1.1 Systems of Linear Equations

We will dive into the math and agree upon some language. Application come soon...

Ex Consider the following system.

$$\left.\begin{aligned} x + 2y &= 1 \\ 3x + 4y &= -1 \end{aligned}\right\} \text{ system of 2 eq. in 2 unknowns}$$

x & y

(a) Is $(0,0)$ a solution to the system? What about $(3,-1)$?

- $(0,0)$ is not, b/c $0+2(0)=1$ is <u>NOT</u> true.
- $(3,-1)$ is not. It is a solution to the first, but not both.

(b) Find all solutions to the linear system

<u>We use elimination</u>

- Idea: transform the system into an easier system by "eliminating" some variables.

equivalent: same solution set

$$\begin{aligned} x+2y&=1 \\ 3x+4y&=-1 \end{aligned} \quad \underset{-3r_1+r_2 \to r_2}{\sim} \quad \begin{aligned} x+2y&=1 \\ 0+-2y&=-4 \end{aligned} \quad \underset{-\frac{1}{2}r_2 \to r_2}{\sim} \quad \begin{aligned} x+2y&=1 \\ y&=2 \end{aligned}$$

these imply other true equations

$$-3x-6y=-3$$

AND

$$\begin{array}{rll} & -3x-6y=-3 & (-3r_1) \\ + & \underline{3x+4y=-1} & (r_2) \\ & 0-2y=-4 & (-3r_1+r_2) \end{array}$$

∘ now "back substitute" to find x

there is only 1 solution: $x = -3$, $y = 2$ or $\boxed{(-3, 2)}$

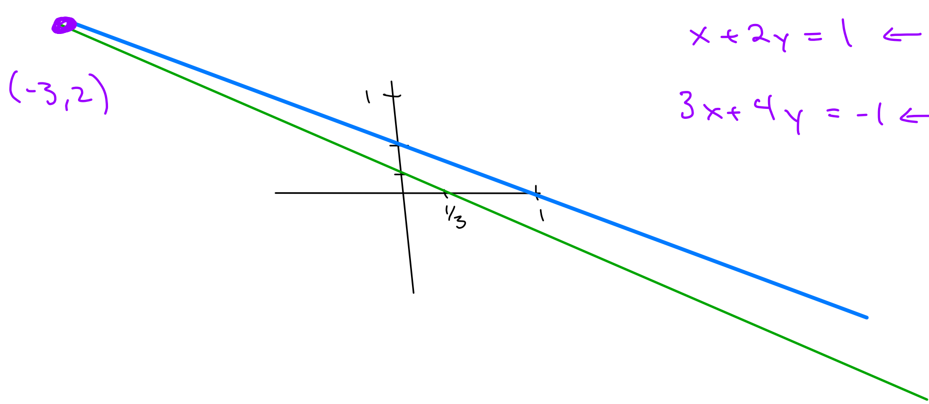
(c) Interpret the solution set graphically.

∘ solutions to $x + 2y = 1$ are the points on this line.

∘ " " $3x + 4y = -1$ " " " " " "

∘ Thus, the solutions to the system, lie on the intersection of the lines.

Graphical interpretation



$x + 2y = 1 \leftarrow$ line: $\boxed{y = -\frac{1}{2}x + \frac{1}{2}}$

$3x + 4y = -1 \leftarrow$ line: $\boxed{y = \frac{-3}{4}x - \frac{1}{4}}$

When manipulating linear systems, we do <u>not</u> want to change the solution sets. What are the allowed operations?

Def Elementary Row Operation (see book too)

1. (Replacement) Replace a row by itself plus any multiple of another row. $r_i + c r_j \to r_i$
2. (Interchange) swap any two rows. $r_i \leftrightarrow r_j$
3. (Scaling) Multiply any row by a nonzero number. $c r_i \to r_i$
   $c \neq 0$

* In first example, we used replacement and scaling.
* If one system can be transformed into another using a series of row operations, we say that the systems are row equivalent.

Theorem If two systems are row equivalent, then they have the same solution set.

Matrix Notation

Let's do this by example.

Ex Convert the following to augmented matrix form and then solve.

$$\begin{aligned} x_1 - 3x_2 &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

book does not use

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] \underset{r_1+r_2 \to r_2}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\underset{r_2 \leftrightarrow r_3}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{array}\right]$$

$$\underset{2r_2+r_3 \to r_3}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{array}\right]$$

$$\underset{\frac{1}{7}r_3 \to r_3}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus, original system is row equivalent to

$$\begin{aligned} x_1 - 3x_2 \quad &= 5 \\ x_2 + x_3 &= 0 \\ x_3 &= 1 \end{aligned}$$

Now, we can back substitute.

$x_3 = 1$

$x_2 = -x_3 = -1$ So, only one solution: $\boxed{(2,-1,1)}$

$x_1 = 5 + 3x_2 = 2$

Q: Can you interpret this problem geometrically?

A: There are 3 planes and we are looking for common points of intersection

$\boxed{\text{use geogebra.org!}}$

How many solutions can linear systems have?

Think geometrically. In the case of 2 variables we are thinking of something like...

$$\begin{aligned} x + 2y &= 1 \\ 3x + 4y &= -1 \end{aligned} \qquad \underline{\text{OR}} \qquad \begin{aligned} x + y &= 1 \\ x + 2y &= 2 \\ -x + 3y &= 7 \end{aligned}$$

Then, the question about solutions is the same as asking about where the lines simultateously intersect. What are the possibilities:

- intersect in just one point (like the one on left)
- they have no common points of intersection
  - eg. the one on left
  - e.g. parallel lines
- they have infinitely many points of intersection
  - e.g. two lines that are the same
  - e.g. with 3 variables, we could have 2 planes intersecting in a line.

<u>Summary</u> Linear systems must have 0, 1, or ∞-many solutions. If it has at least 1, we say the system is <u>consistent</u>. Otherwise, it is <u>inconsistent</u>.

<u>Ex</u> Determine if the following system is consistent.

$$\begin{aligned} x + y &= 1 \\ x + 2y &= 2 \\ -x + 3y &= 7 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ -1 & 3 & 7 \end{array}\right] \underset{\substack{-r_1+r_2 \to r_2 \\ r_1+r_3 \to r_3}}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 8 \end{array}\right] \underset{-4r_2+r_3 \to r_3}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{array}\right]$$

so system is row equiv. to

$$\begin{aligned} x + y &= 1 \\ 0 + y &= 1 \\ 0 + 0 &= 4 \end{aligned}$$

no solution!

$\boxed{\text{Inconsistent}}$

Q: Can you interpret this problem geometrically?

A: There are three lines with no common point of intersection.

$\boxed{\text{Use geogebra.org!}}$

Ex Find all solutions to the system

$$\left.\begin{array}{l} 2x - y + 3z = 4 \\ 2x + 3y - 5z = 0 \end{array}\right\}$$

2 eqs. in 3 unknowns

sol

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 2 & 3 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 4 & -8 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$$\begin{array}{r} 2x - y + 3z = 4 \\ y - 2z = -1 \end{array} \implies \begin{array}{l} x = \frac{y}{2} - \frac{3}{2}z + 2 \\ y = 2z - 1 \end{array} \implies \boxed{\begin{array}{l} x = -\frac{1}{2}z + \frac{3}{2} \\ y = 2z - 1 \\ z \text{ is free} \end{array}}$$

every choice for z yields a different solution!

parametric form

$$\begin{cases} x = -\frac{1}{2}t + 3\,2 \\ y = 2t - 1 \\ z = t \end{cases}$$

t is any number

what does this represent geometrically?

Q: Can you interpret this problem geometrically?

A: There are 2 planes that intersect in a line.

Use geogebra.org!

1.2 Row Reduction & Echelon Forms

Goal: Develop an algorithm for solving linear systems.

Suppose you know ...

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ -3 & 2 & 0 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

which is easiest to solve?

2nd is not bad

3rd is easiest

$x_1 = -2$
$x_2 = -1$
$x_3 = 3$

Echelon forms

$\boxed{\text{Handout 01}}$

01 – Row Echelon Form

Definition: Row Echelon Forms

A matrix A is in **row echelon form (REF)** if

1. all nonzero rows lie above any rows of all zeros;

2. the leading entry (from the left) of each nonzero row is strictly to the right of the leading entry of the row above it.

If, additionally, A satisfies

3. the leading entry (from the left) of each nonzero row is a 1 (called the **leading one**);

4. each leading one is the only nonzero entry in its column

then A is in **reduced row echelon form (RREF)**.

1. Determine if each of the following are in REF or RREF.

(a) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 7 & 6 \\ 2 & 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 6 & 7 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 0 & 1 & 6 \\ 0 & 2 & 4 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Definition: Elementary Row Operations

An **elementary row operation** on a matrix is any of the following.

Replacement: add to one row any multiple of another row $(cr_i + r_j \to r_j)$

Interchange: interchange two rows $(r_i \leftrightarrow r_j)$

Scale: multiply a row by a nonzero scalar $(cr_i \to r_i)$

2. Look back at the matrices in the previous example.

(a) For each matrix that was not in REF, find a sequence of elementary row operations that could be used to transform it into REF.

(b) For each matrix that was already in REF, find a sequence of elementary row operations that could be used to transform it into RREF.

Row reduction Algorithm

$\boxed{\text{HO-02}}$ all (see next page)

<u>Ex</u> Determine how many solutions the corresponding systems have

(a) $\left[\begin{array}{ccc|c} 1 & 7 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$ $\quad$ 1

(b) $\left[\begin{array}{ccc|c} 1 & 7 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\quad$ None

pivot in last column so inconsistent

(c) $\left[\begin{array}{ccc|c} 1 & 7 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad$ ∞-many

consistent with free variable

<u>Ex</u> (From WeBWork)

For what value of k is the linear system inconsistent

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & -2 \\ 1 & 2 & -4 & 2 \\ 3 & 9 & k & 19 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & -2 \\ 1 & 2 & -4 & 2 \\ 3 & 9 & k & 19 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 1 & 4 & -2 \\ 0 & 1 & -8 & 4 \\ 0 & 6 & k-12 & 25 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 1 & 4 & -2 \\ 0 & 1 & -8 & 4 \\ 0 & 0 & k+36 & 1 \end{array}\right]$$

inconsistent if and only if $k+36=0$ $\quad \boxed{k=-36}$

02 – Row Reduction

Strategy: Row Reduction

To transform a matrix to REF or REF, use the following algorithm.

1. Find the leftmost nonzero column—this is called the **pivot column**.
2. Choose a nonzero entry in the pivot column—this will be called the **pivot**. If necessary, use INTERCHANGE operations to make sure the pivot is in the top row.
3. Use REPLACEMENT operations to create zeros *below* the pivot.
4. Cover up (or ignore) the row containing the pivot, and repeat steps 1–3 on the smaller matrix below. Continue repeating until there are no more nonzero rows to modify.

At this point, the matrix is in REF. To transform to RREF, continue with the process below.

5. Beginning with the rightmost pivot, working up and to the left, use REPLACEMENT operations to create zeros *above* each pivot. If a pivot is not 1, use a SCALING operation to make it 1.

At this point, the matrix is in RREF.

1. Row reduce the following matrix to RREF, and determine if the corresponding linear system corresponding is consistent or not.

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

RREF

corresponding system is

$$x_1 - x_3 = 0$$
$$x_2 + 2x_3 = 0$$
$$0 = 1$$

So inconsistent

2. Solve the following linear system by reducing the corresponding augmented matrix to RREF.

$$\begin{aligned} x_2 + x_3 &= 2 \\ -3x_1 + 2x_2 &= 4 \\ x_1 + x_3 &= 1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ -3 & 2 & 0 & 4 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ -3 & 2 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Longrightarrow \begin{aligned} x_1 &= -2 \\ x_2 &= -1 \\ x_3 &= 3 \end{aligned}$$

3. Solve the system given in augmented matrix form as $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$

basic variables

free variables

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\boxed{\begin{aligned} &x_1 = 5 + 7x_2 - 6x_4 \\ &x_2 \text{ is free} \\ &x_3 = -3 + 2x_4 \\ &x_4 \text{ is free} \end{aligned}}$$

$$\begin{aligned} x_1 - 7x_2 + 6x_4 &= 5 \\ x_3 - 2x_4 &= -3 \end{aligned} \qquad \begin{aligned} x_1 &= 5 + 7x_2 - 6x_4 \\ x_3 &= -3 + 2x_4 \end{aligned}$$

solve for basic variables in terms of free variables

Definition: Basic and Free Variables

If A is the augmented matrix of a linear system, then

- the variables corresponding to pivot columns are called **basic variables**, and
- the variables corresponding to columns with NO pivot are called **free variables**.

Theorem: Existence and Uniqueness Theorem

- A linear system is inconsistent if and only if the RREF has a row of the form $\begin{bmatrix} 0 & 0 & \cdots & 0 & b \end{bmatrix}$ with $b \neq 0$.
- If a linear system is consistent, then it has infinitely-many solutions if and only if there are free variables.

1.3 Vector Equations

Goal: to have some other ways to view linear systems.

Notation

- $\mathbb{R}$ denotes the set of real numbers.

  in linear algebra, these are also called scalars.

- $\mathbb{R}^2$ denotes the set of 2×1 matrices

  e.g. $\begin{bmatrix} 7 \\ 2 \end{bmatrix}, \begin{bmatrix} \pi+1 \\ -3 \end{bmatrix}$

  called column vectors (or just vectors)

- $\mathbb{R}^n$ denotes the set of $n\times 1$ matrices

Operations on $\mathbb{R}^n$

(1) Scalar multiplication (by example)

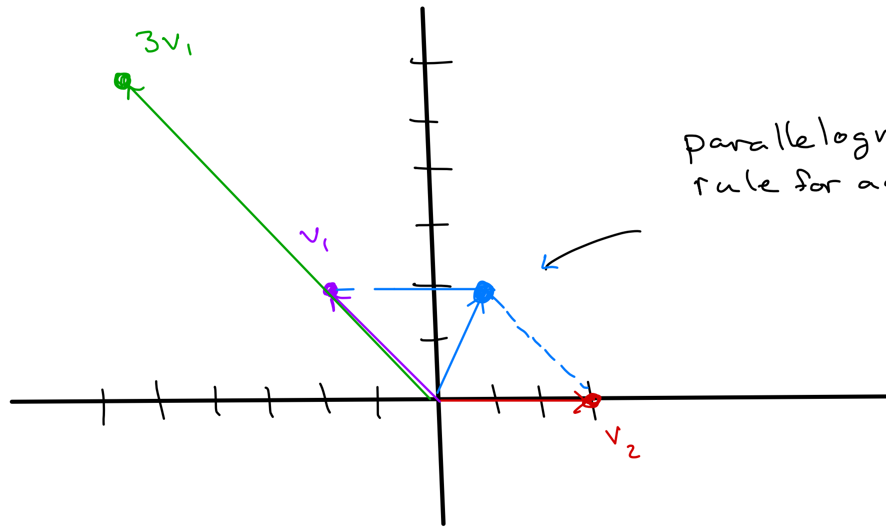
- $7\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -14 \\ 35 \end{bmatrix}$
- $c\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \end{bmatrix}$

(2) vector addition (by example)

- $\begin{bmatrix} 2 \\ -5 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$
- $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix}$

Ex Let $\bar{v}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Graph $\bar{v}_1, \bar{v}_2, \bar{v}_1 + \bar{v}_2, 3v_1$.

think $\begin{bmatrix} x \\ y \end{bmatrix}$



parallelogram rule for addition

Ex Let $\bar{v}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

① compute $-3v_1 - 2v_2$.

$$-3 \begin{bmatrix} -2 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$$

vector equation

② Can you solve $x\bar{v}_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ for x?

Algebraically

$$x\bar{v}_1 = \begin{bmatrix} -2x \\ 2x \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \Longrightarrow \begin{matrix} -2x = 1 \\ -2x = 5 \end{matrix} \Rightarrow \begin{matrix} x = -\frac{1}{2} \\ x = -\frac{5}{2} \end{matrix}$$

No solution!

Graphically

$x\bar{v}_1$ lies on the line through $(0,0)$ and $(-2,2)$. (See picture above.)

But $(1,5)$ is not on this line.

No solution!

③ Can you solve $x_1\bar{v}_1 + x_2\bar{v}_2 = \begin{bmatrix}1\\5\end{bmatrix}$?

$$x_1\bar{v}_1 + x_2\bar{v}_2 = \begin{bmatrix}1\\5\end{bmatrix} \iff \begin{bmatrix}-2x_1\\2x_1\end{bmatrix} + \begin{bmatrix}3x_2\\0x_2\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}$$

$$\iff \begin{bmatrix}-2x_1+3x_2\\2x_1+0x_2\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}$$

$$\iff \begin{aligned}-2x_1+3x_2&=1\\2x_1+0x_2&=5\end{aligned}$$

$$\left[\begin{array}{cc|c}-2&3&1\\2&0&5\end{array}\right] \sim \left[\begin{array}{cc|c}-2&3&1\\0&3&6\end{array}\right] \sim \left[\begin{array}{cc|c}-2&3&1\\0&1&2\end{array}\right]$$

$$\sim \left[\begin{array}{cc|c}-2&0&-5\\0&1&2\end{array}\right] \sim \left[\begin{array}{cc|c}1&0&5/2\\0&1&2\end{array}\right] \qquad \begin{aligned}x_1&=5/2\\x_2&=2.\end{aligned}$$

important

Fact A vector equation $x_1\bar{v}_1 + x_2\bar{v}_2 + \dots + x_n\bar{v}_n = \bar{b}$ has the same solution set as the linear system with augmented matrix $[\bar{v}_1 \ \ \bar{v}_2 \ \dots \ \bar{v}_n \ \ \bar{b}]$

1^{st} col. is v_1 $\quad$ 2^{nd} col is v_2 ...

Ex Make up a linear system. Then write an equivalent vector equation.

$$\begin{aligned}4x_1 - x_2 \quad &= 7\\ x_1 - x_2 + x_3 &= 0\\ 3x_2 - x_3 &= 1\end{aligned} \quad \rightsquigarrow \quad x_1\begin{bmatrix}4\\1\\0\end{bmatrix} + x_2\begin{bmatrix}-1\\-1\\3\end{bmatrix} + x_3\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}7\\0\\1\end{bmatrix}$$

Linear combinations

Def If $\bar{v}_1, \dots, \bar{v}_k$ are in $\mathbb{R}^n$, then for any scalars $c_1, \dots, c_k$ in $\mathbb{R}$, the new vector $c_1\bar{v}_1 + \cdots + c_k\bar{v}_k$ is called a <u>linear combination</u> of $\bar{v}_1, \dots, \bar{v}_k$ with <u>weights</u> $c_1, \dots, c_k$.

Ex Let $\bar{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$.

① write 2 different linear combinations of $\bar{v}_1, \bar{v}_2, \bar{v}_3$.

Many possibilities, e.g. $1v_1 + 2v_2 + 3v_3 = \begin{bmatrix} 7 \\ 8 \\ 45 \end{bmatrix}$

② Is $\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$ a lin. comb. of $\bar{v}_1, \bar{v}_2, \bar{v}_3$?

Yes! $\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = -1v_1 + 0v_2 + 0v_3$

③ Is $\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ a lin. comb. of $\bar{v}_1, \bar{v}_2, \bar{v}_3$?

... is there a solution to $x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$?

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$\boxed{\text{No}}$

Optional

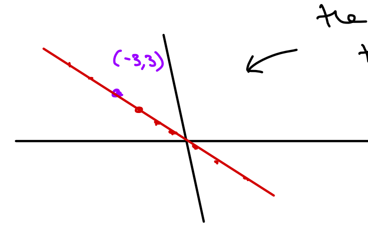
Ex Describe the collection of all linear combinations of $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$ algebraically and geometrically.

<u>Algebraically</u>

$$x \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -3x \\ 3x \end{bmatrix}$$

for any x in $\mathbb{R}$

<u>Geometrically</u>



the line through the origin and (-3,3)

optional

Ex Show that every vector in $\mathbb{R}^3$ is a linear combination of $\bar{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, and $\bar{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Span

Def The collection of all possible linear combinations of $\bar{v}_1, \ldots, \bar{v}_k$ is written $\text{Span}\{\bar{v}_1, \ldots, \bar{v}_k\}$. It is called the subset spanned by $\bar{v}_1, \ldots, \bar{v}_k$.

Ex We have seen that...

optional

① $\text{Span}\left\{\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^3$. (see prev. example)

② $\text{Span}\left\{\begin{bmatrix} -3 \\ 3 \end{bmatrix}\right\}$ can be described as the subset of $\mathbb{R}^2$ lying on the line through $(0,0)$ and $(-3,3)$. (see example 2 back)

③ $\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ is not in $\text{Span}\left\{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}\right\}$.

(see example 3 back)

Ex Suppose you want to determine if $\overset{\bar{b}}{\begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}}$ is in $\text{span}\left\{\overset{\bar{a}_1}{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}}, \overset{\bar{a}_2}{\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}}\right\}$. What is the process?

Thinking to self... need to determine if

$$x_1 \underset{\bar{a}_1}{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}} + x_2 \underset{\bar{a}_2}{\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}} = \underset{\bar{b}}{\begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}}$$

has a solution.

① Create the augmented matrix $[\bar{a}_1 \ \bar{a}_2 \ \bar{b}]$

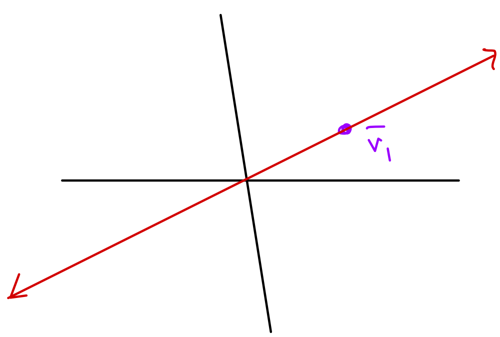
$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 6 & 3 & 1 \\ -2 & 6 & -4 \end{array}\right]$$

② Row reduce and solve the corresponding system.

- $\bar{b}$ is in span$\{\bar{a}_1, \bar{a}_2\}$ if there is a solution.
- otherwise, $\bar{b}$ is not in the span.

<u>Geometric interpretation of span</u>

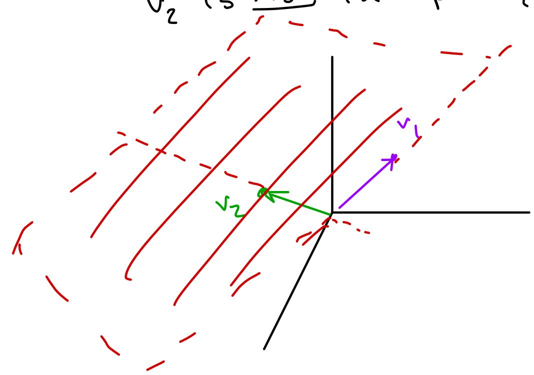
① Assume $\bar{v}_1 \neq \bar{0}$.



Span$\{\bar{v}_1\}$ is all vectors on the line through $\bar{v}_1$ and the origin

Q: what if $\bar{v}_1 = \bar{0}$?

② Assume $\bar{v}_1 \neq 0$, $\bar{v}_2 \neq 0$ <u>AND</u> $\bar{v}_2$ is <u>not</u> in Span$\{v_1\}$



Span$\{\bar{v}_1, \bar{v}_2\}$ is all vectors on the plane determined by $\bar{v}_1, \bar{v}_2$, and $\bar{0}$.

Q: what if $\bar{v}_2$ <u>is</u> in Span$\{\bar{v}_2\}$

1.4 The Matrix Equation $Ax = b$

Goal: another (important) view of linear systems

... first, the beginnings of matrix multiplication

$\boxed{\text{HO-03}}$ Def MVP, # 1 (See next page)

Ex Rewrite the linear combination as a matrix-vector product.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Notice that the following problems all have the same solution sets.

① Solve

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

①′ Solve

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 3 & 1 \end{array}\right]$$

② Solve

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

③ Solve

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$\boxed{\text{HO-03}}$ Theorem 1 (pg 2) (See next page)

03 – Matrix-Vector Products

typo !

Definition: Matrix-Vector Product (MVP)

Suppose that A is an $m \times n$ matrix, and let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be in $\mathbb{R}$. n Let $\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_n$ be the columns of A. Then we define the product $A\mathbf{x}$ by

must match !

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

1. Compute the following.

match!

2×3   3×1

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ 6 \end{bmatrix}}$

don't match

2×3   2×1

(b) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ undefined

match !

4×2   2×1

(c) $\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = -2\begin{bmatrix} 7 \\ 2 \\ 9 \\ -3 \end{bmatrix} - 5\begin{bmatrix} -3 \\ 1 \\ -6 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}}$

Theorem

Suppose that A is an $m \times n$ matrix, and let $\mathbf{b}$ be in $\mathbb{R}$. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ be the columns of A. Then each of the following have exactly the same solution sets.

- **Matrix equation:** $A\mathbf{x} = \mathbf{b}$
- **Vector equation (with columns of A):** $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$
- **Linear system (as an augmented matrix):** $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \mid \mathbf{b} \end{bmatrix}$

Theorem

Suppose that A is an $m \times n$ matrix. Then the following are logically equivalent. (If one is true, they all are; if one is not true, none are.)

(a) For every $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

- In other words, the system with augmented matrix $\begin{bmatrix} A \mid \mathbf{b} \end{bmatrix}$ always has a solution.

(b) For every $\mathbf{b}$ in $\mathbb{R}^m$, $\mathbf{b}$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

- In other words, if $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ are the columns of A, $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n\} = \mathbb{R}^m$.

(d) A has a pivot position in every row.

2. Determine if $A\mathbf{x} = \mathbf{b}$ has a solution for every choice of $\mathbf{b}$ in each case below.

(a) $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ -5 & 6 & 11 \\ 3 & -4 & -2 \\ 3 & 0.5 & 0 \end{bmatrix}$

When is $A x = \bar{b}$ consistent for all $\bar{b}$?

Thus far we have asked something like

Is $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ consistent?

Now, we'll ask $\quad A\bar{x} = \bar{b}$

Is $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ consistent

for every choice of b_1, b_2?

Let's explore this. In this example,

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & 4 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\bar{b}} \iff \left[\begin{array}{ccc|c} 1 & 1 & 0 & b_1 \\ -1 & 3 & 4 & b_2 \end{array}\right]$$

is consistent for all $\bar{b}$ $\iff$ is consistent for all $\bar{b}$

RREF

$$\iff \left[\begin{array}{ccc|c} 1 & 0 & -2 & b_1 - \frac{b_1+b_2}{2} \\ 0 & 1 & 2 & \frac{b_1+b_2}{2} \end{array}\right]$$

is consistent for all $\bar{b}$

so, it <u>is</u> consistent for all $\bar{b}$

b/c pivot in each row

of coeff. matrix A

But, what if it had been $\begin{bmatrix} 1 & -1 & 0 \\ -3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$?

$$\begin{bmatrix} 1 & -1 & 0 \\ -3 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$ is consistent for all $\bar{b}$

$\iff$ $\left[\begin{array}{ccc|c} 1 & -1 & 0 & b_1 \\ -3 & 3 & 0 & b_2 \end{array}\right]$ is consistent for all $\bar{b}$

RREF

$\iff$ $\left[\begin{array}{ccc|c} 1 & -1 & 0 & b_1 \\ 0 & 0 & 0 & b_2+3b_1 \end{array}\right]$ is consistent for all $\bar{b}$

so, it is NOT consistent for all $\bar{b}$

b/c there is NOT a pivot in each row of $\underbrace{\text{coeff. matrix}}_{A}$

$\boxed{\text{HO - 03}}$ Theorem 2 (pg 2), #2 (see 2 pages back)

1.5 Solution Sets of Linear Systems

Homogeneous Systems

<u>Def</u> A linear system of the form $A\bar{x} = \bar{0}$ is called <u>homogeneous</u>.

* Notice that homogeneous systems are always consistent: $\bar{x} = \bar{0}$ is always a solution.

  called the <u>trivial</u> solution

* A homogeneous system has a nontrivial solution precisely when there is at least one free variable.

Parametric Vector Form for solution sets

<u>Ex</u> Show that the following homogeneous system has nontrivial solutions and describe the solution set parametrically.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

Observe that

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - 4/3\, x_3 &= 0 \\ x_2 &= 0 \\ x_3 &\text{ is free} \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 4/3\, x_3 \\ x_2 &= 0 \\ x_3 &\text{ is free} \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 4/3\, s \\ x_2 &= 0 \\ x_3 &= s \end{aligned} \quad \text{for } s \text{ in } \mathbb{R}$$

<u>vector form</u>

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\, s \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \quad \text{so}$$

solution set is

$$\boxed{\bar{x} = s \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \quad (s \text{ in } \mathbb{R})}$$

Notice:

* there are ∞-many solutions.
* the solution set is represented by a line!
* the solution set is $\mathrm{Span}\left\{ \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Ex Describe all solutions to $A\bar{x} = \bar{0}$ in parametric vector form, assuming that

$$A \sim \begin{bmatrix} 1 & -4 & -2 & 0 & 3 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now,

$$[A \mid \bar{0}] \sim \left[\begin{array}{ccccc|c} 1 & -4 & 0 & -14 & 0 & 0 \\ 0 & 0 & 1 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 4x_2 - 2x_4$
x_2 is free
$x_3 = 7x_4$
x_4 is free
$x_5 = 0$

$$\Longrightarrow \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4s - 2t \\ s \\ 7t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 4s \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ 0 \\ 7t \\ t \\ 0 \end{bmatrix}$$

so

$$\boxed{\bar{x} = s \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} \quad (s, t \text{ in } \mathbb{R})}$$

* solution set is a plane (in $\mathbb{R}^5$)
* solution set is $\text{Span}\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} \right\}$

OR s_1 and s_2 (as in WeBWork)

Solutions to Nonhomogeneous Systems

<u>Ex</u> Describe all solutions to $A\bar{x} = \bar{b}$ in parametric vector form where

$$A = \begin{bmatrix} 3 & -4 & 5 & 0 \\ -3 & 4 & -2 & 3 \\ 6 & -8 & 1 & -9 \end{bmatrix} \quad \text{and} \quad \bar{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

As before, observe that

$$A = \left[\begin{array}{cccc|c} 3 & -4 & 5 & 0 & 7 \\ -3 & 4 & -2 & 3 & -1 \\ 6 & -8 & 1 & -9 & -4 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & -4/3 & 0 & -5/3 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -1 + 4/3\, x_2 + 5/3\, x_4$

s_1 — x_2 free

$x_3 = 2 - x_4$

s_2 — x_4 free

$$\bar{x} = \begin{bmatrix} -1 + 4/3\, s_1 + 5/3\, s_2 \\ s_1 \\ 2 - s_2 \\ s_2 \end{bmatrix} = \boxed{\underbrace{\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}}_{\bar{p}} + \underbrace{s_1 \begin{bmatrix} 4/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 5/3 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{these are all solutions to } A\bar{x} = \bar{0}}}$$

$\ast$ The solutions to $A\bar{x} = \bar{b}$ are of the form $\bar{p} + \bar{v}_h$ where $\bar{v}_h$ is a solution to $A\bar{x} = \bar{0}$.

* Graphically the solution set is a plane through the origin shifted by the vector $\overline{p}$

<u>Theorem</u> Suppose that $A\overline{x} = \overline{b}$ is consistent and that $\overline{p}$ is any one particular solution. Then the set of all solutions to $A\overline{x} = \overline{b}$ are the vectors of the form $\overline{w} = \overline{p} + \overline{v}_h$ where $\overline{v}_h$ is any solution to the homogeneous equation $A\overline{x} = \overline{0}$.

1.6 Applications

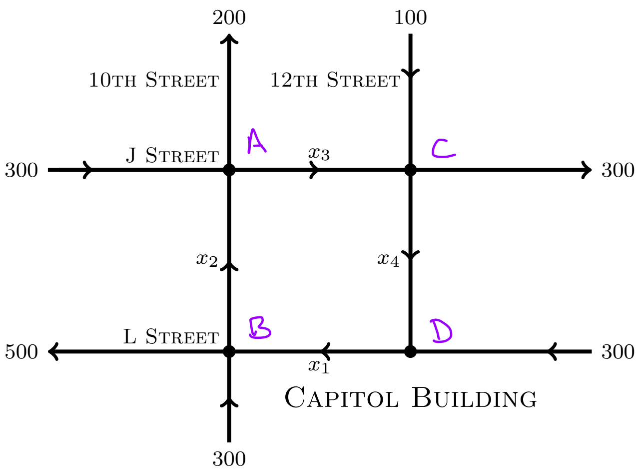
Several applications are presented. We will only look at network flow.

HO-04

See next page.

04 – Applications

1. The network below shows the approximate traffic flow in vehicles per hour over various one-way streets in downtown Sacramento near the capitol building.



Main Idea:

Flow in = Flow out at each intersection.

A $\quad x_2 + 300 = x_3 + 200$

B $\quad x_1 + 300 = x_2 + 500$

C $\quad x_3 + 100 = x_4 + 300$

D $\quad x_4 + 300 = x_1$

(a) Determine the general flow pattern.

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & -100 \\ 1 & -1 & 0 & 0 & 200 \\ 0 & 0 & 1 & -1 & 200 \\ -1 & 0 & 0 & 1 & -300 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 200 \\ 0 & 1 & -1 & 0 & -100 \\ 0 & 0 & 1 & -1 & 200 \\ -1 & 0 & 0 & 1 & -300 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 200 \\ 0 & 1 & -1 & 0 & -100 \\ 0 & 0 & 1 & -1 & 200 \\ 0 & -1 & 0 & 1 & -100 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 100 \\ 0 & 1 & -1 & 0 & -100 \\ 0 & 0 & 1 & -1 & 200 \\ 0 & 0 & -1 & 1 & -200 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 300 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 1 & -1 & 200 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 300 + x_4$
$x_2 = 100 + x_4$
$x_3 = 200 + x_4$
x_4 free

(b) What is the smallest possible value for x_1? Why?

- $x_1 \geq 300$ b/c $x_4 \geq 0$

(c) Suppose that $x_4 = 150$. Determine the values for the remaining roads.

$x_1 = 450$
$x_2 = 250$
$x_3 = 350$

1.7 Linear Independence

Ex Consider the vectors:

$$\bar{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \bar{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}.$$

(a) How many possible solutions are there to

$$x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 = \bar{0}$$

Ans. There is at least 1, since it is homogeneous.

Thus $\boxed{1 \text{ or } \infty\text{-many}}$

(b) Determine if $x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 = \bar{0}$ has a nontrivial solution.

$$\left[\begin{array}{ccc|c} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

no free variables so only one solution which must be $\bar{x} = \bar{0}$.

Thus, in this case, there are no nontrivial solutions.

Def The vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ are called linearly independent if $x_1\bar{v}_1 + x_2\bar{v}_2 + \cdots + x_k\bar{v}_k$ has only the trivial solution. If it has a nontrivial solution, they are called linearly dependent.

Ex the vectors in the 1st example are linearly independent.

Ex Show that the following vectors are linearly dependent and find a linear dependence relation.

$$\bar{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} -4 \\ 4 \\ -8 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 3 \\ -9 \end{bmatrix}$$

* Need to find a non trivial sol. to $x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 + x_4\bar{v}_4 = \bar{0}$

$$\left[\begin{array}{cccc|c} 3 & -4 & 5 & 0 & 0 \\ -3 & 4 & -2 & 3 & 0 \\ 6 & -8 & 1 & -9 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -4/3 & 0 & -5/3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 4/3\, x_2 + 5/3\, x_4$

x_2 free

$x_3 = -x_4$

x_4 free

$\rightsquigarrow$ we need 1 nontrivial sol., so let's pick $x_2 = 1$, $x_4 = 1$. Thus

$$\boxed{3\bar{v}_1 + \bar{v}_2 - \bar{v}_3 + \bar{v}_4 = 0}$$

Ex In the previous example, how could we have known that $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$ were L.D. without doing any work?

Note: the coeff. matrix of the system

$$\left[\begin{array}{cccc|c} 3 & -4 & 5 & 0 & 0 \\ -3 & 4 & -2 & 3 & 0 \\ 6 & -8 & 1 & -9 & 0 \end{array}\right]$$

is 3×4 so not every column of the coeff. can have a pivot. Since the system is consistent (b/c it in homogeneous), there will be at least one free variable — hence nontrivial solutions.

i.e. if more vectors than length of vectors

<u>Theorem</u> Suppose that $\bar{v}_1, \dots, \bar{v}_k$ are vectors in $\mathbb{R}^n$.

If $k > n$, then $\bar{v}_1, \dots, \bar{v}_k$ must be linearly dependent.

* If $k \le n$, the vectors may or may not be L.D.

in previous example $\bar{v}_1, \bar{v}_2$ are still L.D.

Another observation...

<u>Theorem</u> If at least one of $\bar{v}_1, \dots, \bar{v}_k$ is the $\bar{0}$ vector than they are L.D.

* why? Suppose $\bar{v}_1 = \bar{0}$. Then, $1 \cdot \bar{v}_1 + 0\bar{v}_2 + \dots + 0\bar{v}_k = \bar{0}$.

Linear Dep. for 1 or 2 vectors

<u>Theorem</u> One vector $\bar{v}_1$ is linearly dependent if and only if ... $\bar{v}_1 = \bar{0}$.

<u>Pf</u> not 0

$$\underset{\text{not } 0}{x_1} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{\bar{v}_1} = x_1 \bar{v}_1 = \bar{0} \text{ has a nontrivial sol.} \iff \bar{v}_1 = \bar{0}$$

* complete the sentence: $\bar{v}_1$ is L.I. iff ______

* Geometrically: $\bar{v}_1$ is L.I. iff Span$\{\bar{v}_1\}$ is a line.

<u>Theorem</u> Two vectors $\bar{v}_1, \bar{v}_2$ are L.D. if and only if ... one vector is a scalar multiple of the other.

<u>Pf</u> Suppose x_1, x_2 are not both zero but

$$x_1\bar{v}_1 + x_2\bar{v}_2 = \bar{0}.$$

If $x_1 \neq 0$, then mult. both sides by x_1^{-1}, we get

$$\frac{x_1}{x_1}\bar{v}_1 + \frac{x_2}{x_1}\bar{v}_2 = \bar{0} \implies \bar{v}_1 = -\frac{x_2}{x_1}\bar{v}_2$$

$\implies \bar{v}_1$ is a scalar mult. of $\bar{v}_2$.

similarly, if $x_2 \neq 0$, $\bar{v}_2$ is a multiple of $\bar{v}_1$.

Also, if one is a multiple of the other (e.g. $\bar{v}_1 = c\bar{v}_2$), then $\bar{v}_1, \bar{v}_2$ are L.D. (e.g. $1v_1 - c\bar{v}_2 = \bar{0}$). □

if and only if

* Geometrically: $\bar{v}_1, \bar{v}_2$ are L.I. iff Span$\{\bar{v}_1, \bar{v}_2\}$ is a plane (not a line or point).

* If we have more than 2 vectors a similar argument shows that we can always write one of them as a linear comb. of the others.

<u>Theorem</u> Vectors $\bar{v}_1, \ldots, \bar{v}_k$ are L.D. iff at least one of the vectors is a linear comb. of the others (i.e. if one vector is in the subset spanned by the others).

* complete the sentence: $\bar{v}_1, \ldots, \bar{v}_k$ are L.I. iff ______

Ex Determine if the following are L.I.

(a) $\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\boxed{\text{L.D.}}$ b/c $\bar{0}$ is included

(b) $\begin{bmatrix}3\\-5\\-2\end{bmatrix}, \begin{bmatrix}-6\\10\\4\end{bmatrix}$

$\boxed{\text{L.D.}}$ b/c $-2\bar{v}_1 = \bar{v}_2$

(c) $\begin{bmatrix}3\\-5\\-6\end{bmatrix}, \begin{bmatrix}-6\\10\\4\end{bmatrix}$

$$\left[\begin{array}{cc|c}3 & -6 & 0\\ -5 & 10 & 0\\ -6 & 4 & 0\end{array}\right] \sim \left[\begin{array}{cc|c}1 & -2 & 0\\ -1 & 2 & 0\\ -3 & 2 & 0\end{array}\right] \sim \left[\begin{array}{cc|c}1 & -2 & 0\\ 0 & 0 & 0\\ 0 & -4 & 0\end{array}\right]$$

No free var. $\Rightarrow$ only triv. sol.

$\Rightarrow \boxed{\text{L.I.}}$

(d) $\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\5\\2\end{bmatrix}, \begin{bmatrix}-7\\3\\4\end{bmatrix}$

$$\left[\begin{array}{ccc|c}0 & 0 & -7 & 0\\ 1 & 5 & 3 & 0\\ 0 & 2 & 4 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 5 & 3 & 0\\ 0 & 2 & 4 & 0\\ 0 & 0 & -7 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 5 & 3 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

No free var. $\Rightarrow$ only triv. sol.

$\Rightarrow \boxed{\text{L.I.}}$

(e) $\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}7\\7\\1\end{bmatrix}, \begin{bmatrix}-2\\0\\0\end{bmatrix}, \begin{bmatrix}5\\4\\0\end{bmatrix}$

4 vectors in $\mathbb{R}^3$. $4>3$ so must be $\boxed{\text{L.D.}}$

(there will be free variables)

Often Lin. ind./dep. comes up when talking about the columns of a matrix. Notice that...

<u>Theorem</u> The columns of A are lin. ind. iff ...

$A\bar{x} = \bar{0}$ has only the trivial solution.

1.8 Intro to Linear Transformations

* You're very familiar with functions from $\mathbb{R}$ to $\mathbb{R}$,
  e.g. $f(x) = e^x$.

* It's not hard to make up functions from say $\mathbb{R}^2$ to $\mathbb{R}^2$ or $\mathbb{R}^2$ to $\mathbb{R}^3$ or...

domain codomain

$$g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \sin x + y \\ y^2 \end{bmatrix} \qquad \text{here } g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix} \qquad \text{here } T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

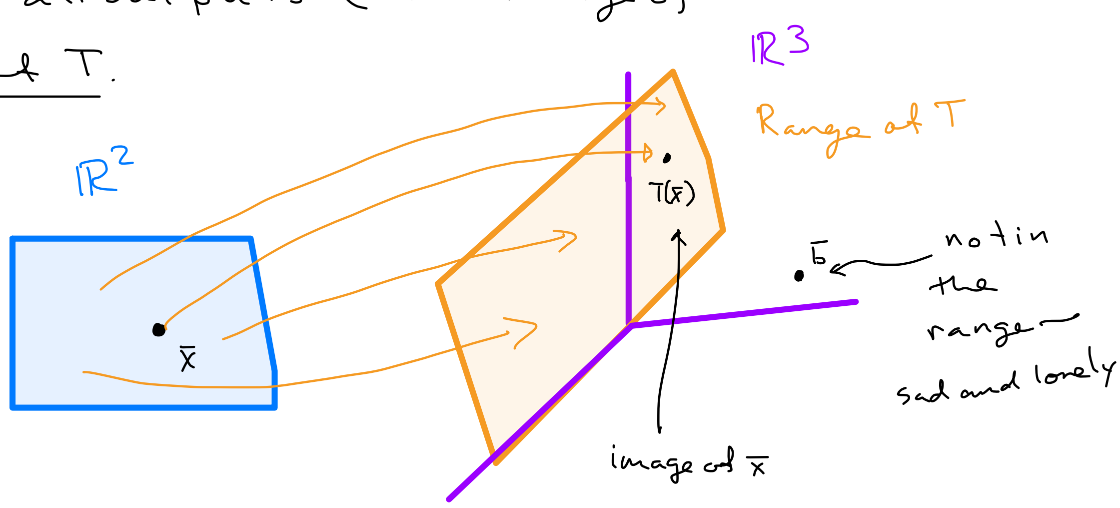
- Recall: the domain of a function is the set of allowable inputs.

- The codomain is a set that contains all outputs, but it may be larger than the collection of all outputs.

domain codomain

Def A function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ will be called a transformation. If $\bar{x}$ is in $\mathbb{R}^n$, the output $T(\bar{x})$ is called the image of $\bar{x}$ under T. The collection of all outputs (i.e. all images) is called the range of T.

A picture



Matrix Transformation

... roughly, these are transformations defined by a matrix.

Def A <u>matrix transformation</u> is any transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ for which there is an $m \times n$ matrix A such that

matrix vector product

$$T(\bar{x}) = A\bar{x}$$

Ex Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, and define $T: \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T(\bar{x}) = A\bar{x}.$$

(a) compute the image of $\bar{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$$T(\bar{u}) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 3\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \boxed{\begin{bmatrix} -7 \\ 21 \\ 19 \end{bmatrix}}$$

(b) Determine if $\bar{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ is in the range. If it is, find a vector $\bar{x}$ in $\mathbb{R}^2$ s.t. $T(\bar{x}) = \bar{b}$.

$\bar{b}$ is in the range $\iff$ $T(\bar{x}) = \bar{b}$ has a sol. $\iff$ $A\bar{x} = \bar{b}$ is consistent

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{array}\right]$$

so, $\boxed{\text{yes,}}$ $\bar{b}$ is in the range since there is a solution. Specifically, $T\left(\begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, or in other words

$$\bar{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \text{ is the image of } \boxed{\bar{x} = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}.}$$

(c) Do you expect that every vector in $\mathbb{R}^3$ is in the range of T? Why or why not?

$\bar{b}$ is in the range $\iff$ $T(\bar{x}) = \bar{b}$ has a sol. $\iff$ $A\bar{x} = \bar{b}$ is consistent

but...

$$\left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 3 & 5 & b_2 \\ -1 & 7 & b_3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & -4 & b_2 - 3b_1 \\ 0 & 4 & b_1 + b_3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & -4 & b_2 - 3b_1 \\ 0 & 0 & -2b_1 + b_2 + b_3 \end{array}\right]$$

so this is inconsistent whenever $-2b_1 + b_2 + b_3 \neq 0$.

Thus, $\boxed{\text{not}}$ every vector in $\mathbb{R}^3$ is in the range.

For example, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in the range.

(d) Show that $\bar{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ is NOT in the range.

could use previous part or start from beginning...

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{array}\right]$$

Inconsistent, so $A\bar{x} = \bar{b}$ has no sol. Thus, $T(\bar{x}) = \bar{b}$ has no sol., so $\bar{b}$ is not in the range.

Ex Invesigate the following matrix transformations.

For each,

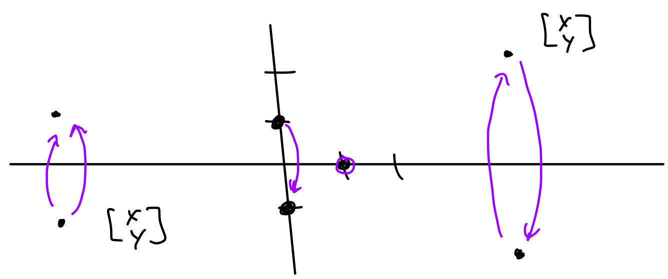
- find the images of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.
- find the image of an arbitrary vector $\begin{bmatrix}x\\y\end{bmatrix}$.
- try to describe the transformation geometrically.

(a) $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ def. by $T(\bar{x}) = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\bar{x}$.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$$

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\-y\end{bmatrix}$$



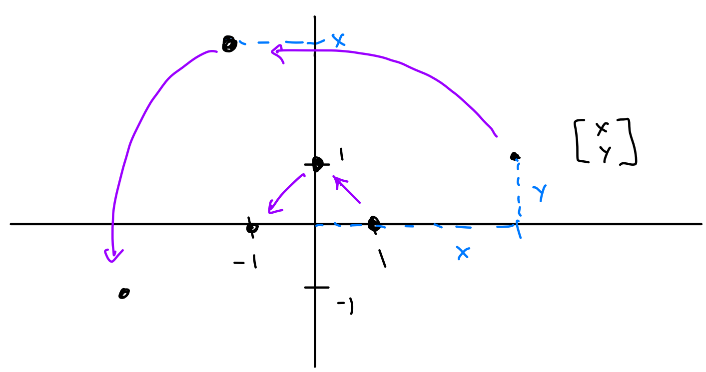
Reflection over x-axis

(b) $S:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ " " $S(\bar{x}) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\bar{x}$

$$S\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

$$S\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-y\\x\end{bmatrix}$$



Rotation (CCW) by $\pi/2$

Fact The matrix transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\bar{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \bar{x}$$

performs a rotation of $\mathbb{R}$ by θ (ccw).

* Notice that if $\theta = \pi/2$ then $T(\bar{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \bar{x}$, just like in the last example.

* This is proven in next section.

Linear Transformations

All matrix transformations have some special properties. Notice that if $T(\bar{x}) = A\bar{x}$ then

$$T(\bar{u}+\bar{v}) = A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v} = T(\bar{u}) + T(\bar{v})$$

and

$$T(c\bar{u}) = A(c\bar{u}) = c(Au) = cT(\bar{u}).$$

* You've see this before ... derivative rules ...

Definition A transformation T is called linear if for all $\bar{u}, \bar{v}$ in the domain of T and all scalars c,

(i) $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$

AND

(ii) $T(c\bar{u}) = cT(\bar{u})$.

* Thus, all matrix transformations are linear.

* lin. trans. automatically have other nice properties: $T(\bar{0}) = \bar{0}$ and $T(c\bar{u} + d\bar{v}) = cT(\bar{u}) + dT(\bar{v})$

1.9 The Matrix of a Linear Transformation

We saw that matrix transformations are linear transformation. We now will see that (perhaps suprisingly) every linear transformation can be written as a matrix transformation.

Let's investigate this... but first some notation:

<u>Def</u> (The standard basis) We use $\bar{e}_k$ to denote the vector

$$\bar{e}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{1 in k-th entry. 0s everywhere else.}$$

* Note that in $\mathbb{R}^3$ $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ but in $\mathbb{R}^4$ $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

<u>Ex</u> Suppose $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear transformation. Further, suppose that you know

$$T(\bar{e}_1) = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \quad T(\bar{e}_2) = \begin{bmatrix} 0 \\ 7 \\ 5 \end{bmatrix}.$$

Find a formula for $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

We know that

- $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$
- $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 7 \\ 5 \end{bmatrix}$

The key is that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

so

$$\begin{aligned} T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= T\left(\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} x \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ y \end{bmatrix}\right) \\ &= T\left(x\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= x\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 7 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 3x \\ -2x \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ 7y \\ 5y \end{bmatrix} = \boxed{\begin{bmatrix} 3x \\ -2x+7y \\ x+5y \end{bmatrix}}. \end{aligned}$$

T is linear

T is linear

Also, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & 7 \\ 1 & 5 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix}$

* so T is a <u>matrix</u> transformation.

<u>Theorem</u> If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T is a matrix transformation. If A is the matrix whose j^{th} column is $T(\vec{e}_j)$

$$A = \left[T(\vec{e}_1) \cdots T(\vec{e}_n)\right],$$

then $T(\vec{x}) = A\vec{x}$.

* A is called the <u>standard matrix</u> of T.

Ex Define $T:\mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(x_1,x_2,x_3) = (3x_1-2x_3,\ 4x_1,\ x_1-x_2+x_3)$$

this is alternative notation for $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}3x_1-2x_3\\4x_1\\x_1-x_2+x_3\end{bmatrix}$

T is a linear transformation. Find the standard matrix for T.

$$A = [\,T(\bar{e}_1)\quad T(\bar{e}_2)\quad T(\bar{e}_3)\,]$$

- $T(\bar{e}_1) = T(1,0,0) = (3,4,1) = \begin{bmatrix}3\\4\\1\end{bmatrix}$
- $T(\bar{e}_2) = T(0,1,0) = (0,0,-1) = \begin{bmatrix}0\\0\\-1\end{bmatrix}$
- $T(\bar{e}_3) = T(0,0,1) = (-2,0,1) = \begin{bmatrix}-2\\0\\1\end{bmatrix}$

Thus, $\boxed{A = \begin{bmatrix}3&0&-2\\4&0&0\\1&-1&1\end{bmatrix}}$ so $T(\bar{x}) = \begin{bmatrix}3&0&-2\\4&0&0\\1&-1&1\end{bmatrix}\bar{x}$.

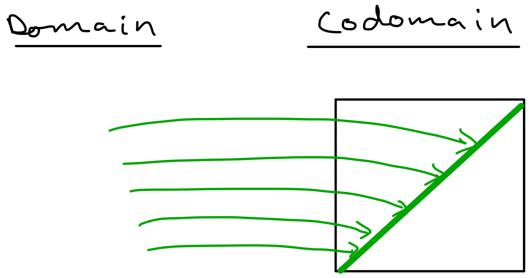
one-to-one and onto

<u>Def</u> Let $T:\mathbb{R}^n \to \mathbb{R}^m$.

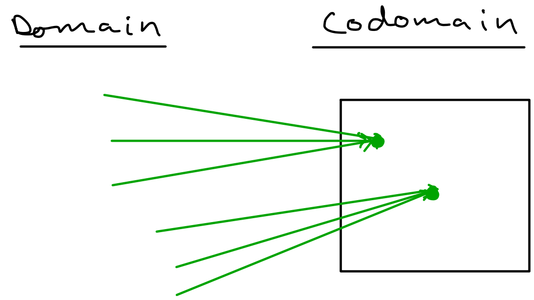
1. we say T is <u>onto $\mathbb{R}^m$</u> if the range of T is all of $\mathbb{R}^m$, i.e. if $T(\bar{x})=\bar{b}$ has a sol. for every $\bar{b}$ in $\mathbb{R}^m$.

2. we say T is <u>one-to-one</u> if every $\bar{b}$ in $\mathbb{R}^m$ is the image of at most one $\bar{x}$ in $\mathbb{R}^n$, i.e. if $T(\bar{x})=\bar{b}$ has at most one solution for every $\bar{b}$ in $\mathbb{R}^m$.

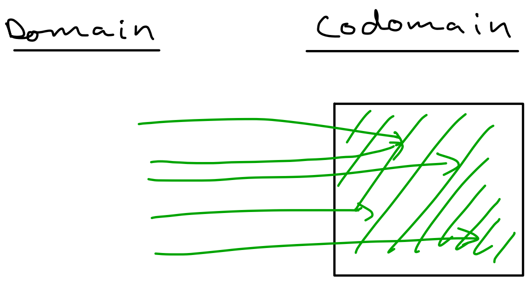
A picture



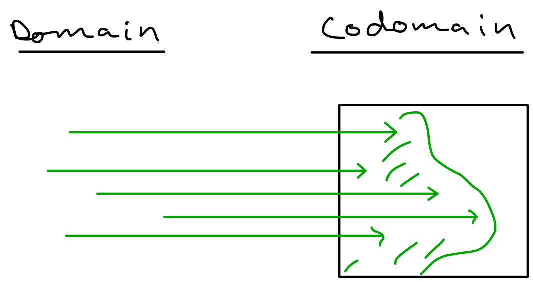
Not onto



Not one-to-one



Onto



one-to-one
(but maybe not onto)

For linear transformations, being onto or one-to-one can be investigated in terms of the standard matrix.

Theorem Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T.

① T maps onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$ (A has a pivot in every row)

② T is one-to-one if and only if the columns of A are linearly independent

(A has a pivot in every column)

Exploring ①: $A\bar{x} = \bar{b}$ has a solution for every $\bar{b}$

$\Longleftrightarrow$ columns of A span $\mathbb{R}^m$

Exploring ②: $A\bar{x} = \bar{b}$ has at most one solution for every $\bar{b}$ $\Longrightarrow$ $A\bar{x} = \bar{0}$ has at most one sol.

optional

also, if $A\bar{x} = \bar{b}$ has 2 sol. $\bar{u}$ and $\bar{v}$ then

$$A\bar{u} = \bar{b},\ A\bar{v} = \bar{b} \Longrightarrow A\bar{u} - A\bar{v} = \bar{0} \Longrightarrow$$

$A(\bar{u} - \bar{v}) = \bar{0} \Longrightarrow A\bar{x} = \bar{0}$ has 2 sol. Thus,

$A\bar{x} = \bar{b}$ has at most one sol. for all $\bar{b}$ $\Longleftrightarrow$ $A\bar{x} = \bar{0}$ has at most one sol. $\Longleftrightarrow$ col. of A are lin. ind.

<u>Ex</u> Define $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by

$$T(x_1, x_2, x_3) = (3x_1 - 2x_3,\ 4x_1,\ x_1 - x_2 + x_3)$$

Show that T is one-to-one and onto $\mathbb{R}^3$.

① Let's find the standard matrix for T.

From a previous ex., the standard matrix is

$$A = \begin{bmatrix} 3 & 0 & -2 \\ 4 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{so} \quad T(\bar{x}) = \begin{bmatrix} 3 & 0 & -2 \\ 4 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \bar{x}.$$

② we now use the theorem.

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -5 \\ 0 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix}$$

- pivot in every row $\Longleftrightarrow$ col. span $\mathbb{R}^3$ $\Longleftrightarrow$ $\boxed{\text{onto } \mathbb{R}^3}$
- pivot in every col $\Longleftrightarrow$ col. are L.I. $\Longleftrightarrow$ $\boxed{\text{one-to-one}}$

<u>Ex</u> Define $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 - x_4,\ 3x_1 - 3x_2 + 4x_3 + 8x_4,\ 2x_1 - 2x_2 + 2x_3 + 5x_4)$$

(a) Show that T is onto $\mathbb{R}^3$ but not one-to-one.

(b) Find two vectors whose image under T is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) ① Let's find the standard matrix for T

$$T(\bar{e}_1) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad T(\bar{e}_2) = \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}, \quad T(\bar{e}_3) = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \quad T(e_4) = \begin{bmatrix} -1 \\ 8 \\ 5 \end{bmatrix}$$

$$\Longrightarrow A = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 3 & -3 & 4 & 8 \\ 2 & -2 & 2 & 5 \end{bmatrix}$$

$$\Longrightarrow T(\bar{x}) = A\bar{x}$$

② Use the theorem: $A \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- pivot in every row $\Longleftrightarrow$ col. span $\mathbb{R}^3$ $\Longleftrightarrow$ $\boxed{\text{onto } \mathbb{R}^3}$
- NOT a pivot in every col $\Longleftrightarrow$ cols. are lin. dep. $\Longleftrightarrow$ $\boxed{\text{\underline{not} one-to-one}}$

(b) Solve $T(\bar{x}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (and we know this is consitent since T is onto $\mathbb{R}^3$)

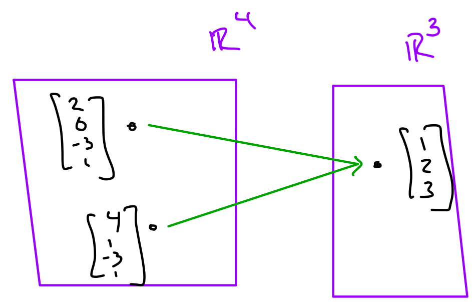
$$T(\bar{x}) = A\bar{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 1 \\ 3 & -3 & 4 & 8 & 2 \\ 2 & -2 & 2 & 5 & 3 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{aligned} x_1 &= 2 + 2x_2 \quad (s) \\ s = x_2 &\text{ free} \\ x_3 &= -3 \\ x_4 &= 1 \end{aligned} \implies \bar{x} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we get 2 sol. choosing $s = 0, s = 1$

$$\boxed{\begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \\ 1 \end{bmatrix}}$$



<u>Ex</u> Explain why a linear transformation $\mathbb{R}^3 \to \mathbb{R}^5$ can not be onto $\mathbb{R}^5$.

The matrix for T is 5×3. Thus, there are at most 3 pivots, so there cannot be a pivot in every row. Thus T is not onto $\mathbb{R}^5$ by the Theorem.

<u>Ex</u> Explain why a linear transformation $\mathbb{R}^3 \to \mathbb{R}^2$ can not be one-to-one

The matrix for T is 2×3 Thus, there are at most 2 pivots, so there cannot be a pivot in every column. Thus T is not one-to-one by the Theorem.