3.1 Intro to Determinants

Recall: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

- $\det A = ad - bc$, and
- A^{-1} exists if and only if $\det A \neq 0$.

Notation: we often write $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ to mean $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$.

We want to generalize this to larger (square) matrices.

We need some definitions...

Def Let A be a $n \times n$ matrix.

1. A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of A.

2. The (i,j)-cofactor, denoted C_{ij}, is

$$\text{a number} \longrightarrow C_{ij} = (-1)^{i+j} \cdot \det A_{ij}$$

Ex Let $A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$. Then $A_{23} = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & 7 \\ 0 & 4 & 0 \end{bmatrix}$.

Ex Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & 7 \\ 0 & 4 & 0 \end{bmatrix}$ Then

- $A_{32} = \begin{bmatrix} 1 & 0 \\ 3 & 7 \end{bmatrix}$

- $C_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 0 \\ 3 & 7 \end{vmatrix} = -1 \cdot (7 - 0) = -7$

3x3 Determinants

Def If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

cofactor expansion along 1^{st} row.

$$\boxed{\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}}$$

$$= a_{11} \cdot \begin{vmatrix} a_{12} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Ex compute $\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 3 \end{vmatrix}$.

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix}$$

$$= (12+2) - 5(6) = 14 - 30 = \boxed{-16}$$

nxn Determinants

Let $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$.

cofactor exp. along 1^{st} row

Def

$$\det A = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

In fact, you can use cofactor expansion along any row or column.

Theorem

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad \leftarrow \text{along } i^{th} \text{ col.}$$

OR

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad \leftarrow \text{along } j^{th} \text{ col.}$$

! usually want to choose a row or col with lots of zeros.

! we will also develope a different (and in many ways better) method for computing det A.

Ex Find det A for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 7 & 8 \end{bmatrix}$$

along 2nd row

$$\det A = 4\cdot C_{21} + 5C_{22} + 0C_{23}$$

$$= 4(-1)^3 \begin{vmatrix} 1 & 2 \\ 6 & 7 \end{vmatrix} + 5(-1)^4 \begin{vmatrix} 1 & 3 \\ 6 & 8 \end{vmatrix} + 0$$

$$= -4(7-12) + 5(8-18) = (-4)(-5) + 5(-10) = \boxed{-30}$$

along 3rd col

$$\det A = 3C_{13} + 0C_{23} + 8C_{33}$$

$$= 3(-1)^4 \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} + 0 + 8(-1)^6 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 3\cdot(-2) + 8(-3) = \boxed{-30}$$

Ex Find $\begin{vmatrix} 0 & 4 & 0 & 5 \\ 2 & 2 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 2 & -1 & -2 & 0 \end{vmatrix}$

A

Along 3^{rd} row

$$|A| = 0 \cdot C_{31} + 0\,C_{32} + 0\,C_{33} - 3\,C_{34}$$
$$= (-3)(-1)^7 \begin{vmatrix} 0 & 4 & 0 \\ 2 & 2 & 3 \\ 2 & -1 & -2 \end{vmatrix}$$
$$= 3 \cdot 4 \begin{vmatrix} 2 & 3 \\ 2 & -2 \end{vmatrix}$$

Def A square matrix A is upper triangular if all of the entries below the main diagonal are 0:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \ddots & \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}.$$

$\leftarrow$ main diagonal

Similarly, we say A is lower triangular if all entries above the main diagonal are 0.

Theorem If A is upper or lower triangular, then det A is the product of the entries on the main diagonal.

* For example,

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \Rightarrow \det A = 2 \cdot (-3) \cdot (7) = \boxed{-42}$$

pf idea

For upper tri., keep doing cofactor exp. along 1^{st} col. □

3.2 Properties of Determinants

Although cofactor expansion is relatively straight forward, it is not very efficient.

Here we study how row operations affect determinants. Let's investigate this in the 2x2 case.

Let $A = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$. Then $\det A = 12$.

Interchanging rows

Consider $B = \begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix}$.

$$\det B = -12$$

Thus,

$$A \underset{r_1 \leftrightarrow r_2}{\sim} B \implies -\underbrace{\det A}_{12} = \underbrace{\det B}_{-12}$$

Scaling a row

Consider $B = \begin{bmatrix} 3 & 3 \\ 2 & 10 \end{bmatrix}$.

$$\det B = 24$$

Thus

$$A \underset{r_2 \to 2r_2}{\sim} B \implies 2\underbrace{\det A}_{12} = \underbrace{\det B}_{24}$$

Replacement

Consider $B = \begin{bmatrix} 3 & 3 \\ 0 & 4 \end{bmatrix}$.

$\det B = 12$.

Thus $A \underset{\frac{1}{3}r_1 + r_2 \to r_2}{\sim} B \implies \det A = \det B$

Theorem Let A be $n \times n$.

① Replacement: If B is the result of applying a replacement row operation to A, then

$$\det A = \det B.$$

② Interchange: If B is the result of interchanging two rows of A, then

$$-\det A = \det B.$$

③ Scaling: If B is the result of scaling a row of A by the number k, then

$$k \det A = \det B$$

$$\left(\det A = \frac{1}{k} \det B, \text{ if } k \neq 0 \right)$$

Reduction to Triangular Form

Ex Compute each of the following determinants.

(a) $$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \underbrace{\frac{1}{\;}}_{\text{just replacement}} \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} = \underbrace{\frac{-1}{\;}}_{\text{swap}} \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix}$$

Δ

$$= (-1)\big(1 \cdot 3 \cdot (-5)\big)$$

$$= \boxed{15}$$

think of factoring 2 out of the row

(b) $$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -1 & 4 & -3 & -6 \\ -3 & 0 & 1 & -2 \end{vmatrix} = \underbrace{\frac{2}{\;}}_{\text{scaling}} \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -1 & 4 & -3 & -6 \\ -3 & 0 & 1 & -2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & 0 & -2 \\ 0 & -12 & 10 & 10 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -6 & 2 \end{vmatrix}$$

just replacement   replacement

$$= -2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & -2 \end{vmatrix} = -2\big(1 \cdot 3 \cdot (-6) \cdot -2\big) = \boxed{-72}$$

Invertibility

Recall: A has an inverse $\iff A \sim I$

$*$ Note that row reduction never scales by 0.

$*$ Also $\det I = 1$.

Thus,

$$A \sim A_1 \sim A_2 \sim \cdots \sim A_n = I$$

$$\Rightarrow \det A = c_1 \det A_1 = c_1 c_2 \det A_2 = \cdots = c_1 c_2 \cdots c_n \overbrace{\cancel{\det I}}^{1}$$

$$c_1 \neq 0,\ c_2 \neq 0, \ldots, c_n \neq 0$$

$$\Rightarrow \det A = c_1 c_2 \cdots c_n$$

So $A \sim I \Rightarrow \det A \neq 0$.

Similarly, $A \nsim I \Rightarrow$ RREF of A has row of $0 \Rightarrow \det A = 0$.

Theorem A is invertible if and only if $\det A \neq 0$.

Other Properties of the Det.

Theorem

- $\det A^T = \det A$
- $\det(AB) = (\det A)(\det B)$. proved using elementary matrices

usually, $\det(A+B) \neq \det A + \det B$

Ex Suppose you know that $\det A = 7$, $\det B = \frac{1}{2}$.

(a) Find $\det(BA)$

$$= \det B \cdot \det A = \frac{1}{2} \cdot 7 = \boxed{\frac{7}{2}}$$

(b) Find $\det(B^3)$

$$= \det(B \cdot B \cdot B) = \det B \cdot \det B \cdot \det B = \left(\frac{1}{2}\right)^3 = \boxed{\frac{1}{8}}$$

(c) Find $\det(A^{-1})$.

* Note A^{-1} exists b/c $\det A \neq 0$.

* Also $A \cdot A^{-1} = I$

$$\det(A A^{-1}) = \det I \Rightarrow \det A \cdot \det A^{-1} = 1$$

$$\Rightarrow 7 \cdot \det A^{-1} = 1$$

$$\Rightarrow \det A^{-1} = \boxed{\frac{1}{7}}$$

Theorem If A is invertible, $\det A^{-1} = (\det A)^{-1}$.

Theorem If $A \cdot B$ is invertible, then both A and B are invertible.

pf

AB invertible $\Rightarrow \det AB \neq 0$

$\Rightarrow \det A \cdot \det B \neq 0$

$\Rightarrow \det A \neq 0$ and $\det B \neq 0$

$\Rightarrow A$ is invertible and B is invertible.

3.3 Cramer's Rule, Volume, and Lin. Trans.

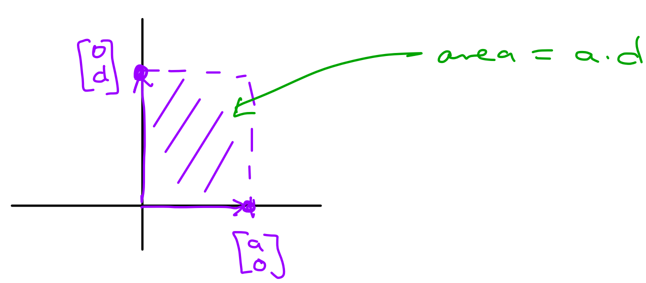
① Cramer's Rule gives an alternative way to solve systems $A\bar{x}=\bar{b}$, but only when A is square and invertible. I recommend reading it. (But row reduction is more efficient.)

② There is a formula for A^{-1} (when it exists) in terms of $\det A$ and cofactors. I recommend reading it. (But row reduction method is more efficient.)

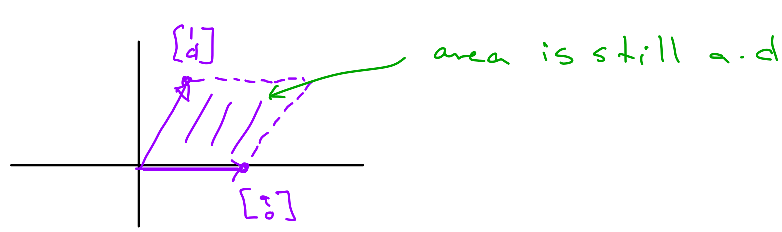
Interpreting the determinant

- Suppose $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. Then $\det A = ad$.

  Now, let's plot the columns of A



- what if $A = \begin{bmatrix} a & 1 \\ 0 & d \end{bmatrix}$. we still have $\det A = ad$.



Theorem

- If A is 2×2, then

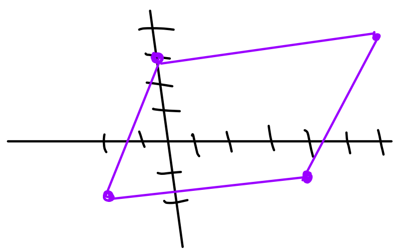
absolute value

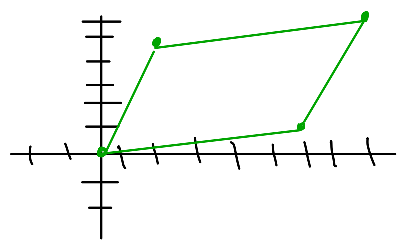
$$|\det A| = \text{area of parallelogram determined by the cols of } A$$

- If A is 3×3, then

$$|\det A| = \text{area of parallelepiped det. by cols of } A$$

Ex Find the area of the parallelogram w/ vertices:
$(-2,-2)$, $(0,3)$, $(4,-1)$, $(6,4)$





- add $(2,2)$ to all vertices to translate it to origin

$(0,0)$, $(2,5)$, $(6,1)$, $(8,6)$

- par. is determined by

$\begin{bmatrix}2\\5\end{bmatrix}$ and $\begin{bmatrix}6\\1\end{bmatrix}$, hence by cols of $A = \begin{bmatrix}2&6\\5&1\end{bmatrix}$

$$\text{Area} = |\det A| = |-28| = \boxed{28}$$

maybe a square or circle or blob

This has an important application to linear transformations.

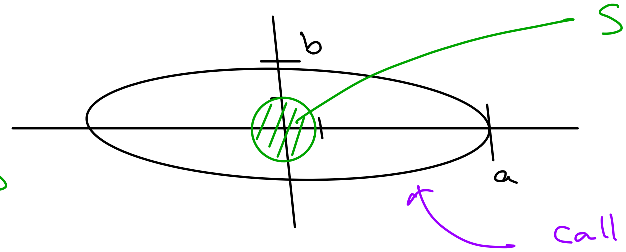
Theorem Suppose $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear trans. Let S be a region in $\mathbb{R}^2$ with finite area. If A is the standard matrix for T, then

image of S

$$\left(\text{area of } T(S)\right) = |\det A| \cdot (\text{area of } S)$$

Does any one rem. the area form. for an ellipse?

<u>Ex</u> Find the area of the ellipse.



call this S

call the ellipse E

This ellipse is the result of stretching the unit circle by a factor of a in x-dir. and b in the y-direction, which can be described by a lin. trans.

Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} a x_1 \\ b x_2 \end{bmatrix}$$

Then T in linear with standard matrix

$$A = \begin{bmatrix} T(\bar{e}_1) & T(\bar{e}_2) \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Then, the theorem says that

$$\begin{pmatrix}\text{area of} \\ E\end{pmatrix} = \begin{pmatrix}\text{area of} \\ T(S)\end{pmatrix} = |\det A| \cdot \begin{pmatrix}\text{area of} \\ S\end{pmatrix}$$

$$= a \cdot b \cdot \pi \cdot 1^2$$

So, area of E is $\boxed{\pi a b}$.