6.1 Inner product, length, orthog.

<u>Motivation</u>: Suppose you are modeling a situation and need to solve a system $A\bar{x} = \bar{b}$, but you find out that the system is inconsistent. what do you do?

<u>option 1</u>: Give up.

<u>Option 2</u>: Try to find the "best possible" approx. solution — that is, find an $\bar{x}$ such that $A\bar{x}$ is a "close as possible" to $\bar{b}$ (even though it may not equal $\bar{b}$).

* To do this, we need to explore "closeness" (i.e. distance).

<u>Inner Product</u>

<u>Def</u> Let $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. The <u>inner product</u> or <u>dot product</u> of $\bar{u}$ and $\bar{v}$ is

$$\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n .$$

* notice that this can be written in terms of matrix multiplication as

$$\bar{u} \cdot \bar{v} = \bar{u}^T \cdot \bar{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

* $\bar{u} \cdot \bar{v}$ is a (single) number.

Ex Let $\bar{u} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$ $\bar{v} = \begin{bmatrix} 6 \\ 7 \\ -2 \end{bmatrix}$.

$$\bar{u} \cdot \bar{v} = \begin{bmatrix} 3 & -1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 7 \\ -2 \end{bmatrix} = 0 - 7 - 10 = -17$$

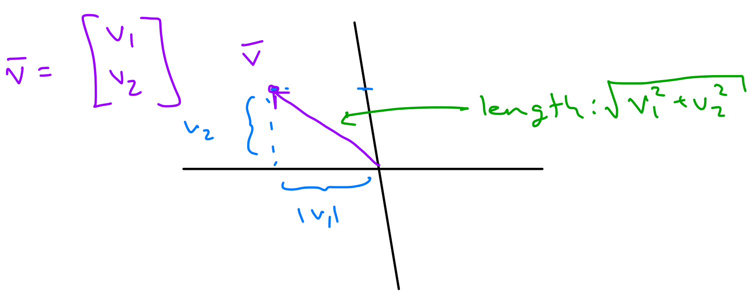
$$\bar{v} \cdot \bar{u} = \begin{bmatrix} 0 & 7 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = 0 - 7 - 10 = -17$$

$$\bar{u} \cdot \bar{u} = (-3)^2 + (-1)^2 + (5)^2 = 9 + 1 + 25 = 35$$

Theorem (Prop. of the inner prod.)

1. $\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$
2. $(\bar{u} + \bar{v}) \cdot w = \bar{u} \cdot \bar{w} + \bar{v} \cdot \bar{w}$
3. $(c\bar{u}) \cdot \bar{v} = c(\bar{u} \cdot \bar{v})$
4. $\bar{u} \cdot \bar{u} \geq 0$ and $\bar{u} \cdot \bar{u} = 0 \iff \bar{u} = \bar{0}$.

Length & Distance



works in higher dimension too

Def The length (or norm) of $\bar{v}$, denoted $\|\bar{v}\|$, is

$$\|\bar{v}\| = \sqrt{\bar{v} \cdot \bar{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

$\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

* Note: $\bar{v} \cdot \bar{v} = \|\bar{v}\|^2$

* $\|c\bar{v}\| = |c| \cdot \|\bar{v}\|$

Def We say $\bar{v}$ is a unit vector if $\|\bar{v}\| = 1$.

Fact $\frac{1}{\|\bar{v}\|} \cdot \bar{v}$ is always a unit vector in the same dir. as $\bar{v}$.

Def The distance b/w $\bar{u}$ and $\bar{v}$, denoted $\text{dist}(\bar{u}, \bar{v})$, is

$$\text{dist}(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2}$$

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Ex Let $\bar{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\bar{w} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$.

1. Find $\text{dist}(\bar{v}, \bar{w})$.
2. Find a unit vector, $\bar{u}$, in same direction as $\bar{v}$.
3. Graph $\bar{v}, \bar{w}, \bar{u}$.

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1. $\text{dist}(\bar{v}, \bar{w}) = \|\bar{v} - \bar{w}\| = \left\| \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \right\| = \sqrt{1^2 + 1^2 + (-3)^2} = \boxed{\sqrt{11}}$

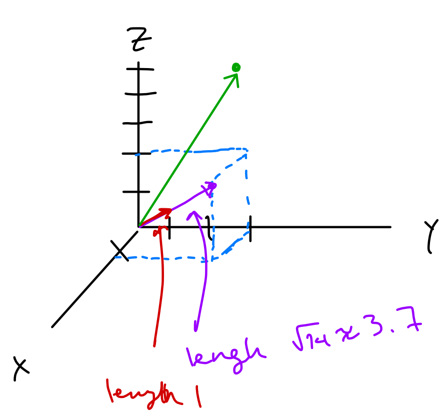
2. $\bar{u} = \frac{1}{\|\bar{v}\|} \cdot \bar{v}$.

 * $\|\bar{v}\| = \sqrt{1 + 9 + 4} = \sqrt{14}$.

 * $\bar{u} = \frac{1}{\sqrt{14}} \cdot \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 1/\sqrt{14} \\ 3/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix}}$.

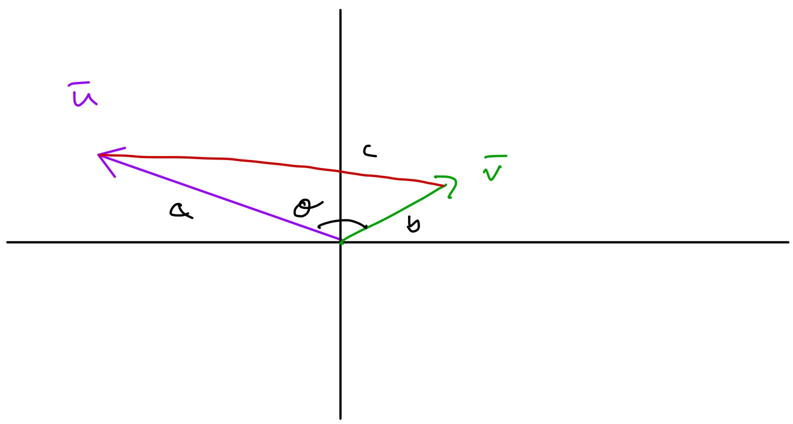
 * Note that $\|\bar{u}\| = \sqrt{\frac{1}{14} + \frac{9}{14} + \frac{4}{14}} = \sqrt{1} = 1$ ✓

3.



Othogonality (i.e. perpendicularity)

we start by looking at the angle b/w two vectors.



By the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$

Thus,

$$\|\bar{u}-\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\|u\|\,\|v\| \cdot \cos\theta.$$

Also,

$$\begin{aligned}\|\bar{u}-\bar{v}\|^2 &= (u-\bar{v})\cdot(\bar{u}-\bar{v}) \\ &= \bar{u}\cdot\bar{u} - 2\,\bar{u}\cdot\bar{v} + \bar{v}\cdot\bar{v} \\ &= \|\bar{u}\|^2 - 2\,\bar{u}\cdot\bar{v} + \|\bar{v}\|^2\end{aligned}$$

dist. and comm.

combining these,

$$-2\,\bar{u}\cdot\bar{v} = -2\,\|\bar{u}\|\,\|\bar{v}\|\cos\theta$$

$$\bar{u}\cdot\bar{v} = \|\bar{u}\|.\|\bar{v}\|\cos\theta$$

Theorem If θ is the angle b/w $\bar{u}$ and $\bar{v}$, then

$$\bar{u}\cdot\bar{v} = \|\bar{u}\|\cdot\|\bar{v}\|\cdot\cos\theta.$$

Def We say $\bar{u}$ and $\bar{v}$ are <u>orthogonal</u> if $\bar{u}\cdot\bar{v}=0$.

* this means that the (smallest) angle b/w. them is $90°$.

Ex Let $\bar{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

(a) show that $\bar{w} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is not orthog. to $\bar{v}$

$\bar{v}\cdot\bar{w} = 1 \neq 0$

(b) Find three different vectors in $\mathbb{R}^3$ that are orthog. to $\bar{v}$. How many possible answers are there?

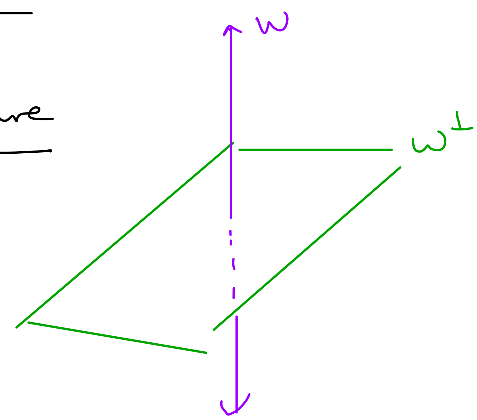
want $\bar{u}$, s.t. $\bar{u}\cdot\bar{v} = 0$. write $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

$$\bar{u}\cdot\bar{v} = 3u_1 + u_2 + u_3 = 0$$

possible answers: $\bar{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}, \ldots$

Def If W is a subspace of $\mathbb{R}^n$, then $W^\perp$ is the collection of all vectors that are orthog. to every vector in W. $W^\perp$ is called the <u>orthogonal complement</u> of W.

Picture



<u>Theorem</u> Let W be a subspace of $\mathbb{R}^n$.

(1) $W^\perp$ is a subspace of $\mathbb{R}^n$.

(2) If $W = \text{span}\{\bar{w}_1, \ldots, \bar{w}_k\}$, then

$\bar{v}$ is in $W^\perp \iff \bar{v}$ is orthog. to $\bar{w}_1, \bar{w}_2, \ldots,$ and $\bar{w}_k$.

6.2 Orthogonal Sets

Def A set of vectors $\{\bar{u}_1, \ldots, \bar{u}_k\}$ is an orthogonal set if each pair of (distinct) vectors is orthogonal. If $\{\bar{u}_1, \ldots, \bar{u}_k\}$ is an orthogonal set AND all vectors are unit vectors, then it is called an orthonormal set.

Ex Let $\bar{u}_1 = \begin{bmatrix} 1 \\ 1/3 \\ 1/3 \end{bmatrix}$, $\bar{u}_2 = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}$, $\bar{u}_3 = \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}$. verify that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is an orthogonal set.

$$\bar{u}_1 \cdot \bar{u}_2 = -2 + 4/3 + \frac{2}{3} = 0$$

$$\bar{u}_1 \cdot \bar{u}_3 = -1 - 4/3 + \frac{7}{3} = 0$$

$$\bar{u}_2 \cdot \bar{u}_3 = 2 - 16 + 14 = 0$$

* Note that this is not an orthonormal set b/c $\|\bar{u}_1\| = \sqrt{1 + 1/9 + 1/9} \neq 1$.

* However, $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ is an orthonormal subset of $\mathbb{R}^3$.

Theorem (orthog. $\Rightarrow$ L.I.) If $\{\bar{u}_1, \ldots, \bar{u}_k\}$ is an orthog. set of nonzero vectors in $\mathbb{R}^n$, then it is automatically lin. independent.

Pf Suppose $c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_k\bar{u}_k = \bar{0}$. we must show that $c_1 = 0$, $c_2 = 0, \ldots, c_k = 0$.

First,

$$(c_1\bar{u}_1 + \cdots + c_k\bar{u}_k) \cdot \bar{u}_1 = \bar{0} \cdot \bar{u}_1$$

$$c_1(\bar{u}_1 \cdot \bar{u}_1) + \cdots + c_k(\bar{u}_k \cdot \bar{u}_1) = 0$$

$$c_1 \|\bar{u}_1\|^2 + 0 + \cdots + 0 = 0$$

since $\bar{u}_1 \neq \bar{0}$, $\|\bar{u}_1\| \neq 0$, so

$$\boxed{c_1 = 0}$$

Next,

$$(c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_k\bar{u}_k)\cdot \bar{u}_2 = \bar{0}\cdot\bar{u}_2$$

$$\Rightarrow \quad c_1(\bar{u}_1\cdot\bar{u}_2) + c_2(\bar{u}_2\cdot\bar{u}_2) + \cdots + c_k(\bar{u}_k\cdot\bar{u}_2) = 0$$

$$\Rightarrow \quad 0 + c_2\|u_2\|^2 + 0 + \cdots + 0 = 0$$

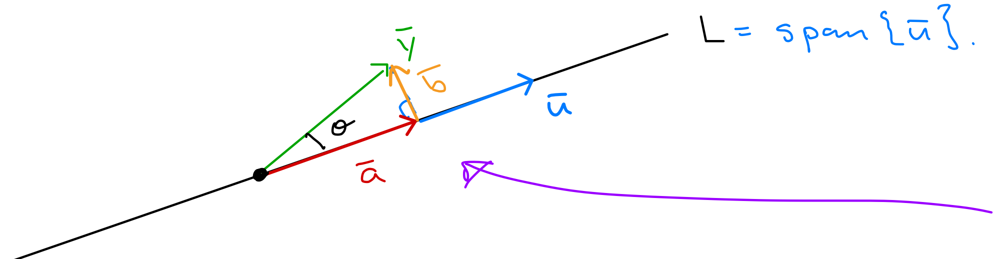
$$\Rightarrow \quad \boxed{c_2 = 0}$$

Continuing in this fashion, we find that all $c_i = 0$. $\square$

The idea in the previous proof leads to another concept.

Projections

L is a line



Notice that $\bar{a}$ is the closest point on L to $\bar{y}$

want $\bar{y} = \bar{a} + \bar{b}$ where

* $\bar{a}$ is in L

AND

* $\bar{a}$ and $\bar{b}$ are orthogonal

Idea: $\bar{a} = (\text{length of } \bar{a}) \cdot \underbrace{\left(\frac{1}{\|\bar{u}\|}\cdot \bar{u}\right)}$ unit vector in dir. of $\bar{u}$

so what is the length of $\bar{a}$?

$$\frac{\|\bar{a}\|}{\|\bar{y}\|} = \cos\theta = \frac{\bar{y}\cdot\bar{u}}{\|\bar{y}\|\cdot\|\bar{u}\|} \Rightarrow \|\bar{a}\| = \frac{\bar{y}\cdot\bar{u}}{\|\bar{u}\|}$$

Thus,

$$\bar{a} = \frac{\bar{y}\cdot\bar{u}}{\|\bar{u}\|} \cdot \frac{1}{\|u\|} \cdot \bar{u} = \frac{\bar{y}\cdot\bar{u}}{\bar{u}\cdot\bar{u}} \cdot \bar{u}$$

Also, note that $\bar{b} = \bar{y} - \bar{a}$, so once we know $\bar{a}$, we can find $\bar{b}$.

<u>Def</u> (Projection onto a line) If $L = \text{Span}\{\bar{u}\}$ for $\bar{u} \neq \bar{0}$, then we define

$$\bar{a} = \text{proj}_L \bar{y} = \frac{\bar{y}\cdot\bar{u}}{\bar{u}\cdot\bar{u}} \cdot \bar{u}$$

This is called the <u>orthogonal projection of $\bar{y}$ onto L</u> (or onto $\bar{u}$).

* $\text{proj}_L \bar{y}$ is the vector on L that is closest to $\bar{y}$.

<u>Ex</u> Let $\bar{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $L = \text{Span}\left\{\underbrace{\begin{bmatrix} 8 \\ 6 \end{bmatrix}}_{=\bar{u}}\right\}$.

(a) Find $\text{proj}_L \bar{y}$.

(b) Write $\bar{y} = \bar{a} + \bar{b}$ where

* $\bar{a}$ is in L

<u>AND</u>

* $\{\bar{a}, \bar{b}\}$ is an orthogonal set.

(a) $$\text{proj}_L \bar{y} = \frac{\bar{y}\cdot\bar{u}}{\bar{u}\cdot\bar{u}} \cdot \bar{u} = \frac{24+6}{64+36} \cdot \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \frac{3}{10} \cdot \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 2.4 \\ 1.8 \end{bmatrix}$$

(b) Let $\boxed{\bar{a} = \begin{bmatrix} 2.4 \\ 1.8 \end{bmatrix}}$, $\boxed{\bar{b} = \bar{y} - \bar{a} = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}}$

Then,

$\bar{y} = \bar{a} + \bar{b}$ and $\bar{a} \cdot \bar{b} = 1.44 - 1.44 = 0$

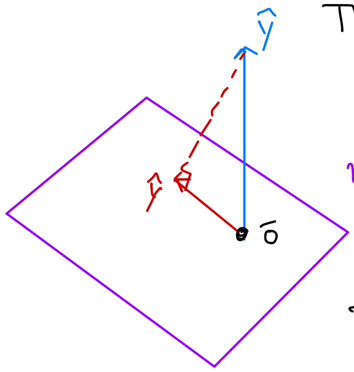
so $\{\bar{a}, \bar{b}\}$ is an orthog. set.

* What if W is an arbitrary subspace?

can we still define $\text{proj}_W \bar{y}$?

6.3 Orthogonal Projections

Def Let W be a subspace of $\mathbb{R}^n$, and let $\{\bar{u}_1, \dots, \bar{u}_k\}$ be any orthogonal basis for W.



Then we define

$$\text{proj}_W \bar{y} = \text{proj}_{\bar{u}_1} \bar{y} + \dots + \text{proj}_{\bar{u}_k} \bar{y}$$

OR

$$= \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1} \bar{y} + \dots + \frac{\bar{y} \cdot \bar{u}_k}{\bar{u}_k \cdot \bar{u}_k} \bar{u}_k$$

This is the orthogonal projection of $\bar{y}$ onto W.

! $\text{proj}_W \bar{y}$ gives the same answer no matter which orthogonal basis you use.

Ex Let $W = \text{Span}\{\bar{w}_1, \bar{w}_2\}$ where

$$\bar{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{w}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

(a) Find $\text{proj}_W \bar{y}$ where $\bar{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$

(b) Graph $\bar{y}$, W, and $\text{proj}_W \bar{y}$ on $\boxed{\text{Geogebra}}$

(a) Note that $\bar{w}_1, \bar{w}_2$ are orthogonal b/c $\bar{w}_1 \cdot \bar{w}_2 = 0$. Thus, they are L.I., so they are an ortog. basis for W. Now,

$$\text{proj}_W \bar{y} = \text{proj}_{\bar{w}_1} \bar{y} + \text{proj}_{\bar{w}_2} \bar{y}$$

$$= \frac{\bar{w}_1 \cdot \bar{y}}{\bar{w}_1 \cdot \bar{w}_1}\bar{w}_1 + \frac{\bar{w}_2 \cdot \bar{y}}{\bar{w}_2 \cdot \bar{w}_2}\bar{w}_2$$

$$= \frac{6}{3} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{7}{14}\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 3/2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}$$

$\bar{w}_1 \cdot \bar{y} = 6$

$\bar{w}_1 \cdot \bar{w}_1 = 3$

$\bar{w}_2 \cdot \bar{y} = 7$

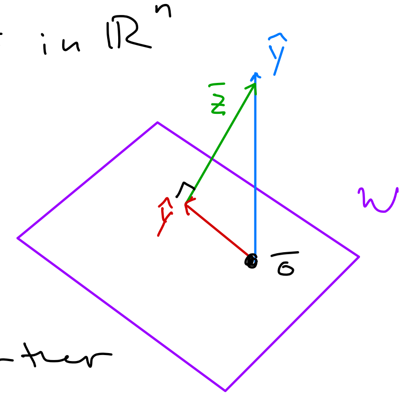
$\bar{w}_2 \cdot \bar{w}_2 = 14$

(b) Geogebra

<u>Theorem</u> (Orthog. Decomp. Theorem) Let W be a subspace of $\mathbb{R}^n$. Then every $\bar{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\bar{y} = \hat{y} + \bar{z}$$

where $\hat{y}$ is in W and $\bar{z}$ is in $\widehat{W}$. Further $\hat{y} = \text{proj}_W \bar{y}$.



<u>Theorem</u> (Best Approx. Theorem) Let W be a subspace of $\mathbb{R}^n$. If $\hat{y} = \text{proj}_W \bar{y}$, then $\hat{y}$ is the point of W closest to $\bar{y}$, i.e.

$$\| y - \hat{y} \| < \| \bar{y} - \bar{w} \|$$

for all $\bar{w}$ in W with $\bar{w} \neq \hat{y}$.

6.5 Least Squares

We know $A\bar{x}=\bar{b}$ may be inconsistent, but we want a process for finding a good approximate solution.

<u>Def</u> If A is $m \times n$ and $\bar{b}$ is in $\mathbb{R}^m$, then we say that $\hat{x}$ is a <u>least squares solution</u> to $A\bar{x}=\bar{b}$ if

$$\|\bar{b} - A\hat{x}\| \leq \|\bar{b} - A\bar{x}\|$$

for all $\bar{x}$ in $\mathbb{R}^n$.

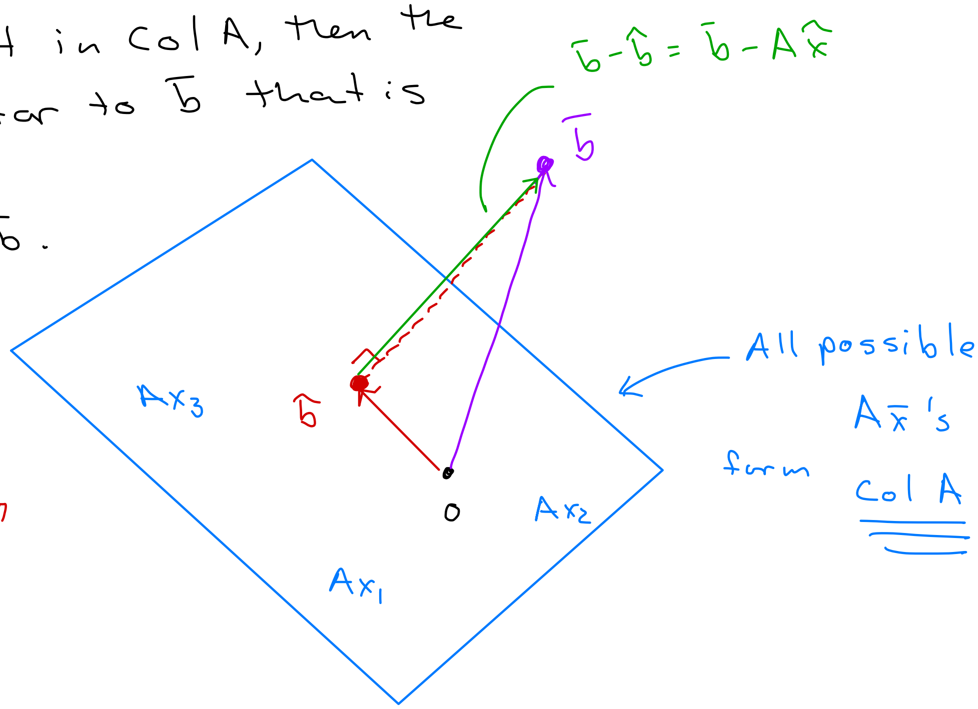
* $\|\bar{b} - A\hat{x}\|$ is called the <u>least squares error</u>.

* $\hat{x}$ is an actual solution if $\|\bar{b} - A\hat{x}\| = 0$.

Q: how might we find a least squares solution?

Idea:

- we can solve $A\bar{x}=\bar{b}$ precisely when $\bar{b}$ is in Col A.
- If $\bar{b}$ is not in Col A, then the closest vector to $\bar{b}$ that is in Col A is $\hat{b} = \text{proj}_{\text{Col}A}\,\bar{b}$.
- So, we solve $A\bar{x} = \hat{b}$
- Picture



- It must be that $\hat{b} = A\hat{x}$ for some $\hat{x}$...

How do we solve for $\hat{x}$?

- Note: $\bar{b}-\hat{b}$ is orthog. to Col A so

  $\bar{b}-\hat{b}$ is orthog to every column of A.

- write $A = [\bar{a}_1 \; \bar{a}_2 \cdots \bar{a}_n]$. Then

$$\bar{a}_1 \cdot (\bar{b}-\hat{b}) = 0 \implies \bar{a}_1^T \cdot (\bar{b}-\hat{b}) = 0$$

$$a_2 \cdot (b-\hat{b}) = 0 \implies \bar{a}_2^T \cdot (\bar{b}-\hat{b}) = 0$$

$$\vdots$$

- thus $A^T \cdot (\bar{b}-\hat{b}) = 0$.

$$\implies A^T\bar{b} - A^T\hat{b} = 0$$

$$\implies A^T\bar{b} = A^T\hat{b} = A^T A\hat{x}$$

Theorem

$\hat{x}$ is a least squares sol. to $A\bar{x}=\bar{b}$ $\iff$ $\hat{x}$ is a (honest) solution to $A^TA\bar{x} = A^T\bar{b}$.

- To find least square sol. to $A\bar{x}=\bar{b}$, we should solve $A^TA\bar{x} = A^T\bar{b}$.

Ex Consider the system $A\bar{x} = \bar{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$$

(a) Show that $A\bar{x} = \bar{b}$ is inconsistent

(b) Find a least squares solution to $A\bar{x} = \bar{b}$

(c) What is the least squares error.

(a) The system of eq. corresponding to $A\bar{x} = \bar{b}$ is

$$\left.\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 &= 3 \end{aligned}\right\} \Rightarrow 0 = -2 \Rightarrow \text{clearly inconsistent}$$

$$x_1 + x_3 = 8$$
$$x_1 + x_3 = 2$$

(b) We solve $A^T A \bar{x} = A^T \bar{b}$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$A^T \bar{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix}$$

Solve

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 14 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 5 - x_3$

$x_2 = -3 + x_3$

x_3 free

All least squares sols: $\hat{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

So, one least squares sol. is

$s=0$

$$\boxed{\hat{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}}$$

(c) Least squares error

$$A\hat{x} = \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix} \quad \text{so}$$

$$\text{error} = \|\bar{b} - A\hat{x}\| = \left\| \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} -1 \\ 1 \\ 3 \\ -3 \end{bmatrix} \right\| = \sqrt{1+1+9+9} = \sqrt{20} \approx 4.5$$

Ex Suppose you obtain the following data points

$(0,2)$, $(-3,5)$, $(2,3)$, $(4,12)$

and want to model the data using a quadratic function of the form

$$f(t) = c_0 + c_1 t + c_2 t^2.$$

Find a best fit quadratic to the data using least squares.

$$f(0)=2 \Rightarrow c_0 + c_1(0) + c_2(0)^2 = 2$$
$$f(-3)=5 \Rightarrow c_0 + c_1(-3) + c_2(-3)^2 = 5$$
$$f(2)=3 \Rightarrow c_0 + c_1(2) + c_2(2)^2 = 3$$
$$f(4)=12 \Rightarrow c_0 + c_1(4) + c_2(4)^2 = 12$$

so

$$\begin{aligned} c_0 + 0c_1 + 0c_2 &= 2 \\ c_0 - 3c_1 + 9c_2 &= 5 \\ c_0 + 2c_1 + 4c_2 &= 3 \\ c_0 + 4c_1 + 16c_2 &= 12 \end{aligned} \Rightarrow \begin{array}{c} \qquad \underline{A} \qquad\qquad \underline{\bar{b}} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 1 & -3 & 9 & 5 \\ 1 & 2 & 4 & 3 \\ 1 & 4 & 16 & 12 \end{array}\right] \end{array}$$

<u>least squares:</u>

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 2 & 4 \\ 0 & 9 & 4 & 16 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -3 & 9 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 29 \\ 3 & 29 & 45 \\ 29 & 45 & 353 \end{bmatrix}$$

$$A^T \bar{b} = \begin{bmatrix} 22 \\ 39 \\ 249 \end{bmatrix}$$

approx.

solve $\left[\begin{array}{ccc|c} 4 & 3 & 29 & 22 \\ 3 & 29 & 45 & 39 \\ 29 & 45 & 353 & 249 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.103 \\ 0 & 1 & 0 & 0.345 \\ 0 & 0 & 1 & 0.571 \end{array}\right]$

so, our best fit quadratic is

$$\boxed{f(t) = 1.103 + 0.345t + 0.571t^2}$$

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