EXAM 2 - REVIEW QUESTIONS

LINEAR ALGEBRA

Questions (answers are below)

Examples. For each of the following, either provide a specific example which satisfies the given description, or if no such example exists, briefly explain why not.

(1) (AP) A linearly independent set of at least 5 vectors in $\mathbb{R}^4$.
(2) (BS) A nonsingular $n \times n$ matrix with determinant 0 that also has a trace of 0.
(3) (MH) A matrix A that has two equal rows or columns where the $\det(A) \neq 0$.
(4) (OS) A subspace of $\mathbb{R}_3$ with dimension 2.
(5) (MJ) A basis for $\mathbb{R}^3$ with 4 vectors in it.
(6) (RS) The coordinate vector of v with respect to the subset of vectors S that spans $\mathbb{R}^3$ where

$$S = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}.$$

(7) (RJT) A transition matrix from the T basis for P_2 to the S basis where $T = \{t^2+4, t^2+2t+1, t+4\}$ and $S = \{t+2, 2t^2+2t, t^2+1\}$.
(8) (JG) Give an example of a vector space with two bases of different sizes.
(9) (JRS) A vector space with only one subspace.
(10) (KG) A set of vectors $\{v_1, v_2, \ldots, v_n\} \in \mathbb{R}^n$ such that a basis for $\mathbb{R}^n$ is obtained by removing one of the vectors from the set.
(11) (TD) A set of vectors that spans R^3.
(12) (JAW) A subspace of $\mathbb{R}$ that consists entirely of negative whole numbers.
(13) (QH) Give an example of a vector in the vector space V that is spanned by the set $S = \{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix} \}$.
(14) (JS) Two vector spaces that have different dimensions and are isomorphic.
(15) (RYT) A basis for $\mathbb{R}^n$ such that $\mathbf{v}_k = \begin{bmatrix} a^k \\ a^{k+1} \\ \vdots \\ a^{k+n-1} \end{bmatrix}$ for $1 \leq k \leq n$ and some nonzero $a \in \mathbb{R}$.
(16) (AV) A permutation that has a number repeated within it.
(17) (GD) An nonsingular matrix with a determinate of 0
(18) (MH) A linearly dependent set in which only one co-efficient is not zero.
(19) (AR) A 3×3 symmetric matrix with determinant 0.
(20) (AM) A linearly dependent set of four vectors in $\mathbb{R}^4$.
(21) (KH) A matrix transformation that is not an isomorphism.
(22) (MRR) A basis for S where S is a subspace of $\mathbb{R}^3$ and $S = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \right)$.
(23) (DW) A linearly dependent set of three vectors that spans $\mathbb{R}^3$

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True or False.

(a) (JR) It is possible to find a finite set of vectors which spans P, the set of all polynomials.
(b) (BS) The permutation 1234 is considered odd because there are no inversions.
(c) (MH) Let V be an n-dimensional vector space. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans V, then S is a basis for V.
(d) (OS) Let S_1 and S_2 be finite subsets of V where S_1 is a subset of S_2. If S_1 spans V, then S_2 also spans V.
(e) (RS) Let A be a $n \times n$ matrix. If another $n \times n$ matrix B is obtained by multiplying all n rows of A by a real number k, then the determinant of B is equal to $k\det(A)$.
(f) (MJ) $B \subseteq V$ is a basis if the span of B is V and if B is linearly dependent.
(g) (RJT) Let S be a linearly independent set of vectors in the vector space $\mathbb{R}^n$. There exists a basis for $\mathbb{R}^n$ that includes S.
(h) (JG) If V is the set of all polynomials of degree exactly 2 (so V is a subset of P, the vector space of all polynomials), then V is also a subspace of P.
(i) (JAW) Every row or column operation on a matrix affects the determinant.
(j) (JRS) Every basis for $\mathbb{R}^n$ has size n, and every basis for P_n has size n.
(k) (KG) Let $P_{S \leftarrow T}$ be the transition matrix from the T to the S basis for a vector space V, and let $[\mathbf{v}]_S$ be the coordinate vector $\mathbf{v}$ with respect to S. Then $P_{S \leftarrow T}[\mathbf{v}]_S = [\mathbf{v}]_T$
(l) (TD) A set of vectors is a basis for a vector space if the set of vectors spans said vector space.
(m) (JAW) Does not exist, for it to be a real vector space there has to be a -u for each u in V, such that -u + u = 0.
(n) (QH) For all matrices that have a row or a column of zeros, their determinants are 0.
(o) (JS) Let S_1 and S_2 be finite subsets of a vector space, and let S_1 be a subset of S_2. Then if S_2 is linearly dependent, so is S_1.
(p) (RYT) Let W be a subspace of a vector space V. If W is a finite-dimensional vector space, then so is V.
(q) (BYS) For an n-dimensional vector space V, if a set of $m < n$ vectors is a basis for V, then it is linearly dependent.
(r) (AV) The permutation 4312 is even.
(s) (GD) A basis for a vector space can contain a zero vector
(t) (AR) The determinant of a skew-symmetric matrix is always zero.
(u) (AM) A subset of $\mathbb{R}^4$ that contains a single vector can be expanded to a basis for $\mathbb{R}^4$.
(v) (MRR) Let A be an $n \times n$ matrix and let A^T be its transpose. Then, $\det(A^T A) = \det(A^2)$.
(w) (KH) If U is isomorphic to W and W is isomorphic to V, then it cannot be determined if U is isomorphic to V.
(x) (DW) If A is an upper triangular 3×3 matrix, and has only 5 non-zero entries, then it's determinant must be zero.

Answers

Examples - Answers.

(1) (AP) Impossible. By Theorem 4.10, a linearly independent set of vectors in a vector space V cannot have more elements than a basis for V.
(2) (BS) Impossible. By theorem 3.8, an $n \times n$ matrix is nonsingular if and only if its determinant does not equal zero. Because of this, you cannot have a nonsingular matrix with a determinant of 0.
(3) (MH) Impossible. By Theorem 3.3, if two rows (columns) of A are equal, then $\det(A) = 0$.
(4) (OS) Possible. Subspaces need not have to have the same dimensions as the vector space. An example is anything of the form $\begin{bmatrix} a & b & 0 \end{bmatrix}$.
(5) (MJ) Impossible. A basis for $\mathbb{R}^n$ always has exactly n vectors in it.
(6) (RS) Impossible. Since S is linearly dependent, S is not a basis for $\mathbb{R}^3$. Hence, since a coordinate vector of v with respect to S requires S to be an ordered basis, such a coordinate vector does not exist.

(7) (RJT) Possible. First we form the augmented matrix

$$\left[\begin{array}{ccc|c|c|c} 0 & 2 & 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 2 & 0 & 1 & 4 & 1 & 4 \end{array}\right],$$

then we find the RREF version which is

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 2/3 & 5/3 \\ 0 & 1 & 0 & -1/2 & 2/3 & -1/3 \\ 0 & 0 & 1 & 2 & -1/3 & 2/3 \end{array}\right].$$

From this we know that

$$P_{S\leftarrow T} = \begin{bmatrix} 1 & 2/3 & 5/3 \\ -1/2 & 2/3 & -1/3 \\ 2 & -1/3 & 2/3 \end{bmatrix}.$$

(8) (JG) Impossible. All bases for a vector space have the same size (which is the dimension of the vector space), so a vector space cannot have two bases of different sizes.

(9) (JRS) Impossible. All vector spaces have at least two subspaces: itself, and the zero vector space.

(10) (KG) Impossible. We have previously proven that the dimension of $\mathbb{R}^n$ is n; therefore there cannot be $n-1$ vectors in a basis for $\mathbb{R}^n$.

(11) (TD) Possible. Consider $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$.

(12) (JAW) Impossible.

(13) (QH) One example is $\{ \begin{bmatrix} 5 \\ 11 \end{bmatrix} \}$. Remember $a_1\mathbf{s_1} + a_2\mathbf{s_2} = v$ (a_1, a_2 are real numbers and $\mathbf{s_1}, \mathbf{s_2}$ are vectors of set S).

(14) (JS) Impossible. By Theorem 4.16, two finite-dimensional vector spaces are isomorphic if and only if their dimensions are equal.

(15) (RYT) Impossible. Consider the fact that $a^{n-1}\mathbf{v}_1 = \mathbf{v}_n$. Since a is nonzero, the set of vectors $\mathbf{v}_i$ is not linearly independent.

(16) (AV) Impossible. A permutation of S is considered to be a one-to-one mapping of S onto itself so no number within S can be repeated in a permutation.

(17) (GD) Impossible. By the invertibility theorem, a square matrix cannot be invertible if its determinate is 0.

(18) (MH) One example would be $\{1, 2, 0\}$. 0(1)+0(2)+5(0)=0.

(19) (AR) !!This is possible. For example, the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. is a 3×3 symmetric matrix with determinant 0.

(20) (AM) It is possible if, for example, two vectors in the set are identical.

(21) (KH) Consider the discussion question that showed a noninvertible matrix A in $f(\vec{v}) = A\vec{v}$ is not an isomorphism.

(22) (MRR) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$. After placing the vectors as columns of a matrix and row reducing, the resulting matrix is $\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The first and third columns in the row reduced matrix are linearly independent and columns 2 and 4 are linearly dependent with the first and third columns. Thus, only the first and third columns are necessary to form a basis for the subspace.

(23) (DW) This is not possible because a set of three vectors spanning $\mathbb{R}^3$ would have to be a basis, and a basis must be linearly independent.

True or False - Answers.

(a) (JR) False. P is infinite dimensional, so its basis must contain infinitely many vectors.

(b) (BS) False. A permutation is odd when there are an odd number of inversions, and is even when there are an even number of inversions. When a set has no inversions, it is said to be even.

(c) (MH) True. Originally, you may expect the answer to be false because it doesn't say S is also linearly independent, but by Theorem 4.12, the statement is actually true.

(d) (OS) True. If S_1 is smaller than, and contained in, S_2, then S_2 contains those vectors that span V as well.

(e) (RS) False. If all n rows of A are multiplied by k to get B, then the determinant of B will equal $k^n \det(A)$.

(f) (MJ) False. $B \subseteq V$ is a basis if the span of B is V and if B is linearly INDEPENDENT.

(g) (RJT) True.

(h) (JG) False. Consider the sum of the two polynomials $2t^2 + 3t + 4$ and $-2t^2 + t + 8$, which is not in V because it is a polynomial of degree 1.

(i) (JAW) False. The third type of row operation leaves the determinant unchanged.

(j) (JRS) False. The first part is true, but every basis for P_n has size $n + 1$.

(k) (KG) True see definition of transition matrix

(l) (TD) False. The set of vectors would also have to be linearly independent.

(m) (QH) True. By theorem 3.4, if a matrix has a row or a column of zeros, its determinant equals to 0.

(n) (JS) False. However, if S_1 is linearly dependent, so is S_2. Also, if S_2 is linearly independent, so is S_1.

(o) (RYT) False. Consider $W = P_2$ and $V = P$, the set of all polynomials.

(p) (BYS) True. Logic rules! The first part is never true, so the statement is true. Isn't math the greatest?

(q) (AV) False. The permutation 4312 has 5 inversions so is therefore odd.

(r) (GD) False. If a basis has a zero vector, the basis it is not linearly independent. By definition, a set of vectors is linearly independent if, whenever $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots\cdots + a_n\mathbf{v}_n = 0, a_1 = a_2 = \cdots\cdots = a_n = 0$. Because you can multiply any constant to the zero vector and still yield the zero vector as a result, the condition $a_1 = a_2 = \cdots\cdots = a_n = 0$ will not be fulfilled. Since the set will be linearly dependent, a set with a zero vector cannot be a basis.

(s) (AR) False. Consider the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(t) (AM) True as long as the element contained in the set is not the zero vector. Then you have a linearly independent set that exists within $\mathbb{R}^4$.

(u) (MRR) True. Applying theorems 3.1 and 3.9, $\det(A^T A) = \det(A^T)\det(A) = \det(A)\det(A) = \det(A^2)$.

(v) (KH) False. By Theorem 4.15, U will be isomorphic to V.

(w) (DW) False. Consider the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.