewise" action: $\alpha \cdot (i, j) = (\alpha(i), \alpha(j))$. For example, $(153) \cdot (5, 1) = (3, 5)$ and $(153) \cdot (1, 7) = (5, 7)$. We could also consider unordered pairs: $\{ \overline{\underline{\ i\ \ j\ }} \mid 1 \le i \ne j \le n\}$, where we are using the notation $\overline{\underline{\ i\ \ j\ }}$ in place of $\{i, j\}$. The action is again coordinatewise. Notice that here we have

$$(12) \cdot \overline{\underline{\ 1\ \ 2\ }} = \overline{\underline{\ 2\ \ 1\ }} = \overline{\underline{\ 1\ \ 2\ }}$$

since the order does not matter. These lead to new modules.

Example 5.1. Consider $\mathrm{Sym}(5)$. There are $\binom{5}{2} = 10$ unordered pairs $\{ \overline{\underline{\ i\ \ j\ }} | 1 \le i \ne j \le 5\}$. Let $V = \mathbb{R}^{10}$, and label a basis by the 10 unordered pairs. When doing this, we simply write $\overline{\underline{\ i\ \ j\ }}$ in place of $\overline{e}_{\overline{\underline{\ i\ j\ }}}$, so here we are thinking of $\overline{\underline{\ i\ \ j\ }}$ as a vector. Thus, the elements of V can be written in terms of this basis as

$$\begin{aligned} \overline{v} = a_{12} \overline{\underline{\ 1\ \ 2\ }} + a_{13} \overline{\underline{\ 1\ \ 3\ }} + a_{14} \overline{\underline{\ 1\ \ 4\ }} + a_{15} \overline{\underline{\ 1\ \ 5\ }} + a_{23} \overline{\underline{\ 2\ \ 3\ }} \\ + a_{24} \overline{\underline{\ 2\ \ 4\ }} + a_{25} \overline{\underline{\ 2\ \ 5\ }} + a_{34} \overline{\underline{\ 3\ \ 4\ }} + a_{35} \overline{\underline{\ 3\ \ 5\ }} + a_{45} \overline{\underline{\ 4\ \ 5\ }} \end{aligned}$$

for some $a_{ij} \in \mathbb{R}$. One concrete element of V is

$$\overline{w} = 7 \cdot \overline{\underline{\ 1\ \ 2\ }} - 3 \cdot \overline{\underline{\ 3\ \ 4\ }} + \pi \cdot \overline{\underline{\ 4\ \ 5\ }}.$$

With the $\mathrm{Sym}(n)$-multiplication defined above, V is a module of dimension 10.

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Remark 5.2. If $\mathrm{Sym}(n)$ permutes the elements of some set $X = \{x_1, \ldots, x_d\}$, then we may make F^d (for F any field) a module as follows:

- label a basis for F^d as $\{\overline{e}_{x_1}, \ldots, \overline{e}_{x_d}\}$;
- define a $\mathrm{Sym}(n)$-multiplication via $\alpha \cdot \overline{e}_{x_i} = \overline{e}_{\alpha \cdot x_i}$.

5.1. Tableaux and tabloids.

We now develop a general setting that captures all previous examples of $\mathrm{Sym}(n)$-modules. We first introduce (integer) **partitions**, **Young tableaux**, and **Young tabloids**.

Definition 5.3. Let n be a positive integer. A **partition** of n is a non-increasing sequence of positive integers that sum to n.

For example, there are 7 partitions of 5. They are (5), $(4, 1)$, $(3, 2)$, $(3, 1, 1)$, $(2, 2, 1)$, $(2, 1, 1, 1)$, and $(1, 1, 1, 1, 1)$.

We now introduce, by example, **λ-tableaux** and **λ-tabloids**.

Example 5.4. Let $\lambda = (3, 2)$. Here are three examples of **λ-tableaux**:

$$t_1 = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} \quad t_2 = \begin{array}{|c|c|c|} \hline 2 & 3 & 5 \\ \hline 1 & 4 \\ \cline{1-2} \end{array} \quad t_3 = \begin{array}{|c|c|c|} \hline 5 & 3 & 1 \\ \hline 4 & 2 \\ \cline{1-2} \end{array}.$$

Notice how the numbers in $\lambda = (3,2)$ indicate the number of boxes in each row of a λ-tableau. All numbers from 1 to n (in this case $n = 5$) are then used once when filling in the boxes.

Continuing on, each λ-tableau t has a corresponding **λ-tabloid** denoted $\{t\}$:

$$\{t_1\} = \begin{array}{ccc} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} \quad \{t_2\} = \begin{array}{ccc} \hline 2 & 3 & 5 \\ \hline 1 & 4 \\ \cline{1-2} \end{array} \quad \{t_3\} = \begin{array}{ccc} \hline 5 & 3 & 1 \\ \hline 4 & 2 \\ \cline{1-2} \end{array}.$$

The notation for λ-tabloid is meant to highlight that the order of the numbers in a row of a λ-tabloid does *not* matter; whereas, it does matter for tableau. However, the column order matters for both. Thus, in this example, we have $t_1 \neq t_3$, $\{t_1\} = \{t_3\}$, and $\{t_1\} \neq \{t_2\}$.

Remark 5.5. If λ is a partition of n, then $\mathrm{Sym}(n)$ permutes the λ-tabloids by "acting coordinatewise." For example, if $\alpha = (14)(23)$, then

$$\alpha \cdot \begin{array}{ccc} \hline 5 & 3 & 1 \\ \hline 4 & 2 \\ \cline{1-2} \end{array} = \begin{array}{ccc} \hline 5 & 2 & 4 \\ \hline 1 & 3 \\ \cline{1-2} \end{array}.$$

Definition 5.6. Let λ be a partition of n, and let d be the number of λ-tabloids. Label a basis for F^d by the λ-tabloids. Define the module

$$M_F^\lambda = \mathrm{span}(\{t_1\}, \dots, \{t_d\}) = F^d$$

with coordinatewise $\mathrm{Sym}(n)$-multiplication.

Example 5.7. Consider $\lambda = (4,1)$. There are 5 λ-tabloids:

$$\begin{array}{cccc} \hline 2 & 3 & 4 & 5 \\ \hline 1 \\ \cline{1-1} \end{array} \quad \begin{array}{cccc} \hline 1 & 3 & 4 & 5 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{cccc} \hline 1 & 2 & 4 & 5 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{cccc} \hline 1 & 2 & 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \quad \begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array},$$

so $M_F^{(4,1)}$ is a 5-dimensional vector space. Notice how each tabloid above is completely determined by the one number in the second row, which gives a correspondence

$$\begin{array}{cccc} \hline * & * & * & * \\ \hline i \\ \cline{1-1} \end{array} \longleftrightarrow \bar{e}_i.$$

Then, thinking about how the $\mathrm{Sym}(n)$-multiplication is defined, we find that $M_F^{(4,1)}$ is really just perm_F^5 in disguise, i.e. $M_F^{(4,1)} \cong \mathrm{perm}_F^5$ as modules.

Example 5.8. Consider $\lambda = (3,2)$. There are 10 λ-tabloids:

$$\begin{array}{ccc} \hline 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2} \end{array} \quad \begin{array}{ccc} \hline 2 & 4 & 5 \\ \hline 1 & 3 \\ \cline{1-2} \end{array} \quad \begin{array}{ccc} \hline 2 & 3 & 5 \\ \hline 1 & 4 \\ \cline{1-2} \end{array} \quad \cdots \quad \begin{array}{ccc} \hline * & * & * \\ \hline i & j \\ \cline{1-2} \end{array} \quad \cdots \quad \begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}.$$

As before, our tabloids only have two rows, so each one is completely determined by the second row. Thus, in this case, we find that $M_F^{(3,2)}$ is really the same as the 10-dimensional module we constructed in Example 5.1.

5.2. **Specht modules and the main theorem.** So, the M_F^λ construction yields many modules, capturing familiar ones from before, but how can we use this to find *irreducible* modules? The answer is to look inside of M_F^λ for an irreducible module in a way analogous to how we found std_F^n inside of perm_F^n.

Remark 5.9. Recall that permutations of the form (ij) are called **transpositions**. It is not difficult to see that an arbitrary permutation $\alpha \in \mathrm{Sym}(n)$ can be written as a product of transpositions. This can be done in *many* different ways, but it turns out that for a given α every possible way of writing α as a product of transpositions will either require an even number of transpositions or every possible way will require an odd number. This leads to the definition of the **sign** of α:

$$\text{sgn} = \begin{cases} +1 & \text{if } \alpha \text{ can be written as a product of an even number of transpositions} \\ -1 & \text{otherwise.} \end{cases}$$

For example,

$$\begin{aligned} \text{sgn}((1234)) &= \text{sgn}((12)(23)(34)) = -1 \\ \text{sgn}((123)(456)) &= \text{sgn}((12)(23)(45)(56) = 1 \end{aligned}$$

Definition 5.10. Let t be a λ-tableau. We define $e_t \in M_F^\lambda$ as follows.

- Let C_t be the collection of permutations that preserve (as a set) every column of t.
- Write $C_t = \{\alpha_1, \dots, \alpha_k\}$, and define $e_t = \text{sgn}(\alpha_1)\alpha \cdot \{t\} + \text{sgn}(\alpha_2)\alpha \cdot \{t\} + \cdots + \text{sgn}(\alpha_k)\alpha \cdot \{t\}$.

Let's see a couple of examples of constructing e_t.

Example 5.11. Let $t = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 5 & 2 \\ \cline{1-2}\end{array}$. Then $C_t = \{1, (15), (23), (15)(23)\}$. Thus,

$$\begin{aligned} e_t &= 1 \cdot \{t\} + \text{sgn}((15)) \cdot (15) \cdot \{t\} + \text{sgn}((23)) \cdot (23) \cdot \{t\} + \text{sgn}((15)(23)) \cdot (15)(23) \cdot \{t\} \\ &= 1 \cdot \{t\} - (15) \cdot \{t\} - (23) \cdot \{t\} + (15)(23) \cdot \{t\} \\ &= 1 \cdot \begin{array}{ccc}\hline 1 & 3 & 4 \\ \hline 5 & 2 \\ \cline{1-2}\end{array} - (15) \cdot \begin{array}{ccc}\hline 1 & 3 & 4 \\ \hline 5 & 2 \\ \cline{1-2}\end{array} - (23) \cdot \begin{array}{ccc}\hline 1 & 3 & 4 \\ \hline 5 & 2 \\ \cline{1-2}\end{array} + (15)(23) \cdot \begin{array}{ccc}\hline 1 & 3 & 4 \\ \hline 5 & 2 \\ \cline{1-2}\end{array} \\ &= \begin{array}{ccc}\hline 1 & 3 & 4 \\ \hline 5 & 2 \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 5 & 3 & 4 \\ \hline 1 & 2 \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 1 & 2 & 4 \\ \hline 5 & 3 \\ \cline{1-2}\end{array} + \begin{array}{ccc}\hline 5 & 2 & 4 \\ \hline 1 & 3 \\ \cline{1-2}\end{array} \end{aligned}$$

Example 5.12. Let $t = \begin{array}{|c|c|c|c|}\hline 5 & 2 & 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}$. Then $C_t = \{1, (15)\}$. Thus,

$$\begin{aligned} e_t &= 1 \cdot \{t\} + \text{sgn}((15)) \cdot (15) \cdot \{t\} \\ &= 1 \cdot \{t\} - (15) \cdot \{t\} \\ &= 1 \cdot \begin{array}{cccc}\hline 5 & 2 & 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array} - (15) \cdot \begin{array}{cccc}\hline 5 & 2 & 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array} \\ &= \begin{array}{cccc}\hline 5 & 2 & 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array} - \begin{array}{cccc}\hline 1 & 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \end{aligned}$$

Recall that $M_F^{(4,1)} \cong \text{perm}_F^5$ via $\begin{array}{cccc}\hline * & * & * & * \\ \hline i \\ \cline{1-1}\end{array} \mapsto \overline{e}_i$. Using this identification, we have that $e_t = \overline{e}_1 - \overline{e}_5 = f_1$, which in turn leads to an identification of $\text{span}(\{e_t \mid t \text{ is a } (4,1)\text{-tableau}\})$ with std_F^5. (See Fact 4.12 and the example preceding it.) We elaborate a bit on this below in Remark 5.14.

Definition 5.13. Let λ be a partition. The **Specht module** S_F^λ is defined to be

$$S_F^\lambda = \text{span}\left(\{e_t \mid t \text{ is a } \lambda\text{-tableau}\}\right) \leq M_F^\lambda.$$

Remark 5.14. We indicated in Example 5.7 how $M_F^{(n-1,1)} \cong \mathrm{perm}_F^n$ via $\begin{array}{cccc}\hline * & * & * & * \\ \hline i & & & \\ \cline{1-1}\end{array} \mapsto \overline{e}_i$. Building off of this, we find that

$$\begin{aligned} S_F^{(n-1,1)} &= \operatorname{span}\left(\{e_t \mid t \text{ is a } (n-1,1)\text{-tableau}\}\right) \\ &= \operatorname{span}\left(\left\{ \begin{array}{cccc}\hline * & * & * & * \\ \hline i & & & \\ \cline{1-1}\end{array} - \begin{array}{cccc}\hline * & * & * & * \\ \hline j & & & \\ \cline{1-1}\end{array} \;\middle|\; 1 \le i, j \le n \right\}\right) \\ &\cong \operatorname{span}\left(\{\overline{e}_i - \overline{e}_j \mid 1 \le i, j \le n\}\right) \\ &\cong \mathrm{std}_F^n\,. \end{aligned}$$

Theorem 5.15. Assume F has characteristic 0 (e.g. F is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}$). Then

(1) S_F^λ is irreducible for every λ, and
(2) every irreducible $\mathrm{Sym}(n)$-module is isomorphic to S_F^λ for some λ.

So, in the case that F has characteristic 0, this shows how to describe all of the irreducible modules. And that is where the story ends for this seminar.

— The End —

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