A SHORT COURSE ON GROUPS OF FINITE MORLEY RANK

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Contents

1. Introduction

These notes were prepared for a short course on groups of finite Morley rank (fMr) as a part of the Trimester Program *Logic and Algorithms in Group Theory* at the Hausdorff Institute for Mathematics in November 2018.

Familiarity with groups (and permutation groups) are assumed, especially those of a finite or algebraic nature. No experience with model theory is required for the course. The goals are to, beginning from first principles,

(1) build intuition for and comfort with the basics of the theory of groups of fMr,
(2) review the ongoing classification of the simple groups of fMr,
(3) describe current efforts to apply the existing theory to *permutation* groups of fMr, and

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(4) highlight several related threads or research (e.g. representations or fMr).

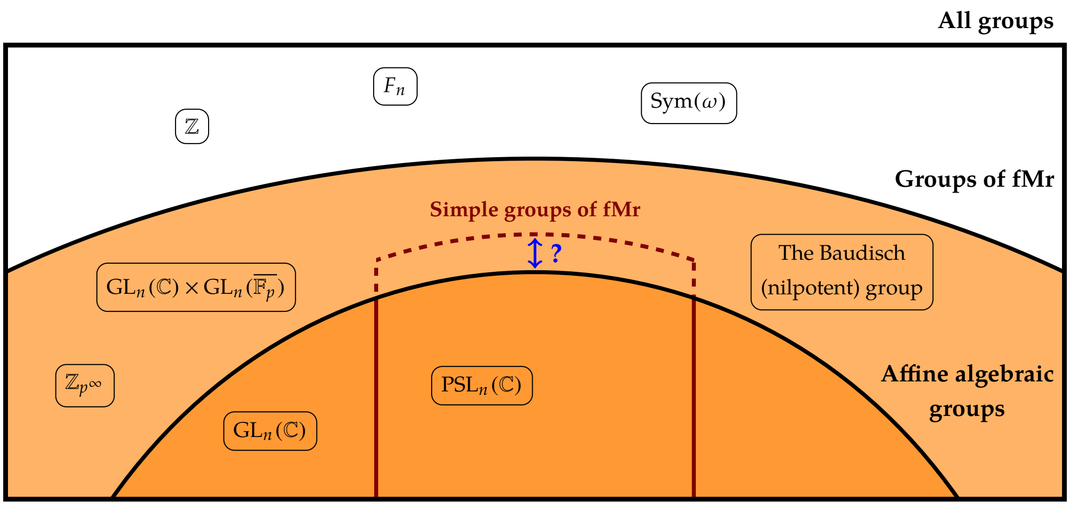
Let us set the stage with some general remarks, only some of which will be properly dealt with below.

Morley rank arose in model theory out of the study of uncountably categorical theories. A theory is κ-*categorical* if it has exactly one model of cardinality κ; the theory of algebraically closed fields of a given characteristic is the classic example when κ is uncountable. Remarkably, Michael Morley showed that if a countable theory is categorical in *some* uncountable cardinal then it must be categorical in *every* uncountable cardinal [Mor65]. His proof of this result introduced a model-theoretic notion of dimension, now called Morley rank. And crucially, it has been since shown by Zilber that the fine structure of uncountably categorical theories naturally involves certain binding groups, analogous to Galois groups, that have finite Morley rank [Zb80].

The most important examples of groups of fMr are the linear algebraic groups over algebraically closed fields, and in this case, Morley rank corresponds to the usual Zariski dimension. In general, groups of fMr share a wealth of similarities with algebraic groups, a point which is emphasized by the *Algebraicity Conjecture* of Cherlin and Zil'ber.

Algebraicity Conjecture ([Che79, Zb77])**.** Every infinite *simple* group of fMr is isomorphic to an algebraic group over an algebraically closed field.

This, of course, should be compared with the classification of the finite simple groups. In the fMr context, all fields are algebraically closed, and conjecturally, there are no sporadic groups. However, if we drop the simplicity requirement, there are plenty of nonalgebraic groups of fMr. The picture is a bit like this—it is not drawn to scale.



Though the conjure remains wide open, it has impressively, after three-decades of effort, been solved for those groups that contain an infinite elementary abelian 2-group [ABC08]. And, just as the classification of the finite simple groups has had far-reaching consequences for the theory of finite permutation groups, this more limited classification result makes it possible to address previously inaccessible problems about permutation groups of finite Morley rank, in full generality [BC08].

2. Ranked structures

We approach groups of fMr via the Borovik-Poizat axioms for ranked structures. This has the effect removing much of model theory from the development of the theory, with the exception of definability (and interpretability). The gain in accessibility to group theorists comes with a loss of both perspective as well as model-theoretic techniques.

2.1. **Definability.** Fix a first-order language $\mathcal{L}$. Good examples to keep in mind are (expansions of) $\mathcal{L}_{\text{GROUP}} = (\cdot, ^{-1}, 1)$ and $\mathcal{L}_{\text{RING}} = (+, \cdot, -, 0)$, both of which implicitly include other familiar symbols like $\forall$, $\exists$, $\wedge$, $\vee$, $\neg$, parentheses, and variables.

An **$\mathcal{L}$-structure**, is a set together with interpretations of the function, relation, and constant symbols (and the "usual" interpretations of the quantifiers and logical symbols). A group is then an $\mathcal{L}_{\text{GROUP}}$-structure for which the group axioms are true. Actually, we use the term "group" (and ring, field, etc.) a bit more broadly.

⚠ Any $\mathcal{L}$-structure with $\mathcal{L}_{\text{GROUP}} \subseteq \mathcal{L}$ for which the group axioms are true will be called a group.

Also, if $\mathcal{M}$ is an $\mathcal{L}$-structure with underlying set M, we can make a new language $\mathcal{L}(M)$ by adding a constant symbol for every element of M; $\mathcal{M}$ becomes an $\mathcal{L}(M)$-structure in the obvious way (by having the constant symbol point to the element it names).

An **$\mathcal{L}$-formula** is a "well-formed" finite sequence of symbols from $\mathcal{L}$ that is expresses a statement that is either true or false for each $\mathcal{L}$-structure. Actually, formulas may have "free" variables, which are not bound by a quantifier, and the truth of the formula may then depend on which elements of the structure are substituted in for the variables. For example, one of $(\forall y)(x^{-1}y^{-1}xy = 1)$ and $x\exists^{-1}x\forall == y$ is an $\mathcal{L}_{\text{GROUP}}$-formula; the other is not. We often denote a formula like $(\forall y)(x^{-1}y^{-1}xy = 1)$ by $\varphi(x)$ to highlight that x is a free variable.

Just like polynomial equations, formulas with free variables naturally define sets via their solutions. In the case of φ, we would write the solutions in G as $\varphi(G) = \{g \in G \mid \varphi(g)\}$. Notice that φ defines $Z(G)$ in any group G. If $h \in G$, then $(\exists y)(x = y^{-1}hy)$ is an $\mathcal{L}_{\text{GROUP}}(G)$-formula that defines the conjugacy class of h. The set of solutions in $\mathbb{R}^2$ to the $\mathcal{L}_{\text{RING}}$-formula $(\exists z)(y - x = z \cdot z)$ is the usual relation $\leq$.

Definition 2.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure.

(1) A set A is **$\mathcal{M}$-definable** (with parameters) if $A = \varphi(M^n)$ for some $\mathcal{L}(M)$-formula $\varphi(\bar{x})$.
(2) A set A is **$\mathcal{M}$-interpretable** if it is "definable modulo a definable equivalence relation":
   - there is a definable $B \subseteq M^n$,
   - there is a definable $E \subseteq B \times B$ giving an equivalence relation on A, and
   - $A = B/E$

⚠ *In what follows (and as is usual), our use of definable includes interpretable.* A *structure* $\mathcal{N}$ is **$\mathcal{M}$-definable** if its underlying set is $\mathcal{M}$-definable as well as (the graphs of) its functions, relations, and constants. Also, let $\text{Def}(\mathcal{M})$ denote the collection of $\mathcal{M}$-definable sets.

Example 2.2. If G is a group (in $\mathcal{L}_{\text{GROUP}}$) and $g \in G$, then the following are G-definable (which you'll remember means *interpretable with parameters*): $Z(G)$, $G/Z(G)$, $C_G(g)$, and g^G (i.e. the set of G-conjugates of g).

Exercise 2.3. Show that $C_G(g)$ is G-definable (with parameters). What about the joint centralizer $C_G(X)$ for X an arbitrary finite subset of G—is it G-definable? Make a conjecture about the G-definability of $C_G(X)$ for X an arbitrary infinite subset of G, and do the same for $[G, G]$.

Example 2.4. If $\mathbb{K}$ is a field, then (a group isomorphic to) $\text{GL}_n(\mathbb{K})$ is $\mathbb{K}$-definable. For example, when $n = 2$,

- $\text{GL}_2(\mathbb{K}) = \{(k_{11}, k_{12}, k_{21}, k_{22}) \in \mathbb{K}^4 \mid k_{11}k_{22} - k_{12}k_{21} \neq 0\}$;
- matrix multiplication and inversion can be defined as well.

Exercise 2.5. Finish showing that $\text{GL}_n(\mathbb{K})$ is $\mathbb{K}$-definable by writing down formulas that define matrix multiplication and inversion. The first formula will have 8 free variables, the second 4.

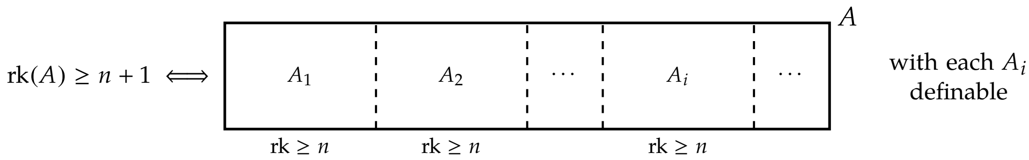
Fact 2.6. *If $\mathbb{K}$ is an algebraically closed field (considered as a $\mathcal{L}_{\text{RING}}$-structure), then every $\mathbb{K}$-definable set is (in definable bijection with) a constructible set, as defined in algebraic geometry.*

2.2. Rank and degree.

Definition 2.7. An $\mathcal{L}$-structure $\mathcal{M}$ is called a **ranked structure** if there is a function rk from $\mathrm{Def}(\mathcal{M}) - \{\emptyset\}$ to $\mathbb{N}$ satisfying the following four axioms (for all $A, B \in \mathrm{Def}(\mathcal{M}) - \emptyset$).

- (*Monotonicity*) $\mathrm{rk}(A) \geq n + 1$ if and only if there exists infinitely many, pairwise disjoint, nonempty, definable subsets of A each of rank at least n.
- (*Additivity*) If $f : A \twoheadrightarrow B$ is a definable surjection with all fibers $f^{-1}(b)$ of constant rank r, then $\mathrm{rk}(A) = \mathrm{rk}(B) + r$.
- (*Definability of rank*) If $f : A \to B$ is definable, then $\{b \in B \mid \mathrm{rk}(f^{-1}(b)) = n\}$ is definable.
- (*Elimination of infinite quantifiers*) If $f : A \to B$ is definable, then there exists a finite n such that for every $b \in B$, $|f^{-1}(b)| \leq n$ or $f^{-1}(b)$ is infinite.

Pictorially, the first axiom is something like this.



It turns out that the ranked *groups* (in a possibly expanded language) are exactly the (ω-stable) groups of finite Morley rank (fMr), as encountered in model theory [Poi01]. We will adopt this terminology.

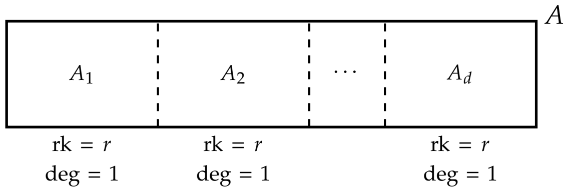
Example 2.8. Here are a handful of examples of groups of fMr.

1. Abelian groups of bounded exponent have fMr.
2. For p a prime, the Prüfer p-group $\mathbb{Z}_{p^\infty} = \{a \in \mathbb{C} \mid a^{p^k} = 1 \text{ for some } k \in \mathbb{N}\}$ have fMr.
3. (Cherlin-Macintyre) An infinite division ring has fMr if and only if it is an algebraically closed field.
4. Any structure $\mathcal{N}$ that is definable over a ranked structure $\mathcal{M}$ is again ranked by the original rank function, but note that definability is still $\mathcal{M}$-definability. In particular, any definable subgroup of group of fMr is again a group of fMr.
5. As a consequence of the previous two points, all affine algebraic groups over algebraically closed fields are groups of fMr.
6. An extremely important example for the model theorist is that of groups definable over any $\aleph_1$-categorical structure in a countable language—this is essentially what launched the study of groups of fMr.

We now look at how one may decompose a set of rank r into subsets of full rank. Assume $\mathrm{rk}(A) = r$. If A cannot be written as a disjoint union of proper definable subsets of rank r, we temporarily say that A is irreducible. Now, by the definition of rank, every decomposition of A into a disjoint union of irreducible subsets of rank r must consist of only a finite number of subsets—it turns out that this number is uniquely determined by X.

Definition 2.9. The **degree** of a definable set A of rank r is the number of subsets occurring in any decomposition of A into a disjoint union of *irreducible* subsets of rank r. (It is well-defined.)

The irreducible sets are those of degree 1. The picture for a set A of rank r and degree d is as follows.



Fact 2.10 (See [BN94, §4.2])**.** *Let A and B be nonempty definable sets over some ranked structure.*

- *(Finite sets) A is finite if and only if* $\operatorname{rk} A = 0$ *and* $\deg A = |A|$.
- *(Monotonicity) If $A \leq B$, then* $\operatorname{rk} A \leq rkB$.
- *(Finite unions)* $\operatorname{rk}(A \cup B) = \max(\operatorname{rk} A, \operatorname{rk} B)$*; if* $\operatorname{rk} A = \operatorname{rk} B > \operatorname{rk}(A \cap B)$*, then* $\deg(A \cup B) = \deg A + \deg B$.
- *(Finite products)* $\operatorname{rk}(A \times B) = \operatorname{rk} A + \operatorname{rk} B$ *and* $\deg(A \times B) = \deg(A) \cdot \deg(B)$.
- *(Invariance under definable bijections) If A and B are in definable bijection, then* $\operatorname{rk} A = \operatorname{rk} B$ *and* $\deg A = \deg B$.

Exercise 2.11. Let H be a definable subgroup of a group G of fMr. Show that the cosets of H in G have constant rank and degree (using invariance). Conclude that if H has finite index in G, then $\operatorname{rk} G = \operatorname{rk} H$ and $\deg G = |G : H| \cdot \deg H$.

Lemma 2.12. *If H is a definable subgroup of a group of fMr G, then* $\operatorname{rk} G = \operatorname{rk}(G/H) + \operatorname{rk} H$.

Proof. The canonical map $G \to G/H$ is definable, and by the previous exercise, the fibers have constant rank $\operatorname{rk} H$. The result follows from additivity of the rank. □

Exercise 2.13. Let $\mathcal{M}$ be a structure for which there is an infinite $\mathcal{M}$-definable set A and an $\mathcal{M}$-definable linear order $<$ on A. Prove that $\mathcal{M}$ is not ranked. Deduce that $\mathbb{R}$ is not ranked as an $\mathcal{L}_{\text{RING}}$-structure.
Hint: consider $A \times A = A_= \sqcup A_< \sqcup A_>$ where $A_= = \{(a,b) \mid a = b\}$, $A_< = \{(a,b) \mid a < b\}$, and $A_> = \{(a,b) \mid b < a\}$. All three sets are definable since $<$ is. Compute ranks.

3. Descending Chain Condition

Groups of fMr are subject to the strongest possible descending chain condition on subgroups that one could reasonably hope for—this has numerous applications.

Proposition 3.1 (Descending Chain Condition on Definable Subgroups—DCC)**.** *If G is a group of fMr, then there are no infinite descending chains of definable subgroups.*

Proof. Suppose $H_0 > H_1 > \cdots$ is a chain of definable subgroups of G. By Exercise 2.11 and Lemma 2.12, either the rank or degree must decrease at each stage, so by finiteness of the rank and degree, the ranks and degrees must eventually stabilize, at which point the chain terminates. □

Exercise 3.2. Show that $\mathbb{Z}$ and $\operatorname{Sym}(\mathbb{Z})$ are not groups of fMr.
Hint: centralizers of elements provide a good source definable subgroups (in the nonabelian case).

Exercise 3.3. Show that a group of fMr has no infinite ascending chains of definable subgroups where each subgroup is of infinite index in the next.

3.1. **Connected components and definable hulls.**

Lemma 3.4. *If G is a group of fMr, then G possesses a smallest definable subgroup of finite index in G; this subgroup is definably characteristic (i.e. invariant under every definable automorphism of G).*

Proof. Let G° denote the intersection of all definable subgroups of finite index in G; clearly G° is definably characteristic. Now, by DCC, G° is equal to a finite intersection of definable subgroup of finite index, so G° is in fact definable and of finite index in G. □

Exercise 3.5. Let X be any subset of a group of fMr G. Show that there is a smallest definable *subgroup* of G containing X.

Definition 3.6. Let G be a group of fMr and X an arbitrary subset.

- The **connected component** of G, denoted G°, is the smallest definable subgroup of finite index in G (which is guaranteed to exist by Lemma 3.4).
- The **definable hull** of X, denoted $\mathrm{d}(X)$, is the smallest definable subgroup containing X (which is guaranteed to exist by Exercise 3.5).

We say G is **connected** if $G = G^\circ$.

If $\langle X\rangle$ denotes the subgroup generated by X, then notice that $\mathrm{d}(X) = \mathrm{d}(\langle X\rangle)$, and $\mathrm{d}(X)$ may be thought of as the *definable* subgroup generated by X.

Lemma 3.7. *If g is any element of a group of fMr G, then $\mathrm{d}(g)$ is abelian.*

Proof. Since $C_G(g)$ is a definable subgroup containing g, $\mathrm{d}(g) \le C_G(g)$, but in fact $g \in Z(C_G(g))$, which is also definable. Thus, $\mathrm{d}(g) \le Z(C_G(g))$, so it is abelian. □

Exercise 3.8. Let g be any element of a group of fMr G. Show that if g centralizes $X \subset G$, then g centralizes $\mathrm{d}(X)$.
Hint: consider both $\mathrm{d}(X)$ and $\mathrm{d}(X)^g = g^{-1}\,\mathrm{d}(X)g$.

Exercise 3.9. Let G be a connected group of fMr, and let $g \in G$. Show that if g has only finitely many conjugates, then $g \in Z(G)$.
Hint: what could be the rank of $C_G(g)$?

Notice that if $\deg(G) = 1$, then as $\deg G = |G : G^\circ| \cdot \deg G^\circ$ (by Exercise 2.11), G is connected. It turns out that the converse is true, which is an incredibly useful fact that we will not prove.

Fact 3.10 (See [BN94, Theorem 5.12])**.** *Let G be a group of fMr. Then G is connected if and only if $\deg(G) = 1$.*

Definition 3.11. If $\mathcal{M}$ is a ranked structure, and $A \subseteq B$ are $\mathcal{M}$-definable sets, we say that A is a **generic** subset of B if $\operatorname{rk} A = \operatorname{rk} B$. We say that A is **strongly generic** in B if $\operatorname{rk} A = \operatorname{rk} B$ and $\deg A = \deg B$ or, equivalently, if $\operatorname{rk} B - A < \operatorname{rk} B$.

Exercise 3.12. Let G be a connected group of fMr, and let $a, b \in G$ be such that a^G and b^G are generic in G. Show that a and b are G-conjugate, i.e. that $a^G = b^G$.
Hint: it suffices to show that a^G and b^G intersect.

4. Semisimplicity, unipotence, and solvable groups of fMr

Inspired by the algebraic theory, we now introduce analogs of semisimplicty (via tori) and unipotence, though in the latter case we only present an analog for characteristic not 0. There is in fact a characteristic 0 theory of unipotence for groups of fMr, but this is not the place to present it. The curious reader is encouraged to start with the PhD thesis of Jeffrey Burdges. It is very important to note that these notions of semisimplicty and unipotence are for connected *subgroups* of elements, and it may be that individual elements fall into both categories.

4.1. **Tori.** A group G is said to be n**-divisible** if, for every $g \in G$, $x^n = g$ has a solution or, put another way, if the map $G \to G : x \mapsto x^n$ is surjective. When G is n-divisible for all natural numbers n, we simply say that G **divisible**.

Example 4.1. The Prüfer p-group $\mathbb{Z}_{p^\infty}$ (see Example 2.8) and $\mathbb{Q}$ (with respect to addition) are divisible groups.

As far as divisible abelian groups are concerned, the previous example more-or-less captures everything.

Fact 4.2. *If D is a divisible group, then $D \cong \left(\bigoplus_\kappa \mathbb{Q}\right) \oplus \left(\bigoplus_{p \text{ a prime}} \left(\bigoplus_{\kappa_p} \mathbb{Z}_{p^\infty}\right)\right)$. The cardinal κ_p is called the **Prüfer** p**-rank** of D, denoted $\mathrm{pr}_p(D)$.*

The following example illustrates a common way that n-divisibility arises when working with groups of fMr. Also, though it is only stated for abelian groups, it is in fact true in general as n-divisibility is a "local" property.

Exercise 4.3. Let G be an abelian group of fMr that has no elements of order dividing n. Show that G is n-divisible. In fact, show that G is *uniquely* n-divisible, i.e. that $x^n = g$ has a *unique* solution for each $g \in G$.
Hint: explain why $x \mapsto x^n$ is a definable bijection between G and $H = \{x^n \mid x \in G\}$. Conclusion?

Definition 4.4. Any divisible abelian p-group (of fMR or not) will be called a p**-torus.**

Note that the multiplicative group $\mathbb{K}^\times$ of an algebraically closed field $\mathbb{K}$ is divisible, and in this case, $\mathrm{char}\,\mathbb{K} = p$ if and only if $\mathbb{K}^\times$ has no p-torus (properly modified in characteristic 0).

Now, though it's true that $\mathbb{Z}_{p^\infty}$ is a group of fMr, it often appears a subgroup of a larger group of fMr in such a way that it is *not* definable. However, by considering it's definable hull, we get a definable subgroup with many desirable properties.

Definition 4.5. A definable subgroup of a group of fMr is called a **decent torus** if it is divisible, abelian, and equal to the definable hull of its torsion subgroup (i.e. the subgroup of all elements of finite order).

Exercise 4.6 (Connectedness of Tori)**.** Show that a divisible group of fMr must be connected.
Hint: suppose not, and note that the class of divisible groups is closed with respect to taking quotients.

Exercise 4.7 (Finiteness of Prüfer Rank of Tori)**.** Show that a divisible abelian group of fMr must have finite Prüfer p-rank for all primes p.
Hint: suppose T is a counterexample and consider the ascending chain of definable subgroups $T_k = \{a \mid a^{p^k} = 1\}$. Since T is a counterexample, T_1 must be infinite—now show that there is a definable isomorphism from T_{k+1}/T_k to T_1 for all k. What could be the rank of T?

Proposition 4.8 (Rigidity of Tori)**.** *If G is a group of fMr and $T \leq G$ is a decent torus, then $N_G^\circ(T) = C_G^\circ(T)$.*

Proof. We want to show T is central in $N = N_G^\circ(T)$. By Exercise 3.8, is suffices to show that the torsion subgroup T_0 of T is central in T. Now, by Exercise 4.7, T, hence T_0, has finite Prüfer p-rank for each prime p. Consequently, T_0 has finitely many elements of each finite order, so as N is connected, Exercise3.9 implies that N centralizes T_0. □

Fact 4.9 (Conjugacy of Maximal Tori, see [Che05] or [ABC08, IV, Proposition 1.15])**.** *Any two maximal decent tori of a group of fMr are conjugate.*

4.2. p**-Unipotent groups.** We now venture to the other end of the torsion spectrum.

Definition 4.10. Let p be a prime. A definable subgroup of a group of fMr G is called p**-unipotent** if it is a connected nilpotent p-group of bounded exponent. We also define $U_p(G)$ to be the subgroup generated by all p-unipotent subgroups, called the p**-unipotent radical**.

Example 4.11. A key example to keep in mind (which is responsible for the choice of terminology) is that of the subgroup of upper-triangular matrices of $\mathrm{GL}_n(\mathbb{K})$, for K an algebraically closed field of characteristic p, with all 1's on the main diagonal—it is p-unipotent in our sense. But, note that $U_p(\mathrm{GL}_n(\mathbb{K})) = \mathrm{SL}_n(\mathbb{K})$ is not p-unipotent.

Fact 4.12 (see [ABC08, I, Lemma 8.3.6])**.** *If G is a solvable group of fMr, then for every prime p, $U_p(G)$ is p-unipotent (and hence also normal).*

The next fact highlights well the interplay between p-unipotence and p-tori.

Fact 4.13 ([BC09])**.** *Let p be a prime. If G is a connected group of fMr with no nontrivial p-unipotent subgroup, then every p-element of G is contained in a p-torus.*

4.3. **Definability of a some good friends.**

Definition 4.14. If G is a group, then the **Fitting subgroup** of G, denoted $F(G)$, is the subgroup generated by all normal nilpotent subgroups. The **solvable radical**, denote $\sigma(G)$, is the subgroup generated by all normal solvable subgroups.

Fact 4.15 (see [BN94, Corollary 5.32, Theorem 7.3])**.** *If G is a group of fMr, then G', $F(G)$, and $\sigma(G)$ are all definable, with the latter two, respectively, nilpotent and solvable.*

4.4. **Structure of solvable groups of fMr.** We now give some structure theorems for nilpotent and solvable groups of fMr; these must be compared with the case of algebraic groups. For their statements, we write $G = A * B$ to denote the **central product** of subgroups A and B, which means that, in addition to generating G, A and B centralize each other.

Fact 4.16 (see [ABC08, I, Propositions 5.8, 5.11])**.** *Let G be a connected nilpotent group of fMr. Then $G = D * U$, with finite intersection, for some definable connected characteristic subgroups $D, U \leq G$ with D divisible and U of bounded exponent. Further, G has a unique maximal decent torus T, and*

- *$D = T * N$ for some not necessarily definable torsion-free divisible subgroup N, and*
- *U is the direct product of its nontrivial p-unipotent radicals.*

Fact 4.17 (see [ABC08, I, Proposition 7.3, Lemma 8.3, Corollary 8.4])**.** *Let G be a connected solvable group of fMr. Then*

- *$F^\circ(G)$ contains G' and all p-unipotent radicals of G,*
- *$G/F^\circ(G)$ is divisible abelian, and*
- *(Fitting's Theorem) $G/Z(F(G)) \leq \mathrm{Aut}(F(G))$.*

In line with the algebraic theory, we cal a maximal connected definable solvable subgroup of a group of fMr a **Borel subgroup**. The previous fact show some structural similarity with algebraic Borel subgroups, but plenty is still unknown. And crucially, it is currently unknown if Borel subgroups must be conjugate—this is a very serious complication.

5. Involutions and 2-groups

Though the previous section illustrated many similarities of groups of fMr with algebraic groups, it also hinted any several quite serious potential deviations. Lacking many of the algebraic tools, the theory of groups of fMr, guided by Borovik, turned to techniques of finite group theory, which consequently lead to a healthy obsession with involutions (i.e. elements of order 2).

Definition 5.1. An **involution** is an element of order 2. An element of a group is called **strongly real** if it is the product of two involutions.

Note that if $x = ij$ is strongly real (for i, j involutions), then $x^i = ixi = iiji = ji = x^{-1}$; that is, every strongly real element is inverted by any involution that "it is based on." The next exercise is essentially the converse.

Exercise 5.2. Let G be any group. If $x \in G$ is inverted by i, show that either $x = i$ or x is strongly real (and based on i).
Hint: $x = xii$.

5.1. **A couple of useful results.** The next few results illustrate some of the many useful properties of involutions in groups of fMr.

Lemma 5.3. *If i, j are involutions in a group of fMr G, then either i and j are conjugate or commute with a third involution.*

Proof. Let $x = ij$, and let $A = \mathrm{d}(x)$. By Lemma 3.7, A is abelian; consequently the (definable) subset $A^- \subseteq A$, consisting of those elements that are inverted by i, is in fact a subgroup. As $x \in A^-$, the definition of definable hull forces $A^- = A$, so i inverts all elements in A.

Now, if A contains an involution k, then k (being inverted by i) is centralized by i. Of course, the argument applies to j as well, so in this case, i and j commute with the involution k. It remains to treat the case when A has no involutions, which by Exercise 4.3 implies that A is uniquely 2-divisible. Let $a = \sqrt{x}$ (i.e. $x = a^2$), and remember that i inverts $a^{-1} \in A$. Now, observe that

$$i^a = a^{-1}ia = iia^{-1}ia = ia^2 = ix = iij = j.$$

□

Exercise 5.4. Let G be a group of fMr. Suppose that some involution $i \in G$ normalizes a definable subgroup H. Show that if $C_H(i)$ is finite, then i inverts H (and H is abelian).
Hint: note that i inverts the set $X := \{h^{-1}h^i : h \in H\}$. Determine the rank of the fibers of the map $H \to X : h \mapsto h^{-1}h^i$, and conclude that $\operatorname{rk} X = \operatorname{rk} H$ *(i.e. that X is* generic *in H). Fix $a \in X$. Show that $X \cap aX$ is contained in $C_H(a)$ and is generic in H. Conclude that $C_H(a) = H$, so $X \subset Z(H)$. After showing that H is abelian, you then have that X is a* subgroup *of H.*

Here is a gem.

Fact 5.5 ("Brauer-Fowler" Theorem, see [BN94, Theorem 10.3])**.** *If i is any involution in a group of fMr G, then there exists a nontrivial strongly real element x for which* $\operatorname{rk} G \le 2 \operatorname{rk} C_G(i) + \operatorname{rk} C_G(x)$.

5.2. **Sylow theory for the prime 2.** Here, as in the finite context, we define a Sylow 2-subgroup of a group to be a maximal 2-subgroup. In groups of fMr, a Sylow 2-subgroup is quite likely *not* definable. The reader is trusted to not underestimate the importance of this result.

Fact 5.6 (Borovik-Poizat—1990)**.** *In any group of fMr, the Sylow 2-subgroups are conjugate.*

This conjugacy result yields, as usual, a Frattini argument. Here is one example.

Exercise 5.7. Let G be any group, and let P be a Sylow 2-subgroup. Show that $G = G^\circ N_G(P)$.

There is also a structure theorem for 2-subgroups. It requires a quick definition: the connected component of a (not necessary definable) subgroup H is defined to be $H^\circ := H \cap d^\circ(H)$.

Fact 5.8. *In any group of fMr, the Sylow 2-subgroups are nilpotent, and the connected component of a Sylow 2-subgroup is of the form $U * T$ where U is 2-unipotent and T is a 2-torus.*

The previous two fact give rise to a case division for analyzing groups of fMr; the names refer to the characteristic of a possible definable field hiding in the group.

Definition 5.9. A group of fMr is said to be of **odd**, **even**, **mixed**, or **degenerate** type according to the structure of a Sylow 2-subgroup P:

Even type: P° is nontrivial and 2-unipotent;
Odd type: P° is a nontrivial 2-tori;
Mixed type: P° contains a nontrivial 2-unipotent subgroup and a nontrivial 2-torus;
Degenerate type: P° is finite.

6. Algebraicity Conjecture

Here it is again, in all it's glory.

Algebraicity Conjecture ([Che79, Zb77])**.** Every infinite *simple* group of fMr is isomorphic to an algebraic group over an algebraically closed field.

Status. . .

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