2.1 Matrix Operations

Addition and Scalar Multiplication

by example...

$$A = \begin{bmatrix} 1 & 3 & 6 \\ -2 & 0 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 2 & 0 \\ 7 & 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

2×3 2×1 1×2

rows x # cols

- $A + B = \begin{bmatrix} 8 & 5 & 6 \\ 5 & 1 & 7 \end{bmatrix}$
- $-3B = \begin{bmatrix} -21 & -6 & 0 \\ -21 & -3 & 3 \end{bmatrix}$
- $A - 3B = \begin{bmatrix} -20 & -3 & 6 \\ -23 & -3 & 11 \end{bmatrix}$
- $A + C$ is undefined
- $C \neq D$

* Addition/subtraction is only defined for matrices of the same dimensions.

* For scalar multiplication, you multiply every entry by the scalar

* Two matrices are equal if and only if they have the same dimensions and same corresponding entries.

Matrix Multiplication

Recall: we know how to multiply a matrix by a vector.

$$\underbrace{\begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}}_{2\times 2} \cdot \underbrace{\begin{bmatrix} 1 \\ 4 \end{bmatrix}}_{2\times 1} = 1\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -17 \\ 6 \end{bmatrix}$$

Def If A is $m \times n$ and B is $n \times p$ with match

$$B = [\bar{b}_1 \ \bar{b}_2 \cdots \bar{b}_p]$$

then

$$AB = [A\bar{b}_1 \ \ A\bar{b}_2 \ \cdots \ A\bar{b}_p].$$

AB is $m \times p$.

* I.e. $\text{col}_j(AB) = A \cdot \text{col}_j(B)$

Ex Let $A = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 0 \\ 4 & 2 & -3 \end{bmatrix}$

2×2 2×3 match!

$$\left.\begin{aligned} A\begin{bmatrix} 1 \\ 4 \end{bmatrix} &= \begin{bmatrix} -17 \\ 6 \end{bmatrix} \\ A\begin{bmatrix} 3 \\ 2 \end{bmatrix} &= \begin{bmatrix} -1 \\ 8 \end{bmatrix} \\ A\begin{bmatrix} 0 \\ -3 \end{bmatrix} &= \begin{bmatrix} 15 \\ -3 \end{bmatrix} \end{aligned}\right\} \Longrightarrow \boxed{AB = \begin{bmatrix} -17 & -1 & 15 \\ 6 & 8 & -3 \end{bmatrix}}$$

! BA is not defined

$2\times 3 \ 2\times 2$ not equal

* Aside: why define matrix mult. this way?

Thinking of the transformations $T_A(\bar{x}) = A\bar{x}$ and $T_B(\bar{x}) = B\bar{x}$, then $T_A(T_B(\bar{x})) = T_{AB}(\bar{x})$.

Another view of multiplication

Ex Let $A = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 0 \\ 4 & 2 & -3 \end{bmatrix}$

$$A \cdot B = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 4 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -17 & -1 & 15 \\ -6 & 8 & -3 \end{bmatrix}$$

(1,3) entry comes from row 1 of A and col 3 of B

$3 \cdot 2 + 2(-1)$

Comments

① $\text{col}_j(AB) = A \cdot \text{col}_j(B)$

② $\text{row}_i(AB) = \text{row}_i A \cdot B$

③ (i,j) entry of AB is $\text{row}_i(A) \cdot \text{col}_j(B)$

this repr. a dot product.

Ex Let $A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \\ 5 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 3 & -1 \\ 4 & -1 & 2 & 0 \end{bmatrix}$

$$AB = \underbrace{\begin{bmatrix} -2 & 1 \\ 3 & 0 \\ 5 & -2 \end{bmatrix}}_{3 \times 2} \cdot \underbrace{\begin{bmatrix} 2 & 0 & 3 & -1 \\ 4 & -1 & 2 & 0 \end{bmatrix}}_{2 \times 4} = \begin{bmatrix} 0 & -1 & 4 & 2 \\ 6 & 0 & 9 & -3 \\ 2 & 2 & 11 & -5 \end{bmatrix}$$

match

should be 3×4

Properties of Matrix arithmetic

Many familiar properties are true: read Theorem 1, 2 pg. 95, 99. For example

- $A(B+C) = AB + AC$.

Def I_n is the $n \times n$ matrix with 1's on the main diagonal and 0's everywhere else.

$$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

* e.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

* I_n is called the identity matrix

* The zero matrix is the matrix with all zeros.

* Notice that if A is $m \times n$, then

$$I_m \cdot A = A \quad \text{and} \quad A \cdot I_n = A.$$

But some "familiar" properties fail.

Ex Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \qquad AC = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

Thus, $AB = AC$ but $B \neq C$.

! so, in general, you can NOT cancel!

Also,

$$AB = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$$

! so, in general, $AB \neq BA$!

Powers of a matrix

Def $A^k = \underbrace{A \cdot A \cdot \cdots A}_{k\text{-times}}$

Ex If $A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

* $A \neq 0$ but $A^3 = 0$!!

zero matrix

! It's possible that $A \neq 0$, $B \neq 0$ but $AB = 0$.

Transpose of a matrix

Def If A is $m \times n$, then the transpose of A, denoted A^T, is the $n \times m$ matrix where

$$\text{row}_i(A^T) = \text{col}_i(A)$$

* interchange rows and columns

Ex If $A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^T = \begin{bmatrix} 1 & 0 & 4 \\ 2 & -3 & 5 \end{bmatrix} \quad B^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Theorem $(A^T)^T = A$, $(A+B)^T = A^T + B^T$, $(AB)^T = B^T A^T$

2.2 Inverse of a matrix

Q: How would you solve

$$5x = 7 \Longrightarrow \frac{1}{5}5 \cdot x = \frac{1}{5} \cdot 7 \Longrightarrow x = \frac{7}{5}$$

$$\underline{\underline{5^{-1}}} \cdot 5x = \underline{\underline{5^{-1}}} \cdot 7$$

Q: Can we apply a similar method to solve $A\bar{x} = b$?

For example,

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \bar{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

__Def__ Let A be $n \times n$. If there exists a matrix A^{-1} such that $A \cdot A^{-1} = I_n$ and $A^{-1} \cdot A = I_n$, then we say that A is __invertible__.

* If A is invertible, there is only one possible choice for A^{-1}.
* we call A^{-1} the inverse of A.
* note that A is the inverse of A^{-1}.

Q: so how can we determine if a matrix is invertible?

Q: if A is invertible, how do we find A^{-1}?

When A is 2×2

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A^{-1} (if it exists) has the property

$$A^{-1}A = I_n \text{ and } A\cdot A^{-1} = I_n.$$

So, if a potential A^{-1} drops from the sky, we just check if it works. Look up!

$$\text{Let } B = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Now,

$$B\cdot A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \checkmark$$

this is important

similarly

$A\cdot B = I_2$. Thus $\underline{\underline{B = A^{-1}}}$.

Def If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det A = ad-bc$ is called the <u>determinant</u> of A.

<u>Theorem</u> Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

1. If $\det A \neq 0$, then $A^{-1} = \frac{1}{\det A}\cdot\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
2. If $\det A = 0$, then A is <u>not</u> invertible. (A^{-1} DNE)

Ex Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ and use it to solve $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \bar{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

(1) • $\det A = 18 - 20 = -2 \longleftarrow$ so A^{-1} exists

• $A^{-1} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1} = \frac{1}{-2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}}$

(2) $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \bar{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \implies \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \bar{x} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$\implies I_2 \bar{x} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$\implies \boxed{\bar{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}}$

properties of the inverse

so A^{-1}, B^{-1} exist.

Theorem Assume A, B are invertible.

(a) $(A^{-1})^{-1} = A$

(b) $(AB)^{-1} = B^{-1}A^{-1} \longleftarrow$ but this may be different than $A^{-1}B^{-1}$

(c) $(A^{T})^{-1} = (A^{-1})^{T}$

Pf of (b)

Let $C = B^{-1}A^{-1}$. Then $ABC = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$.

Also, $CAB = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$. □

When A is $n \times n$

HO-05

05 – Matrix Inverses

Definition: Elementary Matrix

An **elementary matrix** a matrix obtained by performing a single elementary row operation on the identity I.

1. Determine if each of the following are elementary matrices.

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\boxed{\text{Yes}}$ $r_1 \leftrightarrow r_3$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\boxed{\text{Yes}}$ $r_1 \to 3r_1$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ $\boxed{\text{Yes}}$ $r_2 \to 3r_3 + r_2$

(d) $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ No need 2 ops

(e) $\begin{bmatrix} 1 & 2 & 0 \\ -2 & 3 & 0 \\ \frac{1}{2} & 3 & 0 \end{bmatrix}$ No col. of zeros

(f) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ No need 2 ops

2. Let $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Compute EA. What do you notice?

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+3a_{31} & a_{22}+3a_{32} & a_{23}+3a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

mult. A by E performs the row operation associated to E

Theorem

Let A be $m \times n$. If ρ is an elementary row operation and $E = \rho(I)$ is the corresponding elementary matrix, then $\rho(A) = EA$. Moreover, E is invertible with $E^{-1} = \rho^{-1}(I)$.

Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if its RREF is I_n, and this happens if and only if A is a product of elementary matrices. Further, when A is invertible, any sequence of row operations that transforms A to I_n will also transform I_n to A^{-1}.

Theorem: Algorithm for finding A^{-1}

If A is $n \times n$, row reduce the augmented matrix $\left[A \mid I_n\right]$ to RREF.

- If the RREF of $\left[A \mid I_n\right]$ is $\left[I_n \mid B\right]$, then A is invertible, and $B = A^{-1}$.
- If the RREF of $\left[A \mid I_n\right]$ is $\left[\text{“}\textit{not } I_n\text{”} \mid B\right]$, then A is not invertible.

pf idea

- A invertible
  $\Rightarrow A\bar{x} = \bar{b}$ has a unique sol. $A^{-1}\bar{b}$
  $\Rightarrow$ A has pivot in every row + col.
  $\Rightarrow A \sim I_n$
- $A \sim I_n$
  $\Rightarrow \underbrace{E_n \cdots E_1}_{\text{elem. matrices}} A = I_n$
  $\Rightarrow A^{-1} = E_n \cdots E_1$
  (although $A E_n \cdots E_1 = I_n$ should also be checked.)

3. Find the inverse of A, if it exists.

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \text{ so } \boxed{A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{bmatrix}}$$

(b) $\begin{bmatrix} 0 & 3 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 3 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \not\sim I_3$ b/c no pivot in 3rd column. Thus $\boxed{A^{-1} \text{ DNE}}$

(c) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & 3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -9/2 & 7 & 3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}}$$

(d) $\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 4 & -3 & 7 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & -3 & 7 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 0 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & -4 & 0 \end{array}\right]$$

$A \not\sim I_3$ so $\boxed{A^{-1} \text{ DNE}}$

2.3 Characterizations of Invertible Matrices

Invertibe Matrix Theorem Let A be a square $n \times n$ matrix.

Then the following are equivalent.

a. A is invertible

b. $A \sim I_n$

c. A has n pivots

d. $A\bar{x} = \bar{0}$ has only the trivial sol.

e. Columns of A are linearly independent

f. The transformation $T(\bar{x}) = A\bar{x}$ is one-to-one.

g. $A\bar{x} = \bar{b}$ is consistent for every choice of $\bar{b}$

h. The columns of A span $\mathbb{R}^n$

i. The transformation $T(\bar{x}) = A\bar{x}$ is onto

j. There is an $n \times n$ matrix C s.t. $CA = I_n$

k. " " " D s.t. $AD = I_n$

l. A^T is invertible

2.8 Subspaces of $\mathbb{R}^n$

<u>Idea</u>: There are two main operations on $\mathbb{R}^n$: addition + scalar multiplication. Subspaces will be subsets that carry these same operations.

<u>Def</u> A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

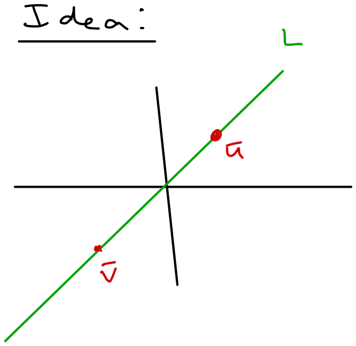
1. $\bar{0}$ is in $\mathbb{R}^n$

closed under addition $\to$ 2. For each $\bar{u}$ and $\bar{v}$ in H, $\bar{u}+\bar{v}$ is also in H.

closed under scalar mult. $\to$ 3. For each $\bar{u}$ in H and each scalar c, $c\bar{u}$ is also in H.

<u>Ex</u> Let L be any line through the origin in $\mathbb{R}^n$. Then L is a subspace.

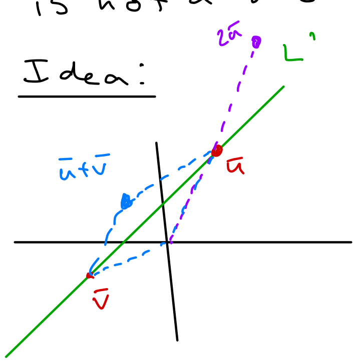
<u>Idea</u>:



1. $\begin{bmatrix}0\\0\end{bmatrix}$ is on L
2. $\bar{u}+\bar{v}$ is on L for all $\bar{u},\bar{v}$ on L
3. $c\bar{u}$ is on L for all $\bar{u}$ on L

<u>Ex</u> If L' is a line not through the origin, then L' is not a subspace.

<u>Idea</u>:



1. $\begin{bmatrix}0\\0\end{bmatrix}$ is Not on L'
2. $\bar{u}+\bar{v}$ is NOT on L'
3. $2\bar{u}$ is NOT on L'

Let treat the first example carefully. Suppose L is a line through the origin. Let $\bar{v}$ be any nonzero vector on L. Then $L = \text{Span}\{\bar{v}\}$.

<u>Ex.</u> If $\bar{v}$ is in $\mathbb{R}^n$, then $\text{Span}\{\bar{v}\}$ is a subspace of $\mathbb{R}^n$.

<u>Recall</u>: $\text{Span}\{\bar{v}\}$ is all linear combinations of $\bar{v}$.

1. $\bar{0}$ is in $\text{Span}\{\bar{v}\}$ b/c $\bar{0} = 0 \cdot \bar{v}$
2. Let $\bar{a}, \bar{b}$ be in $\text{Span}\{\bar{v}\}$. Then $\bar{a} = c_1\bar{v}$, $\bar{b} = c_2\bar{v}$. Thus

$$\bar{a}+\bar{b} = c_1\bar{v} + c_2\bar{v} = (c_1+c_2)\bar{v} \Rightarrow \bar{a}+\bar{b} \text{ is a lin. comb. of } \bar{v}$$
$$\Rightarrow \bar{a}+\bar{b} \text{ is in Span}\{\bar{v}\}$$

3. Let $\bar{a}$ be in $\text{Span}\{\bar{v}\}$. Then $\bar{a} = c_1\bar{v}$. Let c be any scalar. Thus,

$$c\bar{a} = c(c_1\bar{v}) = (cc_1)\bar{v} \Rightarrow c\bar{a} \text{ is a lin. comb. of } \bar{v}$$
$$\Rightarrow c\bar{a} \text{ is in span}\{\bar{v}\}.$$

Thus $\text{Span}\{\bar{v}\}$ is a subspace of $\mathbb{R}^n$

<u>Theorem</u> If $\bar{v}_1, \dots, \bar{v}_k$ are in $\mathbb{R}^n$. Then $\underbrace{\text{Span}\{\bar{v}_1, \dots, \bar{v}_k\}}_{\text{All lin. comb. of } \bar{v}_1, \dots, \bar{v}_k}$ is always a subspace of $\mathbb{R}^n$.

* This implies that lines and planes through the origin are always subspaces.

Subspace associated to a matrix

Def Let A be any matrix. The column space of A is the set, denoted $\operatorname{Col} A$, of all linear comb. of the columns of A.

* $\operatorname{Col} A = \operatorname{Span}\{\bar{a}_1, \dots, \bar{a}_n\}$ where $\bar{a}_1, \dots, \bar{a}_n$ are the columns of A. Thus,

$\operatorname{Col} A$ is a subspace of $\mathbb{R}^m$.

Ex Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$. Determine if $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is in $\operatorname{Col} A$.

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is in $\operatorname{Col} A \iff \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is a linear. comb. of cols. of A

$\iff \left[\begin{array}{ccc|c} 1 & -3 & -4 & 1 \\ -4 & 6 & -2 & 1 \\ -3 & 7 & 6 & 2 \end{array}\right]$ is consistent

Now,

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & 1 \\ -4 & 6 & -2 & 1 \\ -3 & 7 & 6 & 2 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

inconsistent

so $\boxed{\text{NO}}$

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is NOT in $\operatorname{Col} A$

Def Let A be any matrix. The null space of A is the set, denoted $\operatorname{Nul} A$, of all solutions to $A\bar{x} = \bar{0}$.

Theorem If A is $m \times n$, then $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$.

optional

Pf

- $\bar{0}$ is in $\operatorname{Nul} A$ since $A\bar{0} = \bar{0}$

- Let $\bar{a}, \bar{b}$ be in Nul A. We want to show $\bar{a}+\bar{b}$ is in Nul A. Now

$$\begin{aligned} A(\bar{a}+\bar{b}) &= A\bar{a} + A\bar{b} \\ &= \bar{0} + \bar{0} \quad \text{since } \bar{a}, \bar{b} \text{ in Nul A.} \\ &= \bar{0} \end{aligned}$$

Thus, $\bar{a}+\bar{b}$ satisfies the prop. for being in Nul A.

- Similarly, if $\bar{a}$ is in Nul A and c is any scalar, then $A(c\bar{a}) = cA\bar{a} = c\bar{0} = \bar{0}$, so $c\bar{a}$ is in Nul A.

$\square$

Bases: describing subspaces efficiently

Recall: The standard basis for $\mathbb{R}^3$ is $\bar{e}_1, \bar{e}_2, \bar{e}_3$. This is useful, b/c if $\bar{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is any vector in $\mathbb{R}^3$ then

- $\bar{v}$ is a lin. comb. of $\bar{e}_1, \bar{e}_2, \bar{e}_3$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underline{a} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \underline{b} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \underline{c} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so

$$\bar{v} = a\bar{e}_1 + b\bar{e}_2 + c\bar{e}_3 .$$

- also, $\bar{v}$ is a linear comb. of $\bar{e}_1, \bar{e}_2, \bar{e}_3$ in only one way — so it is efficient.

Def Let H be a subspace. A subset of H is called a basis for H if

① the subset spans H, **AND**

② the subset is linearly independent.

* It can be shown that every nonzero subspace of $\mathbb{R}^n$ has a basis with only finitely may vectors.

Ex which of the following are bases for $\mathbb{R}^3$.

(a) $\begin{bmatrix}-1\\-2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\-2\end{bmatrix}, \begin{bmatrix}3\\9\\-6\end{bmatrix}$ $\quad \begin{bmatrix}-1 & 2 & 3\\-2 & 4 & 7\\1 & -2 & -6\end{bmatrix} \sim \begin{bmatrix}-1 & 2 & 3\\0 & 0 & 3\\0 & 0 & 0\end{bmatrix}$

- Not a pivot in every col. ⟹ free var. ⟹ not L.I.
- Not a pivot in every row ⟹ cols. do not span $\mathbb{R}^3$

$\boxed{\text{NO}}$

(b) $\begin{bmatrix}0\\1\\-2\end{bmatrix}, \begin{bmatrix}5\\-7\\4\end{bmatrix}, \begin{bmatrix}6\\3\\5\end{bmatrix}$ $\sim \begin{bmatrix}1 & -7 & 3\\0 & 5 & 6\\0 & 0 & 23\end{bmatrix}$

- Pivot in every col ⟹ L.I.
- Pivot in every row ⟹ cols span $\mathbb{R}^3$

$\boxed{\text{Yes}}$

(c) $\begin{bmatrix}-1\\-2\\1\end{bmatrix}, \begin{bmatrix}3\\9\\-6\end{bmatrix}$

- can not possibly have a pivot in each row ⟹ cols. do NOT span $\mathbb{R}^3$

$\boxed{\text{No}}$

(d) $\begin{bmatrix}0\\1\\-2\end{bmatrix}, \begin{bmatrix}5\\-7\\4\end{bmatrix}, \begin{bmatrix}6\\3\\5\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$

- Not a pivot in every col. ⟹ not. L.I. $\boxed{\text{NO}}$

Finding Bases

$\boxed{\text{HO-06}}$ All.

06 – Null and Column Spaces

Definition: Null Space

The **null space** of a matrix A, is the set of *all* solutions to $A\mathbf{x} = \mathbf{0}$.

Strategy: Basis for $\operatorname{Nul} A$

Let A be any matrix. To find a basis for $\operatorname{Nul} A$, do the following.

- Solve $A\mathbf{x} = \mathbf{0}$ (usually with row reduction).
- Write the solution set in *parametric vector form* (using the process from class).
- The vectors appearing in the parametric vector form are a basis for $\operatorname{Nul} A$.

1. Find a basis for the null space of the following matrix.

$$A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix}$$

You can use the fact that

$$A \sim \begin{array}{c} \begin{array}{ccccc} & & x_3 & & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & -2 & 0 & 7 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

$$A \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 7 \\ 0 & 1 & 5/2 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$x_1 = 2x_3 - 7x_5$

$x_2 = -5/2\, x_3 + 1/2\, x_5$

$s = x_3$ free

$x_4 = -x_5$

$t = x_5$ free

$$\bar{x} = \begin{bmatrix} 2s - 7t \\ -5/2\, s + 1/2\, t \\ s \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} 2 \\ -5/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 1/2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

!! number of vectors in a basis for Nul A is equal to numb. of free variables in $A\bar{x} = \bar{0}$.

Basis for Nul A is $\boxed{\left\{ \begin{bmatrix} 2 \\ -5/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 1/2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}}$

dimension 2

Definition: Column Space

The **column space** of a matrix A, is the set of *all* linear combinations of the columns of A.

Strategy: Basis for $\operatorname{Col} A$

Let A be any matrix. To find a basis for $\operatorname{Col} A$, do the following.

- Row reduce A to REF, and locate the pivots.
- The columns of the *original* matrix A that correspond to the pivots form a basis for $\operatorname{Col} A$.

2. Find a basis for the column space of the matrix in the previous exercise.

Basis for col A is $\boxed{\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix} \right\}}$

dimension 3

Strategy: Basis for $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$

Make a matrix A using $\mathbf{v}_1, \ldots, \mathbf{v}_k$ as the columns, so $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix}$. Then find a basis for $\operatorname{Col} A$.

3. Find a basis for the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix}$.

Create $A = \begin{bmatrix} -1 & 2 & 3 \\ -2 & 4 & 9 \\ 1 & -2 & -6 \end{bmatrix}$.

$A \sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, a basis for

$\operatorname{Span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix} \right\}$ is $\boxed{\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix} \right\}}$

... so the second vector is redundant, which is easy to see.

2.9 Dimension & Rank

This section mostly introduces terminology related to bases.

<u>Note:</u> subspaces have lots of different bases.

For example, we have seen that both of the following are bases for $\mathbb{R}^3$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix} \right\}$$

But, they have one thing in common...

<u>Theorem</u> If H is a nonzero subspace of $\mathbb{R}^n$, then every basis for H has the same number of vectors.

<u>Def</u> The <u>dimension</u> of a nonzero subspace H is the number of vectors in a basis for H. The dimension of $\{\vec{0}\}$ is 0.

↑ has no basis

* dimension of H is denoted $\dim H$
* If A is a matrix,
  - $\dim(\operatorname{Col} A)$ is called the <u>rank</u> of A
  - $\dim(\operatorname{Nul} A)$ is called the <u>nullity</u> of A.

Ex Determine the dimension of each of the following.

(a) $\mathbb{R}^3$

$\bar{e}_1, \bar{e}_2, \bar{e}_3$ is a basis for $\mathbb{R}^3$ so $\dim(\mathbb{R}^3) = \boxed{3}$

(b) $\mathbb{R}^n$

$\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n$ is a basis for $\mathbb{R}^n$, so $\dim(\mathbb{R}^n) = n$

(c) Col(A) where $A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ 4 & 12 & 2 & 7 \end{bmatrix}$

$$A \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Basis for Col A is}$$

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix} \right\}$$

$\Rightarrow \dim(\text{Col}(A)) = \text{rank } A = \boxed{3}$

(d) Nul A where A is as above.

$$A \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 = 3x_2$

x_2 free $\leftarrow s$

$x_3 \quad 0$

$x_4 \quad 0$

$$\bar{x} = \begin{bmatrix} 3s \\ s \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ basis for Nul A is $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\Rightarrow \dim(\text{Nul } A) = \text{nullity } A = \boxed{1}$

(e) $H = \text{Span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix} \right\}$

* we did this before...

Create $A = \begin{bmatrix} -1 & 2 & 3 \\ -2 & 4 & 9 \\ 1 & -2 & -6 \end{bmatrix}$, $A \sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

$\Longrightarrow$ basis for H is $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix} \right\}$

$\Longrightarrow \dim H = \boxed{2}$.

<u>Theorem</u> Let A be $m \times n$. Assume the RREF for A has p many pivots.

① $\text{rank}\, A = p$

② $\text{nullity}\, A = n - p \leftarrow$ # free var. = # var. − # not free var. in $A\vec{x} = \vec{0}$

③ $\text{rank}\, A + \text{nullity}\, A = n$.

Rank-Nullity Theorem