Chapter 2

Mathematics and Logic

Before you get started, make sure you've read Chapter 1, which sets the tone for the work we will begin doing here.

2.1 A Taste of Number Theory

In this section, we will work with the set of integers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. The purpose of this section is to get started with proving some theorems about numbers and study the properties of $\mathbb{Z}$. Because you are so familiar with properties of the integers, one of the issues that we will bump into knowing which facts about the integers we can take for granted. As a general rule of thumb, you should attempt to use the definitions provided without relying too much on your prior knowledge. We will likely need to discuss this further as issues arise.

It is important to note that we are diving in head first here. There are going to be some subtle issues that you will bump into and our goal will be to see what those issues are, and then we will take a step back and start again. See what you can do!

Recall that we use the symbol "$\in$" as an abbreviation for the phrase "is an element of" or sometimes simply "in." For example, the mathematical expression "$n \in \mathbb{Z}$" means "n is an element of the integers."

Definition 2.1. An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$.

Definition 2.2. An integer n is **odd** if $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Notice that we did not define "even" as being divisible by 2. When tackling the next few theorems and problems, you should use the formal definition of even as opposed to the well-known divisibility condition. For the remainder of this section, you may assume that every integer is either even or odd but never both.

Theorem 2.3. The sum of two consecutive integers is odd.

Theorem 2.4. If n is an even integer, then n^2 is an even integer.

Problem 2.5. Prove or provide a counterexample: The sum of an even integer and an odd integer is odd.

Question 2.6. Did Theorem 2.3 need to come before Problem 2.5? Could we have used Problem 2.5 to prove Theorem 2.3? If so, outline how this alternate proof would go. Perhaps your original proof utilized the approach I'm hinting at. If this is true, can you think of a proof that does not rely directly on Problem 2.5? Is one approach better than the other?

Problem 2.7. Prove or provide a counterexample: The product of an odd integer and an even integer is odd.

Problem 2.8. Prove or provide a counterexample: The product of an odd integer and an odd integer is odd.

Problem 2.9. Prove or provide a counterexample: The product of two even integers is even.

Definition 2.10. An integer n **divides** the integer m, written $n|m$, if and only if there exists $k \in \mathbb{Z}$ such that $m = nk$. In the same context, we may also write that m **is divisible by** n.

Question 2.11. For integers n and m, how are following mathematical expressions similar and how are they different?

(a) $m|n$

(b) $\dfrac{m}{n}$

(c) m/n

In this section on number theory, we allow addition, subtraction, and multiplication of integers. In general, division is not allowed since an integer divided by an integer may result in a number that is not an integer. The upshot: don't write $\frac{m}{n}$. When you feel the urge to divide, switch to an equivalent formulation using multiplication. This will make your life much easier when proving statements involving divisibility.

Problem 2.12. Let $n \in \mathbb{Z}$. Prove or provide a counterexample: If 6 divides n, then 3 divides n.

Problem 2.13. Let $n \in \mathbb{Z}$. Prove or provide a counterexample: If 6 divides n, then 4 divides n.

Theorem 2.14. Assume $n, m, a \in \mathbb{Z}$. If $a|n$, then $a|mn$.

A theorem that follows almost immediately from another theorem is called a **corollary** (see Appendix B). See if you can prove the next result quickly using the previous theorem. Be sure to cite the theorem in your proof.

Corollary 2.15. Assume $n, a \in \mathbb{Z}$. If a divides n, then a divides n^2.

Problem 2.16. Assume $n, a \in \mathbb{Z}$. Prove or provide a counterexample: If a divides n^2, then a divides n.

Theorem 2.17. Assume $n, a \in \mathbb{Z}$. If a divides n, then a divides $-n$.

Theorem 2.18. Assume $n, m, a \in \mathbb{Z}$. If a divides m and a divides n, then a divides $m+n$.

Problem 2.19. Is the converse[1] of Theorem 2.18 true? That is, is the following statement true?

> Assume $n, m, a \in \mathbb{Z}$. If a divides $m+n$, then a divides m and a divides n.

If the statement is true, prove it. If the statement is false, provide a counterexample.

Once we've proved a few theorems, we should be on the look out to see if we can utilize any of our current results to prove new results. There's no point in reinventing the wheel if we don't have to. Try to use a couple of our previous results to prove the next theorem.

Theorem 2.20. Assume $n, m, a \in \mathbb{Z}$. If a divides m and a divides n, then a divides $m-n$.

Problem 2.21. Assume $a, b, m \in \mathbb{Z}$. Determine whether the following statement holds sometimes, always, or never. If ab divides m, then a divides m and b divides m. Justify with a proof or counterexample.

Theorem 2.22. If $a, b, c \in \mathbb{Z}$ such that a divides b and b divides c, then a divides c.

The previous theorem is referred to as **transitivity of division of integers**.

Theorem 2.23. The sum of any three consecutive integers is always divisible by three.

2.2 Introduction to Logic

After diving in head first in the last section, we'll take a step back and do a more careful examination of what it is we are actually doing.

Definition 2.24. A **proposition** (or **statement**) is a sentence that is either true or false.

For example, the sentence "All liberals are hippies" is a false proposition. However, the perfectly good sentence "$x = 1$" is *not* a proposition all by itself since we don't actually know what x is.

Exercise 2.25. Determine whether the following are propositions or not. Explain.

(a) All cars are red.

(b) Led Zeppelin is the best band of all time.

(c) If my name starts with the letter J, then my name is Joe.

(d) $x^2 = 4$.

[1]See Definition 2.37 for the formal definition of converse.

(e) There exists an x such that $x^2 = 4$.

(f) For all real numbers x, $x^2 = 4$.

(g) $\sqrt{2}$ is an irrational number.

(h) p is prime.

Given two propositions, we can form more complicated propositions using logical connectives.

Definition 2.26. Let A and B be propositions.

(a) The proposition "**not** A" is true iff[2] A is false; expressed symbolically as $\boxed{\neg A}$ and called the **negation** of A.

(b) The proposition "A **and** B" is true iff both A and B are true; expressed symbolically as $\boxed{A \wedge B}$ and called the **conjunction** of A and B.

(c) The proposition "A **or** B" is true iff at least one of A or B is true; expressed symbolically as $\boxed{A \vee B}$ and called the **disjunction** of A and B.

(d) The proposition "**If** A**, then** B" is true iff both A and B are true, or A is false; expressed symbolically as $\boxed{A \implies B}$ and called an **implication** or **conditional statement**. Note that $A \implies B$ may also be read as "A implies B" or "A only if B".

Exercise 2.27. Describe the meaning of $\neg(A \wedge B)$ and $\neg(A \vee B)$.

Exercise 2.28. Let A represent "6 is an even number" and B represent "6 is a multiple of 4." Express each of the following in ordinary English sentences and state whether the statement is true or false.

(a) $A \wedge B$

(b) $A \vee B$

(c) $\neg A$

(d) $\neg B$

(e) $\neg(A \wedge B)$

(f) $\neg(A \vee B)$

(g) $A \implies B$

Definition 2.29. A **truth table** is a table that illustrates all possible truth values for a proposition.

[2]Throughout mathematics, the phrase "if and only if" is common enough that it is often abbreviated "iff." Roughly speaking, this phrase/word means "exactly when."

Example 2.30. Let A and B be propositions. Then the truth table for the conjunction $A \wedge B$ is given by the following.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Notice that we have columns for each of A and B. The rows for these two columns correspond to all possible combinations for A and B. The third column gives us the truth value of $A \wedge B$ given the possible truth values for A and B.

Note that each proposition has two possible truth values: true or false. Thus, if a compound proposition P is built from n propositions, then the truth table for P will require 2^n rows.

Exercise 2.31. Create a truth table for each of $A \vee B$, $\neg A$, $\neg(A \wedge B)$, and $\neg A \wedge \neg B$. Feel free to add additional columns to your tables to assist you with intermediate steps.

Problem 2.32. A coach promises, "If we win tonight, then I will buy you pizza tomorrow." Determine the case(s) in which the players can rightly claim to have been lied to. Use this to help create a truth table for $A \implies B$.

Definition 2.33. Two statements P and Q are (**logically**) **equivalent**, expressed symbolically as $\boxed{P \iff Q}$ and read "P iff Q", iff they have the same truth table.

Each of the next three facts can be justified using truth tables.

Theorem 2.34. If A is a proposition, then $\neg(\neg A)$ is equivalent to A.

Theorem 2.35 (DeMorgan's Law)**.** If A and B are propositions, then $\neg(A \wedge B) \iff \neg A \vee \neg B$.

Problem 2.36. Let A and B be propositions. Conjecture a statement similar to Theorem 2.35 for the proposition $\neg(A \vee B)$ and then prove it. This is also called DeMorgan's Law.

Definition 2.37. The **converse** of $A \implies B$ is $B \implies A$.

Exercise 2.38. Provide an example of a true conditional proposition whose converse is false.

Definition 2.39. The **contrapositive** of $A \implies B$ is $\neg B \implies \neg A$.

Exercise 2.40. Let A and B represent the statements from Exercise 2.28. Express the following in ordinary English sentences.

(a) The converse of $A \implies B$.

(b) The contrapositive of $A \implies B$.

Exercise 2.41. Find the converse and the contrapositive of the following statement: "If a person lives in Flagstaff, then that person lives in Arizona."

Use truth tables to prove the following theorem.

Theorem 2.42. The implication $A \implies B$ is equivalent to its contrapositive.

The upshot of Theorem 2.42 is that if you want to prove a conditional proposition, you can prove its contrapositive instead. Try proving each of the next three theorems by proving the contrapositive of the given statement instead.

Theorem 2.43. Assume $x, y \in \mathbb{Z}$. If xy is odd, then both x and y are odd.

Theorem 2.44. Assume $x, y \in \mathbb{Z}$. If xy is even, then x or y is even.

Theorem 2.45. Assume $x \in \mathbb{Z}$. If x^2 is even, then x is even.

2.3 Negating Implications and Proof by Contradiction

So far we have discussed how to negate propositions of the form A, $A \wedge B$, and $A \vee B$ for propositions A and B. However, we have yet to discuss how to negate propositions of the form $A \implies B$. To begin, try proving the following result with a truth table.

Theorem 2.46. The implication $A \implies B$ is equivalent to the disjunction $\neg A \vee B$.

The next result follows quickly from Theorem 2.46 together with DeMorgan's Law.

Corollary 2.47. The proposition $\neg(A \implies B)$ is equivalent to $A \wedge \neg B$.

Exercise 2.48. Let A and B be the propositions "Darth Vader is a hippie" and "Sarah Palin is a liberal," respectively.

(a) Express $A \implies B$ as an English sentence involving the disjunction "or."

(b) Express $\neg(A \implies B)$ as an English sentence involving the conjunction "and."

Exercise 2.49. The proposition "If $.\overline{99} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$, then $.\overline{99} \neq 1$" is *false*. Write its (true) negation, as a conjunction.

Recall that a proposition is exclusively either true or false—it never be both.

Definition 2.50. A compound proposition that is always false is called a **contradiction**. A compound proposition that is always true is called a **tautology**.

Theorem 2.51. For any proposition A, the proposition $\neg A \wedge A$ is a contradiction.

Exercise 2.52. Provide an example of a tautology using arbitrary propositions and any of the logical connectives $\neg$, $\wedge$, and $\vee$. Prove that your example is in fact a tautology.

Suppose that we want to prove some proposition P (which might be something like $A \implies B$ or even more complicated). One approach, called **proof by contradiction**, is to assume $\neg P$ and then logically deduce a contradiction of the form $Q \wedge \neg Q$, where Q is some proposition (possibly equal to P). Since this is absurd, the assumption $\neg P$ must have been false, so P is true. The tricky part about a proof by contradiction is that it is not usually obvious what the statement Q should be.

Skeleton Proof 2.53 (Proof of P by contradiction)**.** Here is what the general structure for a proof by contradiction looks like if we are trying to prove the proposition P.

Proof. For sake of a contradiction, assume $\neg P$.

… *[Use definitions and known results to derive some Q and its negation ¬Q.]* …

This is a contradiction. Therefore, P. □

Proof by contradiction can be useful for proving statements of the form $A \implies B$, where $\neg B$ is easier to "get your hands on," because $\neg(A \implies B)$ is equivalent to $A \wedge \neg B$ (see Corollary 2.47).

Skeleton Proof 2.54 (Proof of $A \implies B$ by contradiction)**.** If you want to prove the implication $A \implies B$ via a proof by contradiction, then the structure of the proof is as follows.

Proof. For sake of a contradiction, assume A and $\neg B$.

… *[Use definitions and known results to derive some Q and its negation ¬Q.]* …

This is a contradiction. Therefore, if A, then B. □

Establish the following theorem in two ways: (i) prove the contrapositive, and (ii) prove via contradiction.

Theorem 2.55. Assume that $x \in \mathbb{Z}$. If x is odd, then 2 does not divide x. (Prove in two different ways.)

Prove the following theorem via contradiction. Afterward, consider the difficulties one might encounter when trying to prove the result more directly.

Theorem 2.56. Assume that $x, y \in \mathbb{N}$.[3] If x divides y, then $x \leq y$.

[3] $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of **natural numbers**. Some mathematicians (set theorists, in particular) include 0 in $\mathbb{N}$, but this will not be our convention. The given statement is not true if we replace $\mathbb{N}$ with $\mathbb{Z}$. Do you see why?

2.4 Introduction to Quantification

The sentence "$x > 0$" is not itself a proposition because x is a **free variable**. A sentence with a free variable is a **predicate**. To turn a predicate into a proposition, we must either substitute values for each free variable, or else "quantify" the free variables.

Function notation is a convenient way to represent predicates. For example, each of the following represents a predicate with the indicated free variables.

- $S(x) :=$ "$x^2 - 4 = 0$"
- $L(a,b) :=$ "$a < b$"
- $F(x,y) :=$ "x is friends with y"

The notation := indicates a definition. Also, note that the use of the quotation marks above removed some ambiguity. What would $S(x) = x^2 - 4 = 0$ mean? It looks like $S(x)$ equals 0, but actually we want $S(x)$ to represent the whole sentence "$x^2 - 4 = 0$".

One way we can make propositions out of predicates is by assigning specific values to the free variables. That is, if $P(x)$ is a predicate and x_0 is specific value for x, then $P(x_0)$ is now a proposition (and is either true or false).

Exercise 2.57. Consider $S(x)$ and $L(a,b)$ as defined above. Determine the truth values of $S(0)$, $S(-2)$, $L(2,1)$, and $L(-3,-2)$. Is $L(2,b)$ a proposition or a predicate? Explain.

Besides substituting specific values for free variables in a predicate, we can also make a claim about which values of the free variables apply to the predicate.

Exercise 2.58. Both of the following sentences are propositions. Decide whether each is true or false. What would it take to justify your answers?

(a) For all $x \in \mathbb{R}$, $x^2 - 4 = 0$.[4]

(b) There exists $x \in \mathbb{R}$ such that $x^2 - 4 = 0$.

Definition 2.59. "For all" is the **universal quantifier** and "there exists...such that" is the **existential quantifier**.

We can replace "there exists...such that" with phrases like "for some" (possibly with some other tweaking to the sentence). Similarly, "for all", "for any", "for every" are used interchangeably in mathematics (even though they might convey slightly different meanings in colloquial language). It is important to note that the existential quantifier is making a claim about "at least one" *not* "exactly one."

Variables that are quantified with a universal or existential quantifier are said to be **bound**. To be a proposition, *all* variables must be bound. That is, in a proposition all variables are quantified.

We must take care to specify the universe of acceptable values for the free variables. Consider the sentence "For all x, $x > 0$." Is this sentence true or false? The answer depends on what set the universal quantifier applies to. Certainly, the sentence is false if

[4]The symbol $\mathbb{R}$ denotes the set of all real numbers.

we apply it for all $x \in \mathbb{Z}$. However, the sentence is true for all $x \in \mathbb{N}$. Context may resolve ambiguities, but otherwise, we must write clearly: "For all $x \in \mathbb{Z}$, $x > 0$" or "For all $x \in \mathbb{N}$, $x > 0$." The set of acceptable values for a variable is called the **universe (of discourse)**.

Exercise 2.60. Suppose our universe of discourse is the set of integers.

(a) Provide an example of a predicate $P(x)$ such that "For all x, $P(x)$" is true.

(b) Provide an example of a predicate $Q(x)$ such that "For all x, $Q(x)$" is false, but "There exists x such that $Q(x)$" is true.

If a predicate has more than one free variable, then we can build propositions by quantifying each variable. However, *the order of the quantifiers is extremely important!*

Exercise 2.61. Let $P(x,y)$ be a predicate with free variables x and y in a universe of discourse U. One way to quantify the variables is "For all $x \in U$, there exists $y \in U$ such that $P(x,y)$." How else can the variables be quantified?

The possibilities listed in the previous exercise are *not* equivalent to each other.

Exercise 2.62. Suppose the universe of discourse is the set of people. Consider the predicate $M(x,y) :=$ "x is married to y". Discuss the meaning of each of the following.

(a) For all x, there exists y such that $M(x,y)$.

(b) There exists y such that for all x, $M(x,y)$.

(c) For all x, for all y, $M(x,y)$.

(d) There exists x such that there exists y such that $M(x,y)$.

Exercise 2.63. Consider the predicate $F(x,y) :=$ "$x = y^2$". Discuss the meaning of each of the following.

(a) There exists x such that there exists y such that $F(x,y)$.

(b) There exists y such that there exists x such that $F(x,y)$.

(c) For all $x \in \mathbb{R}$, for all $y \in \mathbb{R}$, $F(x,y)$.

(d) For all $y \in \mathbb{R}$, for all $x \in \mathbb{R}$, $F(x,y)$.

There are a couple of key points to keep in mind about quantification. To be a proposition, all variables must be quantified. This can happen in at least two ways:

- The variables are explicitly bound by quantifiers in the same sentence.
- The variables are implicitly bound by preceding sentences or by context. Statements of the form "Let $x = \ldots$" and "Let $x \in \ldots$" bind the variable x and remove ambiguity.

The order of the quantification is important. Reversing the order of the quantifiers can substantially change the meaning of a proposition.

Quantification and logical connectives ("and," "or," "If..., then...," and "not") enable complex mathematical statements. For example, the formal definition of $\lim_{x\to c} f(x) = L$ is

> For all $\epsilon > 0$, there exists $\delta > 0$ such that for all x, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

In order to study the abstract nature of complicated mathematical statements, it is useful to adopt some notation.

Definition 2.64. The universal quantifier "for all" is denoted $\boxed{\forall}$, and the existential quantifier "there exists... such that" is denoted $\boxed{\exists}$.

Using our abbreviations for the logical connectives and quantifiers, we can symbolically represent mathematical propositions. For example, the (true) proposition "There exists $x \in \mathbb{R}$ such that $x^2 - 1 = 0$" becomes "$(\exists x \in \mathbb{R})(x^2 - 1 = 0)$," while the (false) proposition "For all $x \in \mathbb{N}$, there exists $y \in \mathbb{N}$ such that $y < x$" becomes "$(\forall x)(x \in \mathbb{N} \implies (\exists y)(y \in \mathbb{N} \implies y < x))$" or "$(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y < x)$."

Exercise 2.65. Convert the following statements into statements using only logical symbols. Assume that the universe of discourse is the set of real numbers.

(a) There exists a number x such that $x^2 + 1$ is greater than zero.

(b) There exists a natural number n such that $n^2 = 36$.

(c) For every real number x, x^2 is greater than or equal to zero.

Exercise 2.66. Express the formal definition of a limit (given above Definition 2.64) in logical symbols.

If $A(x)$ and $B(x)$ are predicates, then it is standard practice for the sentence $A(x) \implies B(x)$ to mean $(\forall x)(A(x) \implies B(x))$ (where the universe of discourse for x needs to be made clear). In this case, we say that the universal quantifier is implicit.

Exercise 2.67. Consider the proposition "If $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$." Assume the universe of discourse is the set $\mathbb{R}$.

(a) Express the statement in logical symbols. Is the statement true?

(b) Reverse the order of the quantifiers to get a new statement. Does the meaning change? If so, how? Is the new statement true?

The symbolic expression $(\forall x)(\forall y)$ can be replaced with the simpler expression $(\forall x, y)$ as long as x and y are elements of the same universe.

Exercise 2.68. Express the statement "For all $x, y \in \mathbb{R}$ with $x < y$, there exists $m \in \mathbb{R}$ such $x < m < y$" using logical symbols.

Exercise 2.69. Rewrite the following statements in words and determine whether each is true or false.

(a) $(\forall n \in \mathbb{N})(n^2 \geq 5)$

(b) $(\exists n \in \mathbb{N})(n^2 - 1 = 0)$

(c) $(\exists N \in \mathbb{N})(\forall n > N)(\frac{1}{n} < 0.01)$

(d) $(\forall m, n \in \mathbb{Z})(2|m \wedge 2|n \implies 2|(m+n))$

(e) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(x - 2y = 0)$

(f) $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})(y \leq x)$

To whet your appetite for the next section, consider how you might prove a statement of the form "For all x. . . ." If a statement is false, then its negation is true. How would you go about negating a statement involving quantifiers?

2.5 More About Quantification

Mathematical proofs do not explicitly use the symbolic representation of a given statement in terms of quantifiers and logical connectives. Nonetheless, having this notation at our disposal allows us to compartmentalize the abstract nature of mathematical propositions and will provide us with a way to talk about the meta-concepts surrounding the construction of proofs.

Definition 2.70. Two quantified propositions are **equivalent in a given universe of discourse** iff they have the same truth value in that universe. Two quantified propositions are **equivalent** iff they are equivalent in every universe of discourse.

Exercise 2.71. Consider the propositions $(\forall x)(x > 3)$ and $(\forall x)(x \geq 4)$.

(a) Are these propositions equivalent if the universe of discourse is the set of integers?

(b) Give two different universes of discourse that yield different truth values for these propositions.

(c) What can you conclude about the equivalence of these statements?

It is worth pointing out an important distinction. Consider the propositions "All cars are red" and "All natural numbers are positive". Both of these are instances of the **logical form** $(\forall x)P(x)$. It turns out that the first proposition is false and the second is true; however, it does not make sense to attach a true value to the logical form. A logical form is a blueprint for particular propositions. If we are careful, it makes sense to talk about whether two logical forms are equivalent. For example, $(\forall x)(P(x) \implies Q(x))$ is equivalent to $(\forall x)(\neg Q(x) \implies \neg P(x))$. For fixed $P(x)$ and $Q(x)$, these two forms will always have the same truth value independent of the universe of discourse. If you change $P(x)$ and $Q(x)$, then the truth value may change, but the two forms will still agree.

The next theorem tell us how to negate logical forms involving quantifiers.

Theorem 2.72. Let $P(x)$ be a predicate. Then

(a) $\neg(\forall x)P(x)$ is equivalent to $(\exists x)(\neg P(x))$

(b) $\neg(\exists x)P(x)$ is equivalent to $(\forall x)(\neg P(x))$.

Exercise 2.73. Negate each of the following. Disregard the truth value and the universe of discourse.

(a) $(\forall x)(x > 3)$

(b) $(\exists x)(x \text{ is prime} \wedge x \text{ is even})$

(c) All cars are red.

(d) Every Wookiee is named Chewbacca.

(e) Some hippies are republican.

(f) For all $x \in \mathbb{N}$, $x^2 + x + 41$ is prime.

(g) There exists $x \in \mathbb{Z}$ such that $1/x \notin \mathbb{Z}$.

(h) There is no function f such that if f is continuous, then f is not differentiable.

Using Theorem 2.72 and our previous results involving quantification, we can negate complex mathematical propositions by working from left to right. For example, if we negate the (false) proposition $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x+y = 0)$, we obtain the proposition $\neg(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x+y = 0)$, which is equivalent to $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x+y \neq 0)$.

For a more complicated example, consider the (false) proposition $(\forall x)[x > 0 \implies (\exists y)(y < 0 \wedge xy > 0)]$. Then its negation $\neg(\forall x)[x > 0 \implies (\exists y)(y < 0 \wedge xy > 0)]$ is equivalent to $(\exists x)[x > 0 \wedge \neg(\exists y)(y < 0 \wedge xy > 0)]$, which happens to be equivalent to $(\exists x)[x > 0 \wedge (\forall y)(y \geq 0 \vee xy \leq 0)]$. Can you identify the previous theorems that were used when negating this proposition?

Exercise 2.74. Negate each of the following. Disregard the truth value and the universe of discourse.

(a) $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m < n)$

(b) $(\forall x, y, z \in \mathbb{Z})((xy \text{ is even} \wedge yz \text{ is even}) \implies xy \text{ is even})$

(c) For all positive real numbers x, there exists a real number y such that $y^2 = x$.

(d) There exists a married person x such that for all married people y, x is married to y.

At this point, we should be able to use our understanding of quantification to construct counterexamples to complicated false propositions and proofs of complicated true propositions. Here are some general proof structures for various logical forms.

Skeleton Proof 2.75 (Direct Proof of $(\forall x)P(x)$)**.** Here is the general structure for a direct proof of the proposition $(\forall x)P(x)$.

Proof. Let $x \in U$. *[U is the universe of discourse]*

… *[Use definitions and known results.]* …

Therefore, $P(x)$ is true. Since x was arbitrary, for all x, $P(x)$. □

Skeleton Proof 2.76 (Proof of $(\forall x)P(x)$ by Contradiction)**.** Here is the general structure for a proof of the proposition $(\forall x)P(x)$ via contradiction.

Proof. For sake of a contradiction, assume that there exists $x \in U$ such that $\neg P(x)$. *[U is the universe of discourse]*

… *[Do something to derive a contradiction.]* …

This is a contradiction. Therefore, for all x, $P(x)$ is true. □

Skeleton Proof 2.77 (Direct Proof of $(\exists x)P(x)$)**.** Here is the general structure for a direct proof of the proposition $(\exists x)P(x)$.

Proof. … *[Use definitions and previous results to deduce that an x exists for which P(x) is true; or if you have an x that works, just verify that it does.]* …
Therefore, there exists x such that $P(x)$. □

Skeleton Proof 2.78 (Proof of $(\exists x)P(x)$ by Contradiction)**.** Here is the general structure for a proof of the proposition $(\exists x)P(x)$ via contradiction.

Proof. For sake of a contradiction, assume that for all x, $\neg P(x)$.

… *[Do something to derive a contradiction.]* …

This is a contradiction. Therefore, there exists x such that $P(x)$. □

Note that if $Q(x)$ is a proposition for which $(\forall x)Q(x)$ is false, then a counterexample to this proposition amounts to showing $(\exists x)(\neg Q(x))$, which might be proved via the third scenario above.

It is important to point out that sometimes we will have to combine various proof techniques in a single proof. For example, if you wanted to prove a proposition of the form $(\forall x)(P(x) \implies Q(x))$ by contradiction, we would start by assuming that there exists x such that $P(x)$ and $\neg Q(x)$.

Problem 2.79. For each of the following statements, determine its truth value. If the statement is false, provide a counterexample. Prove at least two of the true statements.

(a) For all $n \in \mathbb{N}$, $n^2 \geq 5$.

(b) There exists $n \in \mathbb{N}$ such that $n^2 - 1 = 0$.

(c) There exists $x \in \mathbb{N}$ such that for all $y \in \mathbb{N}$, $y \leq x$.

(d) For all $x \in \mathbb{Z}$, $x^3 \geq x$.

(e) For all $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$ such that $n + m = 0$.

(f) There exists integers a and b such that $2a + 7b = 1$.

(g) There do not exist integers m and n such that $2m + 4n = 7$.

(h) For all integers a, b, c, if a divides bc, then either a divides b or a divides c.

To prove the next theorem, you might want to consider two different cases.

Theorem 2.80. For all integers, $3n^2 + n + 14$ is even.