Chapter 4

Induction

In this chapter, we introduce mathematical induction, which is a proof technique that is useful for proving statements of the form $(\forall n \in \mathbb{N})P(n)$, or more generally $(\forall n \in \mathbb{Z})(n \geq a \implies P(n))$, where $P(n)$ is some predicate and $a \in \mathbb{Z}$.

4.1 Introduction to Induction

Consider the claims:

(a) For all $n \in \mathbb{N}$, $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

(b) For all $n \in \mathbb{N}$, $n^2 + n + 41$ is prime.

Let's take a look at potential proofs.

"Proof" of (a). If $n = 1$, then $1 = \frac{1(1+1)}{2}$. If $n = 2$, then $1 + 2 = 3 = \frac{2(2+1)}{2}$. If $n = 3$, then $1 + 2 + 3 = 6 = \frac{3(3+1)}{2}$, and so on. □

"Proof" of (b). If $n = 1$, then $n^2 + n + 41 = 43$, which is prime. If $n = 2$, then $n^2 + n + 41 = 47$, which is prime. If $n = 3$, then $n^2 + n + 41 = 53$, which is prime, and so on. □

Are these actual proofs? **NO!** In fact, the second claim isn't even true. If $n = 41$, then $n^2 + n + 41 = 41^2 + 41 + 41 = 41(41 + 1 + 1)$, which is not prime since it has 41 as a factor. It turns out that the first claim is true, but what we wrote cannot be a proof since the same type of reasoning when applied to the second claim seems to prove something that isn't actually true. We need a rigorous way of capturing "and so on" and a way to verify whether it really is "and so on."

Axiom 4.1 (Axiom of Induction)**.** Let $S \subseteq \mathbb{N}$ such that both

(i) $1 \in S$, and

(ii) if $k \in S$, then $k + 1 \in S$.

Then $S = \mathbb{N}$.

Recall that an axiom is a basic mathematical assumption. That is, we are assuming that the Axiom of Induction is true, which I'm hoping that you can agree is a pretty reasonable assumption. We can think of the first hypothesis as saying that we have a first rung of a ladder. The second hypothesis says that if we have any arbitrary rung of the ladder, then we can always get to the next rung. Taken together, this says that we can get from the first rung to the second, from the second to the third, and in general, from any kth rung to the $(k+1)$st rung.

Theorem 4.2 (Principle of Mathematical Induction)**.** Let $P(1), P(2), P(3), \ldots$ be a sequence of statements, one for each natural number.[1] Assume

(i) $P(1)$ is true, and

(ii) if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.[2]

The Principal of Mathematical Induction (PMI) provides us with a process for proving statements of the form: "For all $n \in \mathbb{N}$, $P(n)$," where $P(n)$ is some predicate involving n. Hypothesis (i) above is called the **base step** while (ii) is called the **inductive step**.

You should not confuse *mathematical induction* with *inductive reasoning* associated with the natural sciences. Inductive reasoning is a scientific method whereby one induces general principles from observations. On the other hand, mathematical induction is a deductive form of reasoning used to establish the validity of a proposition.

Skeleton Proof 4.3 (Proof of $(\forall n \in \mathbb{N})P(n)$ by Induction)**.** Here is the general structure for a proof by induction.

Proof. We proceed by induction.

(i) Base step: *[Verify that $P(1)$ is true. This often, but not always, amounts to plugging $n = 1$ into two sides of some claimed equation and verifying that both sides are actually equal.]*

(ii) Inductive step: *[Your goal is to prove "For all $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true."]* Let $k \in \mathbb{N}$ and assume that $P(k)$ is true. *[Do something to derive that $P(k+1)$ is true.]* Therefore, $P(k+1)$ is true.

Thus, by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$. □

Prove the next few theorems using induction.

Theorem 4.4. For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.[3]

[1] Think of $P(n)$ as a predicate, where $P(1)$ is the statement that corresponds to substituting in the value 1 for n.

[2] *Hint:* Let $S = \{k \in \mathbb{N} \mid P(k) \text{ is true}\}$ and use the Axiom of Induction. The set S is sometimes called the **truth set**. Your job is to show that the truth set is all of $\mathbb{N}$.

[3] Recall that $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$, by definition. Also, this theorem should look familiar from calculus.

Theorem 4.5. For all $n \in \mathbb{N}$, 3 divides $4^n - 1$.

Theorem 4.6. For all $n \in \mathbb{N}$, 6 divides $n^3 - n$.

Theorem 4.7. Let $p_1, p_2, \ldots, p_n$ be n distinct points arranged on a circle. Then the number of line segments joining all pairs of points is $\frac{n^2-n}{2}$.

Problem 4.8. A special chessboard is 2 squares wide and n squares long. Using n dominoes that are 1 square by 2 squares, there are many ways to perfectly cover this chessboard with no overlap. How many? Prove your answer.

Problem 4.9. Another chessboard is 2^n squares wide and 2^n squares long. However, one of the squares has been cut out, but you don't know which one! You have a bunch of L-shapes made up of 3 squares. Prove that you can perfectly cover this chessboard with the L-shapes (with no overlap) for any $n \in \mathbb{N}$. Figure 4.1 depicts one possible covering for the case involving $n = 2$.

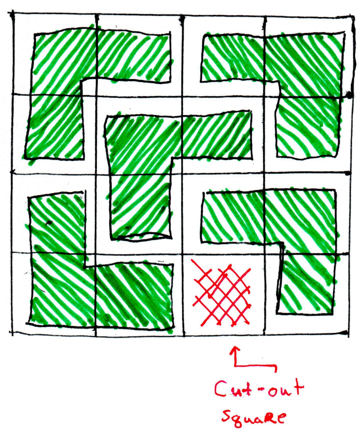

Figure 4.1: One possible covering for the case involving $n = 2$ for Problem 4.9.

4.2 More on Induction

In the previous section, we discussed proving statements of the form $(\forall n \in \mathbb{N})P(n)$. Mathematical induction can actually be used to prove a broader family of results; namely, those of the form

$$(\forall n \in \mathbb{Z})(n \geq a \implies P(n))$$

for any value $a \in \mathbb{Z}$. Theorem 4.2 handles the special case when $a = 1$. The ladder analogy from the previous section holds for this more general situation, too.

Theorem 4.10 (Principle of Mathematical Induction)**.** Let $P(a), P(a+1), P(a+2), \ldots$ be a sequence of statements, one for each integer greater than or equal to a. Assume that

(i) $P(a)$ is true, and

(ii) if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all integers $n \geq a$.[4]

Theorem 4.10 gives a process for proving statements of the form: "For all integers $n \geq a$, $P(n)$." As before, hypothesis (i) is called the **base step**, and (ii) is called the **inductive step**.

Skeleton Proof 4.11 (Proof of $(\forall n \in \mathbb{Z})(n \geq a \implies P(n))$ by Induction)**.** Here is the general structure for a proof by induction when the base case does not necessarily involve $a = 1$.

Proof. We proceed by induction.

(i) Base step: *[Verify that $P(a)$ is true. This often, but not always, amounts to plugging $n = a$ into two sides of some claimed equation and verifying that both sides are actually equal.]*

(ii) Inductive step: *[Your goal is to prove "For all $k \in \mathbb{Z}$, if $P(k)$ is true, then $P(k+1)$ is true."]* Let $k \geq a$ be an integer and assume that $P(k)$ is true. *[Do something to derive that $P(k+1)$ is true.]* Therefore, $P(k+1)$ is true.

Thus, by the PMI, $P(n)$ is true for all integers $n \geq a$. □

Theorem 4.12. Let A be a finite set with n elements. Then $\mathcal{P}(A)$ is a set with 2^n elements.[5]

Theorem 4.13. For all integers $n \geq 0$, 4 divides $9^n - 5$.

Theorem 4.14. For all integers $n \geq 0$, 4 divides $6 \cdot 7^n - 2 \cdot 3^n$.

Theorem 4.15. For all integers $n \geq 2$, $2^n > n + 1$.

Theorem 4.16. For all integers $n \geq 0$, $1 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.

Theorem 4.17. Fix a real number $r \neq 1$. For all integers $n \geq 0$,

$$1 + r^1 + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

Theorem 4.18. For all integers $n \geq 3$, $2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \dfrac{(n-2)(n^2 + 2n + 3)}{3}$.

Theorem 4.19. For all integers $n \geq 1$, $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

Theorem 4.20. For all integers $n \geq 1$, $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$.

[4] *Hint:* Mimic the proof of Theorem 4.2, but this time use the set $S = \{k \in \mathbb{N} \mid P(a - k + 1) \text{ is true}\}$.

[5] We encountered this theorem back in Section 3.2 (see Conjecture 3.24), but we didn't prove it. If you prove this theorem using induction, at some point, you will need to argue that if you add one more element to a finite set, then you end up with twice as many subsets. Also, notice that A may have 0 elements.

Theorem 4.21. For all integers $n \geq 0$, $3^{2n} - 1$ is divisible by 8.

Theorem 4.22. For all integers $n \geq 2$, $2^n < (n+1)!$.

Theorem 4.23. For all integers $n \geq 2$, $2 \cdot 9^n - 10 \cdot 3^n$ is divisible by 4.

Now consider an induction problem of a different sort, where you have to begin with some experimentation.

Problem 4.24. For any $n \in \mathbb{N}$, say that n straight lines are "safely drawn in the plane" if no two of them are parallel and no three of them meet in a single point. Let $S(n)$ be the number of regions formed when n straight lines are safely drawn in the plane.

(a) Compute $S(1)$, $S(2)$, $S(3)$, and $S(4)$.

(b) Conjecture a recursive formula for $S(n)$; that is, a formula for $S(n)$ which may involve some of the previous terms $\{S(n-1), S(n-2), \ldots\}$. (If necessary, first compute a few more values of $S(n)$.)

(c) Prove your conjecture.

4.3 Complete Induction

There is another formulation of induction, where the inductive step begins with a set of assumptions rather than one single assumption. This method is sometimes called **complete induction** or **strong induction**.

Theorem 4.25 (Principle of Complete Mathematical Induction)**.** Let $P(1), P(2), P(3), \ldots$ be a sequence of statements, one for each natural number. Assume that

(i) $P(1)$ is true, and

(ii) For all $k \in \mathbb{N}$, if $P(j)$ is true for all $j \in \mathbb{N}$ such that $j \leq k$, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Note the difference between ordinary induction (Theorems 4.2 and 4.10) and complete induction. For the induction step of complete induction, we are not only assuming that $P(k)$ is true, but rather that $P(j)$ is true for all j from 1 to k. Despite the name, complete induction is not any stronger or more powerful than ordinary induction. It is worth pointing out that anytime ordinary induction is an appropriate proof technique, so is complete induction. So, when should we use complete induction?

In the inductive step, you need to reach $P(k+1)$, and you should ask yourself which of the previous cases you need to get there. If all you need, is the statement $P(k)$, then ordinary induction is the way to go. If two preceding cases, $P(k-1)$ and $P(k)$, are necessary to reach $P(k+1)$, then complete induction is appropriate. In the extreme, if one needs the full range of preceding cases (i.e., all statements $P(1), P(2), \ldots, P(k)$), then again complete induction should be utilized.

Note that in situations where complete induction is appropriate, it might be the case that you need to verify more than one case in the base step. The number of base cases to be checked depends on how one needs to "look back" in the induction step.

Skeleton Proof 4.26 (Proof of $(\forall n \in \mathbb{N})P(n)$ by Complete Induction)**.** Here is the general structure for a proof by complete induction.

Proof. We proceed by induction.

(i) Base step: *[Verify that $P(1)$ is true. Depending on the statement, you may also need to verify that $P(k)$ is true for other specific values of k.]*

(ii) Inductive step: *[Your goal is to prove "For all $k \in \mathbb{N}$, if for each $k \in \mathbb{N}$, $P(j)$ is true for all $j \in \mathbb{N}$ such that $j \leq k$, then $P(k+1)$ is true."]* Let $k \in \mathbb{N}$. Suppose $P(j)$ is true for all $j \leq k$. *[Do something to derive that $P(k+1)$ is true.]* Therefore, $P(k+1)$ is true.

Thus, by the PCMI, $P(n)$ is true for all integers $n \geq a$. □

Recall that Theorem 4.10 generalized Theorem 4.2 and allowed us to handle situations where the base case was something other than $P(1)$. We can generalize complete induction in the same way, but we won't write this down as a formal theorem.

Theorem 4.27. Define a sequence of numbers by $a_1 = 1$, $a_2 = 3$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for all natural numbers $n \geq 3$. Then $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

Theorem 4.28. Define a sequence of numbers by $a_1 = 3, a_2 = 5, a_3 = 9$ and $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for all natural numbers $n \geq 4$. Then $a_n = 2^n + 1$ for all $n \in \mathbb{N}$.

Theorem 4.29. Define a sequence of numbers by $a_1 = 1$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2}$ for all natural numbers $n \geq 3$. Then $a_n < \left(\frac{7}{4}\right)^n$ for all $n \in \mathbb{N}$.

Theorem 4.30. Define a sequence of numbers by $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all natural numbers $n \geq 4$. Then $a_n < 2^n$ for all $n \in \mathbb{N}$.

Theorem 4.31. Define a sequence of numbers by $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for all natural numbers $n \geq 3$. Then $a_n < \left(\frac{5}{3}\right)^n$ for all $n \in \mathbb{N}$.

Problem 4.32. Prove that every amount of postage that is at least 12 cents can be made from 4-cent and 5-cent stamps.

Problem 4.33. Prove that for any $n \geq 4$, one can obtain n dollars using only \$2 bills and \$5 bills.

The final theorem of this chapter is known as the Well-Ordering Principle (WOP). As you shall see, this seemingly obvious theorem requires a bit of work to prove. It is worth noting that in some axiomatic systems, the WOP is sometimes taken as an axiom. However, in our case, the result follows from complete induction.

Theorem 4.34 (Well-Ordering Principle)**.** Every nonempty subset of the natural numbers contains a least element.[6]

[6] *Hint:* Towards a contradiction, suppose S is a nonempty subset of $\mathbb{N}$ that does not have a least element. Define the proposition $P(n)$:="n is not an element of S". Use complete induction.

It turns out that the Well-Ordering Principle (Theorem 4.34) and the Axiom of Induction (Axiom 4.1) are equivalent. In other words, one can prove the Well-Ordering Principle from the Axiom of Induction, as we have done, but one can also prove the Axiom of Induction if the Well-Ordering Principle is assumed.