BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH QUADRATIC GROWTH AND THEIR APPLICATIONS

by

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Chapter 1

Introduction

Backward stochastic differential equations (BSDE's in short) appear in many areas of research including mathematical finance, nonlinear partial differential equations (PDE's), financial economics, stochastic control and game problems of functional diffusions. For the general theory of BSDE's and their applications, we refer the reader to the books by Yong and Zhou (1999), Ma and Yong (1999) and the survey paper by El Karoui et. al. (1997).

The first existence and uniqueness result for nonlinear BSDE's was given by Pardoux and Peng (1990). They studied a BSDE of the type

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t;\ Y_T = \xi \tag{1}$$

where the coefficient $f : [0,T] \times \Re^d \times \Re^{d\times m} \to \Re^d$ (sometimes called the *generator* or the *driver*) is uniformly Lipshitz in state variables Y, Z and the terminal condition ξ is square integrable. The unique solution is a pair $(Y,Z) = (Y_t, Z_t)_{0\le t\le T}$ of square integrable $\mathcal{F}-$adapted processes where $\mathcal{F}$ is the complete $\sigma-$algebra generated by m-dimensional Brownian motion W on a probability space $(\Omega, \mathcal{F}, P)$. Since the uniform Lipshitz condition is too restrictive for many interesting applications, a number of attempts have been made to relax these assumptions on f.

Pardoux and Peng (1994) weakened the assumptions in mainly two ways:

1. f is locally Lipshitz in y but uniformly Lipshitz in z; $f(t,y,z)$ satisfies the linear growth conditions on state variables (y,z) and the terminal condition ξ is bounded.

2. f is continuously differentiable in (y,z) with locally bounded first order derivatives, ξ is a bounded random variable belonging to the Wiener space and its derivatives on this space are bounded.

By studying a more general BSDE, Mao (1995) showed that if

$$f : \Omega \times [0,T] \times \Re^d \times \Re^{d\times m} \to \Re^d$$

satisfies

$$|f(t,y_1,z_1) - f(t,y_2,z_2)|^2 \le \kappa(|y_1-y_2|^2) + c\,|z_1-z_2|^2\,,\ \text{a.s.} \tag{2}$$

where $c > 0$ and $\kappa : [0,\infty) \to [0,\infty)$ is a concave nondecreasing function such that $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$ and $\int\limits_{0^+} \frac{du}{\kappa(u)} = \infty$, then the equation (1) has a unique solution.

Lepeltier and San Martin (1997) obtained the existence of a maximal bounded solution for one-dimensional BSDE's when f is continuous with linear growth in state variables and ξ is square integrable.

Kobylanski (2000) provided existence, comparison and stability results for one dimensional BSDE's when the generator f is continuous and has a quadratic growth in Z and if ξ is bounded. She also discussed the uniqueness of solutions under slightly stronger conditions. The main technique is a transformation of the original equation by means of an exponential change of variable.

Lepeltier and San Martin (1998) generalized Kobylanski's results to the following case

$$|f(t,\omega,y,z)| \leq k(y) + C\left|z^2\right|, \tag{3}$$

where k is a strictly positive function satisfying

$$\int_0^\infty \frac{dx}{k(x)} = \infty = \int_{-\infty}^0 \frac{dx}{k(x)}.$$

However, their emphasis was on the existence of the solutions rather than the uniqueness.

El-Karoui and Hamadene (2003) obtained similar existence and comparison results to solve a risk-sensitive control problem and a zero-sum game problem. They also showed the connections between non-zero sum risk-sensitive games and their associated multidimensional BSDE's.

In many situations, the terminal condition ξ depends explicitly on another state variable X, $\xi = g(X_T)$ where

$$\begin{aligned} dX_t &= b(t,X_t)dt + \sigma(t,X_t)dW_t, \;\; s \leq t \leq T \\ X_s &= x \end{aligned} \tag{4}$$

The system of equations (1) and (4) is called a decoupled FBSDE. The connections between this type of BSDE's and quasilinear PDE's were stated by Pardoux and Peng (1992), Peng (1992) by generalising Feynman-Kac representation of PDE's as follows:

Theorem 1 (Pardoux and Peng, 1992) *Consider the following parabolic PDE:*

$$\begin{aligned} v_t + bv_x + f(t,x,v,\sigma' v_x) + \frac{1}{2}tr(\sigma\sigma' v_{xx}) &= 0 \\ v(T,x) &= g(x) \end{aligned} \tag{5}$$

together with the decoupled system of FBSDE:

$$\begin{aligned} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t && (6) \\ dY_t &= -f(t, Y_t, Z_t)dt + Z_t dW_t && (7) \\ X_s &= x,\ Y_T = g(X_T). \end{aligned}$$

If the PDE (5) has a (classical) solution v, then (Y, Z) with

$$Y_t^{s,x} = v(t, X_t^{s,x}), Z_t^{s,x} = \sigma'(t, X_t^{s,x}) v_x(t, X_t^{s,x})$$

solve the BSDE (7). Conversely, if the system (6)-(7) has a unique (adapted) solution, then $v(t, x) \triangleq Y_t^{t,x}$ is a viscosity solution to the PDE (5). Moreover, this solution is unique if the coefficients involved are uniformly Lipshitz.

El Karoui et. al. (1997) simplified the proof of Theorem 1, and extended the results of Pardoux and Peng (1992) by presenting flow, comparison and regularity properties of BSDE's. They provided some examples of BSDE's in pricing and hedging contingent claims as well as those in stochastic differential utility framework. They also discussed the connections with Malliavin calculus. Zhang (2001), Ma and Zhang (2002) can be referred for new results on the regularity properties and representation theorems for BSDE's using Malliavin calculus.

When the coefficients b, σ, f depend on (x, y, z), the system

$$\begin{aligned} dX_t &= b(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t, Z_t)dW_t,\ s \le t \le T \\ dY_t &= -f(t, X_t, Y_t, Z_t)dt + Z_t dW_t,\ s \le t \le T \\ X_s &= x; Y_T = g(X_T) \end{aligned} \tag{8}$$

is called a coupled Forward-Backward SDE (FBSDE). A solution to this sytem is a set of processes (X, Y, Z) which are $\mathcal{F}_t$-adapted and satisfy some integrability conditions. The first existence and uniqueness result for coupled FBSDE's (when b doesn't depend on Z) was provided by Antonelli (1993). By giving a counterexample, he proved that the Lipshitz condition for the parameters is not enough to guarantee the existence for a large time interval through Ito's rule and related quasilinear PDE's. Later, Peng and Wu (1999) generalized the existence and uniqueness results to an arbitrarily large time duration. Although a considerable number of works are under progress, the general theory of coupled FBSDE's with "reasonable conditions" on coefficients is still lacking.

The "Four Step Scheme" introduced by Ma, Protter and Yong (1994) is one of the first methods to solve FBSDE's in a Markovian setting. Their method uses Ito's rule to transfer the FBSDE system to the corresponding nonlinear PDE and solve this PDE anaytically or numerically under some strong regularity and growth conditions on the coefficients. Although the numerical scheme based on the "Four Step Scheme" is efficient in one dimensional problems, it is quite costly in higher dimensional cases. For some applications of this method, one can refer to the papers by Cvitanic and Ma (1996); Duffie, Ma, Yong (1995); Douglas, Ma, Protter (1996).

Recently, the linear-quadratic regulator (LQR) problems with random coefficients have been the main application of a type of nonlinear BSDE's with quadratic growth, called Ricccati BSDE's. Chen et. al (1998) discussed the solvability of these equations in some special cases. Later, their results were generalized and applied to the problems of mathematical finance, especially to mean-variance hedging and portfolio selection problems. See, for example, Chen and Zhou (2000), Kohlmann and Tang (2000), Lim and Zhou (2002), Lim (2003), Hu and Zhou (2003) and also section 4.2 in this dissertation.

For other approaches and applications of nonlinear BSDE's, the works of Yong and Zhou (1999), Skiadas and Schroder (1999, 2003), Cvitanic et. al. (2001, 2004), Hugonnier and Kaniel (2004), Cetin (2005) can be referred.

1.1 The Description of The Problem

In this dissertation, our emphasis is on the decoupled system of FBSDE's of the form

$$\begin{aligned} dX_t &= b(t,X_t)dt + \sigma(t,X_t)dW_t, \ 0 \le t \le T \\ dY_t &= -f(t,X_t,Y_t,Z_t)dt + Z_t dW_t, \ 0 \le t \le T \\ X_0 &= x; \ \ Y_T = g(X_T) \end{aligned} \tag{9}$$

where b and σ satisfies usual Lipshitz and growth conditions, σ is *non-degenerate*, f is continuous with quadratic growth in Z, and Y_T is square integrable. The next section describes how such a system of FBSDE can be used to solve certain stochastic optimal control problems by focusing on a well known application in engineering, stochastic linear-quadratic regulator (LQR) problem. In low dimensions, the PDE techniques work quite well to get at least a numerical solution to a control problem. When the dimension of a problem increases, the numerical schemes for PDE's becomes very inefficient (from the computational point of view). Therefore, the probabilistic approach using simulation would suggest a more convenient framework for numerical applications. In order to get advantage of this approach, we want to know the 'fine' properties of the solutions as well. We notice that even in the standard case of the deterministic coefficients, the quadratic FBSDE system corresponding to the LQR problem is complicated enough to show the uniqueness of the solutions.

The rest of the dissertation is organized as follows: The next section includes some definitions and notations as well as basic facts from stochastic control theory and Malliavin calculus.

The second Chapter starts with the formulation of stochastic LQR problem for which an explicit solution can be obtained. First, the case of control independent diffusion terms is introduced together with both PDE and FBSDE characterizations. Then the uniqueness to the solutions is discussed for both one and multi-dimensional state processes. After providing some comparison theorems to the solutions of LQR problem and to the associated FBSDE's and PDE's, the Chapter ends with a discussion of the problems with control dependent diffusion terms and a review of the known results concerning Riccati BSDE's.

The third Chapter discusses the numerical solution of the FBSDE's by a discretization algorithm of Bouchard and Touzi (2004) which is based on the Markovian structure of the system. By considering the FBSDE systems arising from LQR problems, some convergence and regularity results for the discretized system of FBSDE's are given. Then a pure simulation scheme is implemented to estimate the conditional expectations by a regression approximation which is based on Malliavin calculus. A three dimensional example is provided and the computational aspects are discussed.

Finally, the fourth Chapter includes some applications of the FBSDE's that are considered in Chapter 2 to economics (impulse control problem of a central bank) and to mathematical finance (mean-variance portfolio selection). The dissertation ends with some concluding remarks on other major applications.

1.2 Preliminaries

The most of the terms and the notations that are used in this thesis are much or less standard in the literature (especially in the areas of stochastic calculus, stochastic optimal control and mathematical finance). However, we fix the notations for the function spaces, derivatives etc. in the next subsection. Throughout the dissertation, we assume a prior knowledge in measure theory, stochastic analysis (including the theory of SDE's, convergence theorems, conditional expectations) and some familiarity with mathematical finance (continuous time complete financial market models and utility maximization). Although we summarize the basics of stochastic optimal control theory and the Malliavin calculus in this section, some other technical results and computations are included in the Appendices.

1.2.1 Definitions and Notations

For a given $T > 0$, a Euclidean space $\Re^k$ with $k \geq 1$, and a probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ is the complete $\sigma-$algebra generated by m-dimensional Brownian motion $W = (W_1, ..., W_m)'$, we define the following spaces which will be used frequently throughout the thesis:

- $C^{p,q}([0,T]; \Re^k)$: The space of all $\Re^k$ valued measurable functions $f : [0,T] \times \Re^k$ such that $f(t,x)$ is p (respectively, q) times continuosly differentiable with respect to t (respectively, x) where p, q are non-negative integers.

- $C_b^\infty(D)$: The set of all infinitely differentiable functions on D with bounded derivatives.

- $L^p_{\mathcal{F}_T}(\Omega; \Re^k)$: The space of all $\Re^k$ valued, $\mathcal{F}_T$-measurable random variables H such that $E[|H|^p] < \infty$ where $|.|$ denotes the Euclidean norm in $\Re^k$.

- $L^\infty_{\mathcal{F}_T}(\Omega; \Re^k)$: The space of all $\Re^k$ valued, $\mathcal{F}_T$-measurable essentially bounded random variables.

- $L^p_{\mathcal{F}}([0,T]; \Re^k)$: The space of all $\Re^k$ valued, $\mathcal{F}$-adapted processes f such that

$$E[\int_0^T |f(t)|^p \, dt] < \infty.$$

  where $|.|$ denote the usual Euclidean norm on $\Re^k$.

- $L^\infty_{\mathcal{F}}([0,T]; \Re^k)$: The space of all $\Re^k$ valued, $\mathcal{F}$-adapted essentially bounded processes.

- $L^p_{\mathcal{F}}(C[0,T]; \Re^k)$: The space of all $\Re^k$ valued, $\mathcal{F}$-adapted continuous processes such that

$$E[\sup_{0 \leq t \leq T} |f(t)|^p \, dt] < \infty.$$

Sometimes, we will write $L^p_{\mathcal{F}_T}(\Omega)$, $L^\infty_{\mathcal{F}_T}(\Omega)$, $L^p_{\mathcal{F}}(\Re^k)$, $L^\infty_{\mathcal{F}}(\Re^k)$, $L^p_{\mathcal{F}}(C; \Re^k)$ by omitting the time (and the space occasionally) components when the context is clear.

Unless otherwise specified, all the vectors are column vectors and the prime $(.')$ denotes the transpose of a matrix so that x' will be a $k \times 1$ row vector for $x \in \Re^k$. The trace of a matrix M will be denoted by $tr(M)$. For a matrix $M \in \Re^{k \times m}$, the matrix norm $\|M\|$ will denote the Euclidean norm on $\Re^{k \times m}$, unless otherwise stated. The identity matrix of dimensions $k \times k$ will be denoted by $I_{k \times k}$ or just by I when the context is clear. A matrix inequality $M \geq N$ ($M > N$, respectively) is a convention for the staement that the matrix difference $M-N$ is nonnegative definite (positive definite, respectively). A matrix valued function $\sigma : [0,T] \to \Re^{k\times m}$ is called *uniformly non-degenerate* if $\sigma\sigma'(t) > \delta I$, some positive constant δ, $\forall\ t \in [0,T]$.

For a deterministic function $h(t,x) : [0,T] \times \Re^k \to \Re$, the subscript notation will usually denote partial derivatives: $h_t(t,x) = \frac{\partial h}{\partial t}(t,x)$, $h_x(t,x) = D_x h(t,x) = \nabla_x h(t,x)$ and $h_{xx}(t,x) = D_x^2 h(t,x)$. In particular, for time dependent ODE's, dot $(\dot{})$ designates the derivative with respect to time parameter t.

The notation $E_t[.]$ will denote the conditional expectation $E[.|\mathcal{F}_t]$. When the initial value of a process X is given at time t, then $E^{t,x}[.]$ refers to $E[.]$ with $X_t = x$.

1.2.2 Basics of Stochastic Control Theory

The most of the following facts are standard results from stochastic optimal control theory and will be summarized in this part without proofs. For the details and the proofs of these arguments, one can refer to the books by Fleming and Rishel (1975), Fleming and Soner, (1993), Yong and Zhou (1999) among other valuable sources.

Let $T > 0$ be given, W be an m dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $x_0 \in \Re^d$. Consider the following control dependent SDE:

$$\begin{aligned} dX(t) &= \alpha(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dW_t \\ X(0) &= x_0 \end{aligned} \tag{10}$$

where the coefficients α and σ are $\Re^d$ and $\Re^{d\times m}$ valued, respectively and satisfy the usual Lipshitz and linear growth conditions with respect to state and control variables. Let $U \subset \Re^m$ and for U-valued, $\mathcal{F}$-adapted control processes $u(.)$, define the *cost functional*

$$J^u(s,x) = E[\int_s^T f(t, X(t), u(t))dt + g(X(T))] \tag{11}$$

where the running cost f, the terminal cost g and the control u satisy

$$\begin{aligned} f(., X(.), u(.)) &\in L^1_{\mathcal{F}}(\Re), \\ g(X(T)) &\in L^1_{\mathcal{F}_T}(\Re) \end{aligned}$$

and $u \in \mathcal{U}^{s,x} = \mathcal{U}^{s,x}([s,T], \Re^m)$ where $\mathcal{U}^{s,x}$ denotes the set of all *admissible control processes*: $u \in \mathcal{U}^{s,x}$ if and only if $u \in L^2_{\mathcal{F}}([s,T]; U)$ such that the SDE (10) with $X(0) = x_0$ replaced by $X(s) = x$ has a unique solution. With the assumptions above, if the cost functional (11) is well defined over $\mathcal{U}$, then we define the *value* *function* as

$$V(s,x) \triangleq \inf_{u \in \mathcal{U}^{s,x}} J^u(s,x) \tag{12}$$

for $(s,x) \in [0,T] \times \Re^d$. Because of the dynamic structure of the optimal control problems, we consider all the pairs $(s,x) \in [0,T] \times \Re^d$ although the original problem concerns only the initial values $(0,x_0)$. Then the *dynamic programming principle (DPP)* for (12) can be stated as

$$V(s,x) = \inf_{u \in \mathcal{U}^{s,x}} E^{s,x}[\int_s^t f(\tau, X(\tau), u(\tau))d\tau + V(t, X^{s,x}(t;u))] \tag{13}$$

for any $0 \le s \le t \le T$.

If a pair (X^*,u^*) is optimal for the control problem (10)-(12), then the value function V satisfies the equation

$$V(t, X^*(t)) = E_t^{s,x}[\int_t^T f(\tau, X^*(\tau), u^*(\tau))d\tau + g(X^*(T))] \tag{14}$$

P-a.s., $\forall\ t \in [s,T]$.

When the value function $V \in C^{1,2}([0,T] \times \Re^d)$, then V solves the following second-order nonlinear PDE:

$$v_t - \sup_{u \in U} H(t,x,u,-v_x,-v_{xx}) = 0,\ \ (t,x) \in [0,T] \times \Re^d \tag{15}$$

with terminal condition $v(T,x) = g(x)$, $x \in \Re^d$ where

$$H(t,x,u,p,z) \triangleq \frac{1}{2} tr(\kappa z) + \alpha'(t,x,u)p - f(t,x,u) \tag{16}$$

with

$$\kappa(t,x,u) = \sigma\sigma'(t,x,u)$$

is called the *(generalized) Hamiltonian* and the equation (15) is called the *Hamilton-Jacobi-Bellman (HJB) equation* (or *HJB PDE*). The HJB PDE is called *uniformly parabolic* if $\exists\ c > 0$ such that

$$\sum_{i,j=1}^{d} \kappa_{ij}(t,x,u) y_i y_j \ge c\,|y|^2 \tag{17}$$

for all $(t,x,u) \in [0,T] \times \Re^d \times U$, and $y \in \Re^d$. If (17) doesn't hold, the HJB equation

(15) is called a *degenerate parabolic PDE*. The uniformly parabolic PDE's of the form (15) are known to have unique classical solutions under some regularity and growth conditions (for a summary of these results, see Fleming and Soner, 1993, IV.4). For minimization problems, the HJB equation can be rephrased as

$$0 = \inf_u \{f(t,x,u) + (\mathsf{L}^u V)(t,x)\} \tag{18}$$

where L is the backward evolution operator

$$\mathsf{L}^u V(s,x) = V_s + \alpha' V_x(s,x) + \frac{1}{2} tr(a V_{xx}(s,x)). \tag{19}$$

Given $(s,x) \in [0,T] \times \Re^d$, a system $\Sigma^{s,x} = (\Omega, \{\mathcal{F}_t : s \le t \le T\}, P, X(.), u(.))$ is called an *admissible control system* if $(\Omega, \mathcal{F}, P)$ is a (complete) probability space, $\{\mathcal{F}_t : s \le t \le T\}$ is a filtration, $X(.)$ and $u(.)$ are $\mathcal{F}$-adapted processes such that $X(s) = x; X(t) \in \Re^d$, $\forall\ t \in [s,T]$, $u \in \mathcal{U}^{s,x}$, $f(.,X(.),u(.)) \in L^1_{\mathcal{F}}([s,T],\Re)$, $g(X(T)) \in L^1_{\mathcal{F}_T}(\Omega,\Re)$.

Notation 2 *We will usually write $u \in \mathcal{U}^{s,x}$ or $u \in \mathcal{U}$ to refer to an admissible control system when the context is clear.*

Remark 3 *The description of the admissible system above is in general a weak formulation which might also involve a reference Brownian motion. However, we omit such extension of the notation to keep the presentation simple.*

Theorem 4 (Verification Theorem) *Let $v \in C^{1,2}([0,T] \times \Re^d)$ be a classical solution to the equation (15). Then for all $(s,x) \in [0,T] \times \Re^d$,*
(a) $v(s,x) \le J^u(s,x)$, $\forall\ u \in \mathcal{U}^{s,x}$
(b) An admissible pair (X^*,u^*) is optimal for (10)-(12) if and only if

$$v_t(t,X^*(t)) = \max_{u \in U}\ H(t,X^*(t),u(t),-v_x(t,X^*(t)),-v_{xx}(t,X^*(t))) \tag{20}$$

$$= H(t,X^*(t),u^*(t),-v_x(t,X^*(t)),-v_{xx}(t,X^*(t))), \forall t \in [s,T],$$

holds P-a.s.

Definition 5 *A pair (τ,ξ) is called an **impulse control** if (τ_n) is a non-negative sequence of increasing stopping times and (ξ_n) is a sequence of random variables such that $\xi_n \in L^1_{\mathcal{F}_{\tau_n}}$, $\forall\ n \ge 1$.*

1.2.3 Some Results From Malliavin Calculus

In this subsection, we review some fundamental facts from Malliavin calculus (sometimes called stochastic calculus of variations or anticipating calculus) including integration by parts formula, Clark-Ocone formula and results for the Malliavin derivatives and Skorohod integrals of smooth functionals. The details and the proofs of these results could be found in Nualart (1995) and Oksendal (1997).

Let Θ be the set of the random variables of the form

$$F = f(\int_0^T \theta_1'(t)dW(t), ..., \int_0^T \theta_k'(t)dW(t))$$

where $f \in C_b^\infty(\Re^k)$ and $\theta_j \in L^2([0,T],\Re^k)$, for some $k \geq 1$. A stochastic process $D : \Theta \to L^2([0,T] \times \Omega)$ is called the *Malliavin derivative operator* if for any $F \in \Theta$, and $t \in [0,T]$, we have

$$\begin{aligned} D_t F &= \theta'(t)\nabla f(\int_0^T \theta_1'(t)dW_t, ..., \int_0^T \theta_k'(t)dW_t) \\ &= \sum_{j=1}^k \frac{\partial f}{\partial x_j}(\int_0^T \theta_1'(t)dW_t, ..., \int_0^T \theta_k'(t)dW_t)\theta_j(t). \end{aligned}$$

Let $D^{1,2}$ denote the (Banach space which is the) completion of Θ in $L^2(\Omega)$ with respect to norm $\|.\|_{1,2}$:

$$\|F\|_{1,2}^2 \triangleq E\,|F|^2 + E\int_0^T |D_t F|^2\, dt.$$

The operator D is known to be a closed linear operator densely defined from $D^{1,2}$ into $L^2([0,T] \times \Omega)$. We now define the *Skorohod integral* δ which is known to be the *adjoint operator* of D.

Let $Dom(\delta)$ be the set of F-adapted stochastic processes in $L^2([0,T] \times \Omega; \Re^m)$ such that

$$\left| E[\int_0^T (D_t F)v_t dt] \right| \leq c\,\|F\|_{1,2}\,,$$

for all $F \in D^{1,2}$. Then the Skorohod integral of the process v is defined by the following duality expression

$$E[F\delta(v)] \triangleq E[\int_0^T (D_t F)v_t dt]. \tag{21}$$

As in Oksendal (1997), the Skorohod integral may also be defined in the framework of the Wiener-Ito chaos expansion. Then the identity (21) is obtained as an integration by parts formula relating the Skorohod integral to Malliavin derivative. For $v \in Dom(\delta)$, we write

$$\delta(v) = \int_0^T v_t' dW_t$$

although this is not an Ito stochastic integral in general (since the integrand v is not necessarily $\mathcal{F}$-adapted). When $v \in L^2_{\mathcal{F}}([0,T];\Re^m)$, $\delta(v)$ represents the usual Ito integral. The following results are well known and summarize the basic properties of the operators D_t and δ.

Lemma 1 (Chain Rule) *Let $F = (F_1, ..., F_n)'$ be a random vector with $F_i \in D^{1,2}$ for $i = 1, ...n$, and $\varphi : \Re^k \to \Re$ be in $C_b^\infty(\Re^k)$. Then $\varphi(F) \in D^{1,2}$ and*

$$D_t\varphi(F) = \sum_{i=1}^{n} \frac{\partial \varphi(F)}{\partial x_i} D_t F_i$$

Lemma 2 *Let $F \in D^{1,2} \cap L^2_{\mathcal{F}_T}$. Then*

(a) (Integration by parts formula) If $v = (v_1, ..., v_n)$ is an $\Re^{n\times m}$ valued random matrix such that $v_i \in Dom(\delta)$ for $i = 1, ...n$ and $Fv \in L^2_{\mathcal{F}}([0,T] \times \Omega; \Re^{n\times m})$, then $Fv_i \in Dom(\delta)$, $\forall\ i = 1, ...n$ and the following identity holds:

$$\delta(Fv) = F\int_0^T v_t dW_t - \int_0^T D_t F v_t dt$$

(b) (Clark-Ocone Formula) F has the following decomposition:

$$F = E[F] + \int_0^T E_t[D_t F] dW_t$$

Lemma 3 *Let $\{X_t : 0 \le t \le T\}$ be the solution to the following $\Re^d$ valued vector SDE:*

$$\begin{aligned} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 &= x \in \Re^d. \end{aligned}$$

where both coefficients b and σ are globally Lipshitz with linear growth and continu-

ously differentiable. Then $X_t \in \mathbb{D}^{1,2}$, $0 \le t \le T$, *and*

$$D_s X_t = \sigma(s, X_s) + \int_s^t \nabla_x b(u, X_u) D_s X_u du + \int_s^t \nabla_x \sigma(u, X_u) D_s X_u dW_u$$

where $\nabla_x f(t,x)$ *denotes the gradient of* f *with respect to the state variable* x.

Chapter 2
The Stochastic LQR Problem

Let $T > 0$ be fixed (deterministic), W be an m dimensional Brownian motion. For a given $\Re^m$-valued control process $u = (u_1, ..., u_m)'$ and for $0 < t \leq T$, consider the following d-dimensional state process

$$\begin{aligned} dX_t &= (A(t)X_t + B(t)u(t) + \alpha(t))dt + (\sum_{j=1}^{m} C_j(t)X_t + D_j(t)u(t) + \sigma^j(t))dW_j(t), \\ X_0 &= x_0 \end{aligned} \tag{22}$$

where the deterministic matrices A, $C_j \in \Re^{d\times d}$; B, $D_j \in \Re^{d\times m}$; α , $\sigma^j \in \Re^{d\times 1}$, $j = 1, ..., m$, are t-continuous. The first part of this work concentrates on minimizing a quadratic cost functional when the diffusion coefficient doesn't depend on the control explicitly, that is, $D_j = 0$, $j = 1, ..m$. In this case, the optimal control is known to be identical to the one in the deterministic LQR problems and is a linear feedback control. In general, the existence of the coefficients C_j introduces some other tecnical difficulties.

There are basically two approaches to solve such control problems: Dynamic Programming Principle (DPP), and Stochastic Maximum Principle. Although they are equivalent for the LQR type control problems, we will appeal to the former one which is also useful to connect certain PDE's with systems of FBSDE's. One may refer to Yong and Zhou (1998) for a discussion of Stochastic Maximum Principle and its connections with DPP approach. The square completion method is also used in the literature for LQR type problems by taking advantage of the quadratic structure of the cost functional. See for instance, Chen et. al (1998).

2.1 Control Independent Diffusion Terms

For $T > 0$ given, the set of all admissible control processes is given by $\mathcal{U}^{0,x_0}([0,T], \Re^d)$ which consists of all $\{\digamma_t, 0 \leq t \leq T\}$ adapted processes $u = \{u(t), 0 \leq t \leq T\}$ with values in $\Re^d$ such that $E^{0,x_0}[\int_0^T |u(t)|^2 dt] < \infty$ and the state process $X^{0,x_0;u}$ in (22) has a unique strong solution. When there is no ambiguity, the notations $X^{s,x;u}$, $\mathcal{U}([0,T], \Re^d)$ will be abbreviated as X and $\mathcal{U}$, respectively. We will often use subscript notation X_t and u_t for the state and control processes when the context is clear.

Throughout this subsection, we will assume that $D_j = 0$, $j = 1, ..m$. Hence, the equation (22) reduces to

$$\begin{aligned} dX_t &= (A(t)X_t + B(t)u(t) + \alpha(t))dt + (\sum_{j=1}^{m} C_j(t)X_t + \sigma^j(t))dW_j(t), \\ X_0 &= x_0. \end{aligned} \tag{23}$$

The primes $(.\prime)$ will denote the transpose of a matrix, as before.

Now, for $(s, x, u) \in [0, T) \times \Re^d \times \Re^m$ given, define the quadratic cost functional as follows:

$$J^u(s,x) = E^{s,x}[\int_s^T (X_t' M(t) X_t + 2X_t' L(t) u_t + u_t' N(t) u_t) dt + X_T' R X_T] \tag{24}$$

where M, N and L are t-continuous, deterministic matrices of sizes $d \times d$, $m \times m$ and $d \times m$, respectively; M, R are symmetric and non-negative definite $d \times d$ matrices; N is symmetric and positive definite. Then the value function V is defined by

$$V(s,x) = \inf_{u \in \Re^m} J^u(s,x)$$

with terminal condition $V(T, x) = x' R x$.

Remark 6 *We can interpret the state variable X and the cost function J as the deviation from a target trajectory and the cost of minimizing this deviation, respectively. Any nonlinear optimization problem would be reasonably approximated by the quadratic cost functional (24) provided $|X_t|$ and $|u_t|$ stays close to zero.*

In this subsection, we will first consider the time dependent diffusion coefficients by assuming $D_j = 0$, $j = 1, ..m$ and discuss the PDE interpretation of this LQR problem. Then our emphasis will be on the well-posedness and qualitative properties of the corresponding FBSDE system(s).

2.1.1 The PDE Characterization.

With $\alpha = 0$, $D_j = 0$, $j = 1, ..., m$, the state process has the following form:

$$\begin{aligned} dX_t &= (A(t)X_t + B(t)u_t)dt + \tilde{\sigma}(t, X_t)dW(t), \ 0 < t \leq T \\ X_0 &= x \end{aligned} \tag{25}$$

where $\tilde{\sigma}$ is the $d \times m$ matrix valued function whose j^{th} column $\tilde{\sigma}^j(t,x)$ is given by $C_j(t)x + \sigma^j(t)$. Set

$$F(t,x,u) = x' M(t) x + u' N(t) u + 2x' L(t) u$$

Then, by (18)-(19), the value function $V(s,x)$ satisfies

$$\begin{aligned} 0 &= \inf_u \{F(s,x,u) + (\mathsf{L}^u V)(s,x)\} \\ &= V_s + x'M(s)x + x'A(s)'V_x(s,x) + \frac{1}{2}tr(\tilde{\sigma}\tilde{\sigma}'(s)V_{xx}(s,x)) \\ &\quad + \inf_u \{u'N(s)u + (B(s)u)'V_x + 2x'L(s)u\} \end{aligned} \tag{26}$$

with the backward evolution operator L^u given by

$$\mathsf{L}^u V(s,x) = V_s + (A(s)x + B(s)u)'V_x(s,x) + \frac{1}{2}tr(\tilde{\sigma}\tilde{\sigma}'(s)V_{xx}(s,x)).$$

Remark 7 *It is well known that if the diffusion term σ doesn't depend on the control variable u, then the condition $N \geq 0$ ($N > 0$, respectively) is a necessary (sufficient, respectively) condition for the LQR problem to be solvable. This condition can be relaxed for the general LQR problems with control dependent diffusion coefficients, as in Section 2.5.*

Clearly, the minimum of

$$u'Nu + (Bu)'V_x + 2u'L'x$$

is

$$\frac{-1}{4}V_x'BN^{-1}B'V_x - x'LN^{-1}BV_x - x'LN^{-1}Lx$$

(suppressing time parameter) with the minimizer

$$u^* = -N^{-1}(\frac{1}{2}B'V_x + L'x). \tag{27}$$

Then the HJB PDE takes the form of

$$\begin{aligned} V_s + \frac{1}{2}tr(\tilde{\sigma}\tilde{\sigma}'V_{xx}) + x'\tilde{M}x + x'\tilde{A}'V_x - \frac{1}{4}V_x'BN^{-1}B'V_x &= 0 \\ V(T,x) &= x'Rx \end{aligned} \tag{28}$$

which can also be written as

$$\begin{aligned} V_s + \frac{1}{2}tr(\tilde{\sigma}\tilde{\sigma}'V_{xx}(s,x)) + f(s,x,\tilde{\sigma}'V_x(s,x)) &= 0 \\ V(T,x) &= x'Rx \end{aligned} \tag{29}$$

with

$$
\begin{aligned}
f(s,x,z) &= x'\tilde{M}x + x'\tilde{A}'(\tilde{\sigma}\tilde{\sigma}')^{-1}\tilde{\sigma}z - \frac{1}{4}((\tilde{\sigma}\tilde{\sigma}')^{-1}\tilde{\sigma}z)'BN^{-1}B'(\tilde{\sigma}\tilde{\sigma}')^{-1}\tilde{\sigma}z \\
V(T,x) &= x'Rx
\end{aligned} \tag{30}
$$

where

$$
\begin{aligned}
\tilde{M} &= M - LN^{-1}L' \\
\tilde{A} &= A - LN^{-1}B'
\end{aligned} \tag{31}
$$

for $(s,x,z) \in [0,T] \times \Re^d \times \Re^m$. In the remaining of this section, we will assume that the condition (17) holds with $\kappa(t,x) = \tilde{\sigma}\tilde{\sigma}'(t,x)$ so that the HJB equation (29) is uniformly parabolic.

Remark 8 *(a) In general, the quasilinear PDE (28) may have multiple solutions. However, the value function is unique. Moreover it is the maximal solution of (26) (and hence of (29)) by Theorem 4.(i)*
(b) The PDE representation given by (29)-(31) is essential for the BSDE interpretation of the problem with a candidate solution $(Y_t, Z_t) = (V(t,X_t), \tilde{\sigma}'(s)V_x(t,X_t))$.
(c) When $\tilde{\sigma}$ *is a square* $(d \times d)$ *invertible matrix uniformly on* $[0,T] \times \Re^d$*, the function* f *in (30) could be further simplified as*

$$
x'\tilde{M}x + x'\tilde{A}'(\tilde{\sigma}')^{-1}z - x'LN^{-1}B(\tilde{\sigma}')^{-1}z - \frac{1}{4}z'(\tilde{\sigma}')^{-1}BN^{-1}B'(\tilde{\sigma}')^{-1}z \tag{32}
$$

with $z = \tilde{\sigma}'(s,x)V_x(s,x)$.

Now, the system (29)-(31) can be solved directly by assuming a solution of the following quadratic form

$$
\begin{aligned}
v(s,x) &= x'S(s)x + K'(s)x + a(s) \\
v(T,x) &= x'Rx, \\
(S(T) &= R, K(T) = 0, a(T) = 0)
\end{aligned} \tag{33}
$$

where $S(.) \in C^1(\Re^{d\times d})$, $K(.) \in C^1(\Re^d)$, $a(.) \in C^1(\Re)$; $S(t)$ is symmetric and uniformly nonegative definite (positive definite if $R > 0$) on $[0,T]$. Note that a solution v of the form (33) has a polynomial (quadratic) growth in x and satisfies $v(t,x) \in C^{1,2}([0,T];\Re^d)$. Then, substituting the trial function (33) back into the equation (29) and using the notation $\dot{f}(t) \equiv \frac{df(t)}{dt}$ for time derivatives, the left hand side of (29) becomes

$$x'(\dot{S} + \tilde{M} - SBN^{-1}B'S + \tilde{A}'S + S\tilde{A} + \sum_{j=1}^{m} C_j'SC_j)x \tag{34}$$
$$+\{\dot{K} + (\tilde{A}' - SBN^{-1}B')K + 2\sum_{j=1}^{m} C_j'S\sigma^j\}x + \dot{a} - \frac{1}{4}K'BN^{-1}B'K + \sum_{j=1}^{m} \sigma^{j'}S\sigma^j.$$

The expression (34) is zero if S, K and a satisfy

$$(\dot{S} + \tilde{M} - SBN^{-1}B'S + \tilde{A}'S + S\tilde{A} + \sum_{j=1}^{m} C_j'SC_j)(t) = 0, \tag{35}$$
$$\{\dot{K} + (\tilde{A}' - SBN^{-1}B')K + 2\sum_{j=1}^{m} C_j'S\sigma^j\}(t) = 0 \tag{36}$$
$$(\dot{a} - \frac{1}{4}K'BN^{-1}B'K + \sum_{j=1}^{m} \sigma^{j'}S\sigma^j)(t) = 0, \tag{37}$$

for $s \le t < T$.

The equation (35) with boundary condition $S(T) = R$ is a Riccati type nonlinear differential equation. When the matrices L and C_j's vanish, (35) becomes

$$\dot{S} + M - SBN^{-1}B'S + A'S + SA = 0 \tag{38}$$

which (sometimes called the *standard or conventional Riccati equation)* is known to have a unique solution $S(.) \in C^1((-\infty, T]; \Re^{d\times d})$ (e.g., Fleming and Rishel, p.89; Davis, Ch. 5). Some basic facts about Riccati type ODE's that will be used to discuss the solutions for more general Riccati ODE's of the type (35) are included in the Appendix B.

Lemma 4 *If $\tilde{M} \ge 0, R \ge 0$ and $N > 0$, then the Riccati equation (35) has a unique solution $S \in C^1([0,T]; \Re^{d\times d})$, which is symmetric and nonnegative on $[0,T]$. Moreover, if $\tilde{M} > 0$ or $R > 0$, then $S > 0$ on $[0,T]$.*

Proof. When C^j's vanish, the result follows from the Lemma B.1 in Appendix. The general case is an extension of Proposition B.5 using Lemma B.4 with $D \equiv 0$.

■

Now, having that (35) has a unique bounded solution, the linear non-homogenous vector ODE (36) with terminal condition $K(T) = 0$ has a unique smooth solution implying also that the scalar ODE (37) with $a(T) = 0$ has the smooth solution

$$a(s) = \int_s^T (\sum_{j=1}^m \sigma^{j'} S \sigma^j - \frac{1}{4} K' B N^{-1} B' K)(t) dt, \ 0 \leq s < T. \tag{39}$$

With these choices of the functions S, K and a, the expression (34) vanishes and the equality in (29) is satisfied. So the trial function $v(t,x)$ given by (33) is a solution to the HJB PDE (29) and satisfies $v(t,x) \leq J^u(t,x)$, for any $u \in \Re^d$, by (the Verification) Theorem 4(a).

The (unique) minimizer (27) is given in the following feedback form:

$$u^*(s,x) = -N^{-1}[(B'S + L')x + \frac{1}{2}K].$$

To show that u^* is an optimal control for this problem, consider the dynamics of the state process X_t under the control $u_t = u^*(t, X_t)$:

$$\begin{aligned} dX_t^* &= (A(t)X_t^* + B(t)u(t, X_t^*))dt + \tilde{\sigma}(t, X_t^*)dW_t, \\ &= \{(A - BN^{-1}(B'S + L'))X_t^* - \frac{1}{2}N^{-1}K\}dt \\ &\quad +(\sum_{j=1}^m C_j(t)X_t^* + \sigma^j(t))dW_j(t). \end{aligned} \tag{40}$$

The linear nonhomogenous SDE (40) above has a unique strong solution $X^* \in L^2_{\mathcal{F}}([0,T]; \Re^d)$ and the adapted process

$$u_t^* = u(t, X_t^*) = -N^{-1}B'S(t)X_t^* - \frac{1}{2}N^{-1}K(t) \tag{41}$$

is an admissible control process in $\mathcal{U}$. It's easy to check that the condition (20) of (the Verification) Theorem 4 (b) is satisfied and u^* is an optimal (Markov) control policy:

$$v(s,x) = \min_u J^u(s,x) = J^{u^*}(s,x).$$

Hence, the (unique) value function $V(s,x)$ is given by

$$\begin{aligned} V(s,x) &= v(s,x) \\ &= x'S(s)x + K'(s)x + \int_s^T (\sum_{j=1}^m \sigma^{j'} S \sigma^j - \frac{1}{4} K' B N^{-1} B' K)(t) dt \end{aligned} \tag{42}$$

Remark 9 *When the forward process X is completely observed, the optimal control $u^*(t, X_t)$ given by (41) is the same for both deterministic and stochastic regulator problems. However, for a partially observable process X, the optimal control (to be adapted to the filtration generated by the noisy observations) is given by $-N(t)^{-1}[B(t)'S(t)\breve{X}_t + \frac{1}{2}K(t)]$ where $\breve{X}_t$ is the (filtered) linear least squares estimate of X_t, given by the Kalman filter (see Davis (1977), Oksendal (2000)). Then the stochastic control problem splits into two separated computations: Estimating $\breve{X}_t$, and computing the optimal control u^* for the completely observable case. This result is known as the Separation Principle (see also Fleming and Rishel (1975)).*

2.1.2 FBSDE Interpretation

For notational convenience, we assume that $m = d$ and $\tilde{\sigma}^{-1}(t, x)$ exists, $\forall (t, x) \in [0, T] \times \Re^d$. In view of Theorem 1 and the equation (29), the triple $(\tilde{X}_t, Y_t^{s,x}, Z_t^{s,x})$ with

$$Y_t = V(t, \tilde{X}_t) = \tilde{X}_t' S(t) \tilde{X}_t + K'(t) \tilde{X}_t + a(t), \tag{43}$$

and

$$Z_t = \tilde{\sigma}'(t, \tilde{X}_t) V_x(t, \tilde{X}_t) = \tilde{\sigma}'(t, \tilde{X}_t)(2S(t)\tilde{X}_t + K(t)) \tag{44}$$

is a solution to the BSDE

$$\begin{aligned} dY_t^{s,x} &= -f(t, \tilde{X}_t, Z_t)dr - Z_t' dW_t \\ Y_T^{s,x} &= g(\tilde{X}_T) = \tilde{X}_T' R \tilde{X}_T \end{aligned} \tag{45}$$

with

$$\tilde{X}_t^{s,x} = x + \int_s^t \tilde{\sigma}(r, \tilde{X}_r^{s,x}) dW_r,$$

and $f(t, x, z)$ as in (32). Note that the function $f(s, x, z)$ is not Lipshitz in state variables x and z. So, the classical results for the existence and uniqueness of the solutions to the BSDE (45) don't apply. Unfortunately, the generator $f(s, x, z)$ doesn't satisfy the non-Lipshitz conditions given by Pardoux and Peng (1994) and by Mao (1995). Similarly, the results of Kobylanski (2000) for one-dimensional BSDE's with quadratic growth are not directly applicable since the terminal condition $g(X_T)$ is not bounded and we are not seeking a bounded solution in general. However, we are going to apply some techniques like exponential change of variable and use the results from Kobylanski (2000), Lepeltier and San Martin (1998) and Mao (1995) to our transformed systems whenever they are applicable. For the FBSDE systems arising from the LQR problem, our emphasis will be on the uniqueness of the solutions since the existence follows from the results above.

Remark 10 .

1. *The representation of (29)-(31) as a FBSDE system is not unique. Another representation may be given by the following system:*

$$\begin{aligned} \hat{X}_t^{s,x} &= x + \int_s^t \hat{b}(r, \hat{X}_r)dr + \int_s^t \tilde{\sigma}(r)dW_r \\ Y_t^{s,x} &= g(\hat{X}_T) + \int_t^T \hat{f}(r, \hat{X}_r, Z_r)dr - \int_t^T Z_r' dW_r \end{aligned} \tag{46}$$

   where

$$\begin{aligned} \hat{f}(t,x,z) &= x'\tilde{M}x - \frac{1}{4}(\sigma'^{-1}z)'BN^{-1}B\sigma'^{-1}z \\ \hat{b}(t,x) &= \tilde{A}(t)x \end{aligned} \tag{47}$$

   Each representation has some advantages depending on the complexity of the forward or backward equations in (45) or (46). In this section, the representation (45) will be used frequently due to the simplicity of the forward state dynamics.

2. *For the notational convenience, we will use $X^{s,x}$ for $\tilde{X}_t^{s,x}$ given by (45).*

3. *The representation by FBSDE's can also be applied to the stochastic optimal control problem of minimizing (24) with more general diffusion processes $\tilde{\sigma} = \tilde{\sigma}(s,x)$ which depends on the state but not on the control process. Although the optimal control strategy doesn't change, in this case, we don't have the advantage of an explicit solution in general (the problem of existence in a suitable space!)*

4. *If there doesn't exist a smooth solution of (29), then generalized or viscosity solutions can be considered for which the uniqueness issue becomes harder.*

Note that the process X with

$$X_t^{s,x} = x + \int_s^t \tilde{\sigma}(r, X_r)dW_r, 0 < t \leq T, \tag{48}$$

has continuous paths, a.s. and is an L^2-continuous martingale with mean x and covariance matrix

$$cov(t_1, t_2) = E[X_{t_1} X'_{t_2}] - xx' = E[\int_{t_1}^{t_2} \tilde{\sigma}\tilde{\sigma}'(r, X_r) dr],$$

for $s \leq t_1 \leq t_2 \leq T$. Since $\tilde{\sigma}(t, x)$ satisfies the global Lipshitz and linear growth conditions for any $t \in (s, T]$, there exist generic constants $c, d > 0$ (depending on t and the matrix norms $\|\sigma(.)\|$, $\|C_j(.)\|$, $j = 1, ...m$) such that

$$\begin{aligned} E[\int_0^t |X_r|^2 \, dr] &\leq c(1 + |x|^2)e^{ct}, \\ E[\sup_{0 \leq r \leq t} |X_r|^p] &\leq d(1 + |x|^p), \; p \geq 1. \end{aligned} \tag{49}$$

implying that $X \in L^p_{\mathcal{F}}(C[0, T]; \Re^d)$ and $g(X_T) = X'_T R X_T \in L^p(\Omega)$, for $p \geq 1$. These identities follow from martingale moment estimates or Burkholder-Davis-Gundy inequalities. See for instance, Karatzas and Shreve (1999), Yong and Zhou (1999). By *(43)* and *(44)*, Y and Z have quadratic growth in X, and by virtue of *(49)*, satisfy the following regularity properties:

$$\begin{aligned} E\left[\sup_{s \leq t \leq T} |Z_t|^p\right] &< \infty, \\ E\left[\sup_{s \leq t \leq T} (Y_t)^p\right] &< \infty, \; p \geq 1. \end{aligned}$$

The identities *(43)* and *(44)* also enable us to set some *a priory* bounds that will be essential in the implementation of the numerical schemes for certain FBSDE systems that will be discussed in Chapter 3. In particular, when $C_j \equiv 0$, $j = 1, ...m$, X is a Gaussian process with simplified covariance function

$$cov(t_1, t_2) = \int_{t_1}^{t_2} \sigma\sigma'(r) dr, 0 \leq t_1 \leq t_2 \leq T$$

and Z is a linear function of X. In the standard case with $L \equiv 0$ (24), we get $K \equiv 0$ and the value function simplifies as

$$V(s, x) = x' S(s) x + a(s)$$

where

$$
\begin{aligned}
a(s) &= \int_s^T (\sum_{j=1}^m \sigma^{j'} S \sigma^j)(t) dt \\
&= \int_s^T tr(\sigma\sigma' S)(t) dt.
\end{aligned}
$$

Hence $V(s,x)$ is nonnegative by Lemma 4. Moreover, if $M > 0$ or $R > 0$, then $V(s,x)$ is positive uniformly on $[0,T] \times \Re^d$. This implies that $Y_t = V(t, X_t)$ where X is given by *(45)* is a positive process with $0 < Y_t \leq c(1 + |X_t|^2)$, a.s. for $t \in [s,T)$. Noting that $V(t,x)$ is a maximal solution of HJB equation *(26)*, we have the following result:

Lemma 5 *Let the pair $(\bar{Y}, \bar{Z})$ be given as in (43) and (44) with $\tilde{M} \geq 0$. Then $\bar{Y}$ is a maximal nonnegative solution to the BSDE (45). Moreover, if $C_j \equiv 0$, $j = 1, ...m$, then*

(i) Any solution $(Y, Z) \in L^2_{\mathcal{F}} \times L^2_{\mathcal{F}}$ to (45) with $Y \geq 0$ a.s. satisfies $\bar{Y} \geq Y$, a.s.

(ii) If $\tilde{M} > 0$ or $R > 0$, then $\bar{Y}$ is positive a.s and hence any nonnegative solution $(Y, Z) \in L^2_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re)$ to (45) satisfies $0 \leq Y_t < c(1 + |X_t|^2)$, a.s. for any $t \in [s,T)$.

Remark 11 *The results above hold for the FBSDE representation (46), too.*

We also provide a regularity result for the maximal solution $(\bar{Y}, \bar{Z})$ for the standard case. A similar result for the general case may be obtained under additional assumptions.

Lemma 6 *Suppose that $C_j \equiv 0$, $j = 1, ...m$, and $L \equiv 0$. Let*

$$
X_t = x + \int_0^t \sigma(r) dW_r, 0 < t \leq T,
$$

and (Y, Z) be the maximal solution

$$
\begin{aligned}
Y_t &= V(t, X_t) = X_t' S(t) X_t + a(t), \\
Z_t &= \sigma(t)' V_x(t, X_t) = 2\sigma(t)' S(t) X_t
\end{aligned}
$$

to the BSDE (45). Then the triple (X, Y, Z) satisfy the following estimate:

$$
\sup_{0 \leq s,t \leq 1} E[|X_t - X_s|^2 + |Y_t - Y_s|^2 + |Z_t - Z_s|^2] \leq k\,|t-s| + l\,|t-s|^2 \tag{50}
$$

for some positive constants c, d.

Proof. Since the matrix and vector valued functions $\sigma(.), S(.)$ and $a(.)$ are uniformly continuous on $[0,1]$ with respect to the corresponding Euclidean norms, there exist constants $c > 0$ and $d \geq 1$ such that for a generic function $\Gamma(.) \equiv \sigma(.), S(.), \sigma' S(.)$ and $a(.)$, the following identities hold:

$$\begin{aligned} \max_{0\leq t\leq 1} \|\Gamma(t)\|^2 &\leq d && (51)\\ \|\Gamma(t)-\Gamma(s)\| &\leq c\,|t-s|\,,\ \forall\ 0\leq s,t\leq 1 && (52)\end{aligned}$$

Without loss of generality, we assume $0 \leq s \leq t \leq 1$. Then by Ito's isometry and *(51),* we get the inequalities

$$\begin{aligned} E[|X_t - X_s|^2] &\leq d(t-s) && (53)\\ E[|X_t|^2] &\leq d(1+|x|^2). \end{aligned}$$

Moreover, by simple manipulations and the inequalities *(51)-(53)*, one can see that the following inequality holds:

$$\begin{aligned} |Y_t - Y_s|^2 &\leq 2\{\|S(t)\|^2 |X_t|^2 |X_t - X_s|^2 + \|S(s)\|^2 |X_s|^2 |X_t - X_s|^2 \\ &\quad + \|S(t)-S(s)\|^2 |X_t|^2 |X_s|^2 + |a(t)-a(s)|^2\} \\ &\leq 4d^2(1+|x|^2)\,|X_t - X_s|^2 + 2c^2d^2(1+|x|^2)(t-s)^2 + 2c^2(t-s)^2. \end{aligned}$$

Similarly, we have:

$$\begin{aligned} \frac{|Z_t - Z_s|^2}{8} &\leq \|\sigma' S(t)\|^2 |X_t - X_s|^2 + \|\sigma' S(t) - \sigma' S(s)\|^2 |X_s|^2 \\ &\leq d\,|X_t - X_s| + c^2 d(1+|x|^2)(t-s)^2. \end{aligned}$$

Hence, taking expectations of both sides in the expressions above and using *(53),* the inequality *(50)* holds with

$$\begin{aligned} k &= 9d + 2d^3(1+|x|^2) \\ l &= c^2d^2(1+|x|^2) + c^2. \end{aligned}$$

■

2.2 Uniqueness of The Solutions

Now, having a maximal solution to the FBSDE system *(45) or (46)* in the form $\bar{Y}_t = V(t, X_t),\ \bar{Z}_t = \tilde{\sigma}(t, X_t)' V_x(t, X_t)$, our emphasis in this part will be on the uniqueness of solutions when the diffusion coefficient depends only on the time parameter: $\tilde{\sigma}(t,x) = \sigma(t)$ such that $\|\sigma^{-1}(.)\| < c$ uniformly on $[0,T]$ for some $c > 0$. With this assumption, the HJB equation is uniformly parabolic and all the results mentioned in the previous subsection hold.

Condition 12 *$\exists c > 0$ with $0 < \|\sigma^{-1}(.)\| < c$, $\tilde{M}(.) > 0$, $B(.) > 0$ or $B(.) < 0$ on $[0,T]$ and $C_j \equiv 0$, $j = 1, ...m$*

For simplicity, we first consider the one dimensional case under the Condition *12*. Then the system *(30)*- *(31)* becomes

$$\begin{aligned} f(t,x,z) &= \tilde{M}(t)x^2 + \frac{\tilde{A}(t)}{\sigma(t)}xz - H(t)z^2 \\ V_x(s,x) &= z/\sigma(t) \\ V(T,x) &= Rx^2 \end{aligned} \tag{54}$$

with $\tilde{M}(t) = (M - L^2/N)(t) > 0$ and

$$H(t) = \frac{B^2(t)}{4N(t)\sigma^2(t)} > 0, \ [0,T].$$

Condition 13 *The time dependent functions $\frac{\dot{H}}{H}(.)$ and $H(.)$ are uniformly bounded on $[0,T]$.*

Define the transformation

$$\begin{aligned} \psi(t,x) &\triangleq \exp(-2H(t).x) \\ U_t &\triangleq \psi(t,Y_t),\ 0 \le t \le T, \end{aligned}$$

for a new (bounded) process U with $0 < U_T = \exp(-2H(T)Y_T) < 1$ a.s. Then, by Ito's lemma, U satisfies

$$\begin{aligned} dU_t &= d\psi(t,Y_t) = -h(t,X_t,U_t,\Lambda_t)dt + \Lambda_t dW_t \\ U_T &= \psi(T,Y_T) = \exp(-2H(T)g(X_T)), \end{aligned} \tag{55}$$

where the driver $h(t,x,u,z)$ and the martingale term Λ_t are given by

$$h(t,x,u,z) = -\frac{\dot{H}}{H}(t)u\log u - 2H(t)\tilde{M}(t)x^2u + \frac{\tilde{A}(t)}{\sigma(t)}xz \tag{56}$$

$$\Lambda_t = -2H(t)U_tZ_t \tag{57}$$

so that (U_t, Λ_t) is a solution to the BSDE *(55)*.

Note that, by construction, the pair (U, Λ) of processes $U = (U_t)_{t \leq T}$ given by

$$\begin{aligned} U_t &= \exp(-2H(t).V(t, X_t)) \\ \Lambda_t &= -2H(t)U_t\sigma(t)V_x(t, X_t) \end{aligned}$$

($0 < U_t < 1$, a.s.) solve *(55)*. If (U_t, Λ_t) is the unique solution to *(55)*, then

$$(Y_t, Z_t) = \left(\frac{-\log U_t}{2H(t)}, \frac{-\Lambda_t}{2H(t)U_t}\right)$$

is (formally) the unique solution for the system *(45)*. The aim of the rest of this section is to make these statements precise by first starting from the uniqueness issue for a backward system which is similar to (1).

Given the random measurable functions $f : \Omega \times [0, T] \times \Re \times \Re^m \rightarrow \Re$, $\phi : \Omega \times [0, T] \times \Re \rightarrow \Re^m$ and $\xi : \Omega \rightarrow \Re$, we will now consider the BSDE's of the form

$$\begin{aligned} dY_t &= -f(t, Y_t, Z_t)dt + (\phi(t, Y_t) + Z_t)dW_t; \\ Y_T &= \xi \end{aligned} \tag{58}$$

under the following assumption:

Condition 14 *The random functions f and ϕ satisfy the followings:*
(a) $f(., 0, 0) \in L^2_{\mathcal{F}}(\Re)$, and $\phi(., 0) \in L^2_{\mathcal{F}}(\Re^m)$, a.s.
(b) For any $y_1, y_2 \in \Re$; $z_1, z_2 \in \Re^m$ and $0 \leq t \leq T$,

$$|f(., y_1, z_1) - f(., y_2, z_2)|^2 \leq \kappa(|y_1 - y_2|^2) + c\,|z_1 - z_2|^2, \ a.s.$$

$$|\phi(., y_1) - \phi(., y_2)|^2 \leq \kappa(|y_1 - y_2|^2), \ a.s.$$

where $c > 0$ and $\kappa : \Re_+ \rightarrow \Re_+$ is a concave increasing function such that $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{du}{\kappa(u)} = \infty$.

The following theorem which is adapted from Mao (1995) will be useful throughout this section. A Picard-Lindelof type approximation procedure and Bihari's inequality (see Appendix A) are key tools in the proof.

Theorem 15 (Mao, 1995) *If the Condition 14 (a)-(b) hold, then there exists a unique solution (Y, Z) to equation (58) in $L^2_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$.*

Note that the function h given by *(56)* is no longer quadratic in z. Morever, it doesn't satisfy the Condition *14* (b) above since the forward state process X is not bounded. We first prove the existence and uniqueness to a BSDE system without involving the forward part. Then, taking advantage of the nice properties of the

L^p martingale X, we get the same uniqueness result in the presence of the forward process X.

Proposition 16 *Let Condition 12 and Condition 13 hold. Then the BSDE*

$$\begin{aligned} dU_t &= -h(t, U_t, \Lambda_t)dt + \Lambda_t dW_t \\ 0 &< U_T = \xi < 1, \ a.s. \end{aligned} \tag{59}$$

with

$$h(t, u, z) = -\frac{\dot{H}}{H} u \log u - 2H(t)\tilde{M}(t)u + \frac{\tilde{A}(t)}{\sigma(t)} z \tag{60}$$

has a unique solution in $L^\infty_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$.

Proof. We may assume WLOG that $h(t, u, z) = -u \log u - \mathrm{u} + z$ since the time dependent coefficients are uniformly bounded on $[0, T]$.

Claim1. The indentities (a)-(b) of Condition *14* are satisfied by $h(t, u, z)$ and $\phi \equiv 0$ on $[0, T] \times [0, l] \times \Re$, for any $M > 0$. This means that $L(u) \triangleq u \log u$ satisfies

$$|L(y_1) - L(y_2)|^2 \leq \kappa(|y_1 - y_2|^2), \text{ a.s.}$$

on $[0, l]$, $l > 0$ where κ is as in Condition *14*.

Claim2. We may assume $0 < U_t < 1$, a.s. since we are seeking a bounded solution. By using a truncation argument to control the growth of U, if necessary, the result of the Proposition *16* holds. In fact, the unique solution is in $L^\infty_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$.

In order to prove Claim1, we introduce the following concave function on $[0, l]$:

$$\kappa(u) = \begin{cases} 2u \log(u^{-1}), \text{ if } 0 \leq u \leq \epsilon \\ 2\epsilon \log(\epsilon^{-1}) + \dot{\kappa}(\epsilon^-)(u - \epsilon), \text{ if } \epsilon \leq u \leq l \end{cases}$$

for a small number ϵ. For example, we may choose $\epsilon < e^{-1}$ so that $u \log(u^{-1})$ is a monotone increasing function on $[0, \epsilon]$ satisfying other conditions above. (One can generalize the definition of κ by introducing $\kappa_1(u) = C_{\epsilon,l}\kappa(u)$ to guarantee that $\kappa_1(u)$ will be bigger than $|u \log(u^{-1})|$ on the set $\epsilon \leq u \leq l$, where $C_{\epsilon,l} \geq 1$ is a constant depending on ϵ and l).

Now, we want to show that

$$|L(u_1) - L(u_2)| \leq \kappa(|u_1 - u_2|) = |u_1 - u_2| \log(|u_1 - u_2|^{-1}) \tag{61}$$

WLOG, we may assume that $0 \leq u_2 < u_1 < 1/e$. We will consider two cases $|u_1 - u_2| \leq u_2$ and $u_2 < |u_1 - u_2|$ separately:

If $0 < |u_1 - u_2| \le u_2$, by Mean Value Theorem, the inequality

$$\begin{aligned} |L(u_1) - L(u_2)| &= |1 + \log \eta| \, |u_1 - u_2| \, , \exists \eta \text{ with } u_2 \le \eta \le u_1 \\ &< |\log \eta| \, |u_1 - u_2| \, , \text{ since } \log \eta < \log(e^{-1}) = -1 \\ &\le |\log(u_1 - u_2)| \, |u_1 - u_2| \, , \; u_1 - u_2 \le u_2 \le \eta \\ &= |u_1 - u_2| \log(|u_1 - u_2|^{-1} \end{aligned}$$

is obtained.

On the other hand, when $u_2 < u_1 - u_2$, by adding and subtracting the term $u_2 \log u_1$, we get

$$\begin{aligned} |L(u_1) - L(u_2)| &\le (u_1 - u_2) \log u_1 + u_2 \, |\log u_1 - \log u_2| \\ &= (u_1 - u_2) \log(u_1^{-1}) + u_2 (\log u_1 - \log u_2) \\ &\le (u_1 - u_2) \log(|u_1 - u_2|^{-1}) - u_2 \log u_2, \end{aligned}$$

since $u_2 \log u_1 < 0$ and $\log(u^{-1})$ is decreasing. The result follows by noting that

$$-u_2 \log u_2 = u_2 \log(u_2^{-1}) \le (u_1 - u_2) \log(|u_1 - u_2|^{-1}).$$

The general case $(0 \le u_2 < u_1 \le l)$ is similar provided $\epsilon < e^{-1}$ holds. ■

Remark 17 *The existence of the solution for (59)-(60) would also follow from Lepeltier and San Martin (1998) by choosing the function k in (3) as*

$$k(y) = \begin{cases} c_1, \; y < 2 \\ c_2 y \ln y, \; y \ge 2 \end{cases}$$

with suitable positive constants c_1 and c_2.
By using a truncation approach to control the growth of the driver in u, Kobilansky's existence results might apply to this truncated process. Then, one should also verify the convergence and stability issues. The following result is adapted from Kobilansky (2000) with some minor changes:

Proposition 18 *Let f and ξ be the generator and terminal condition of a BSDE*

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t; Y_T = \xi \tag{62}$$

and (f^n, ξ^n) be a sequence of generators and terminal conditions such that:

(i) The sequence (f^n) converges to f locally uniformly on $\Re_+ \times \Re \times \Re^m$; $\xi^n \in L^\infty(\Omega)$ and (ξ^n) converges to ξ in $L^\infty(\Omega)$.

(ii) There exists $k : \Re_+ \to \Re_+$ such that for all $T > 0$, $k \in L^1[0,T]$ and $\exists\ C > 0$ such that for $\forall\ n$ and $\forall\ (t,u,z) \in \Re_+ \times \Re \times \Re^m$,

$$|f^n(t,u,z)| \leq k(t) + C\,|z|^2 .$$

(iii) For each n, the BSDE with parameters (f^n, ξ^n) has a solution $(Y^n, Z^n) \in L^\infty_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$ such that the sequence (Y^n) is monotonic and $\exists\ M > 0$ with $\|Y^n\|_\infty \leq M$, $\forall\ n$.
Then $\exists\ (Y,Z) \in L^\infty_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$ such that for all $T > 0$, $\lim_{n\to\infty} Y^n = Y$ uniformly on $[0,T]$; $(Z^n) \to Z$ in $L^2_{\mathcal{F}}(\Re^m)$ and (Y,Z) is a solution of the BSDE (62).
In particular, if for each n, Y^n has continuous paths, so does the process Y.

Theorem 19 *If Condition 12 and Condition 13 hold, then the FBSDE given by (55)-(56) with*

$$X_t = x + \int_0^t \sigma(r)dW_r$$

has a unique solution in $L^\infty_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$.

Proof. For notational convenience, the dependence of integrands on the time variable is suppressed. Let (U_1, Λ_1) and (U_2, Λ_2) be two solutions for the equation *(55)* in $L^\infty_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$. Then by Ito's formula, $|U_1(t) - U_2(t)|^2$ satisfies

$$\begin{aligned} |U_1(t) - U_2(t)|^2 &= \int_t^T \{-2(U_1 - U_2)[U_1 \log U_1 - U_2 \log U_2] - 2x^2(U_1 - U_2)^2 \\ &\quad -2x(U_1 - U_2)(\Lambda_1 - \Lambda_2) - (\Lambda_1 - \Lambda_2)^2\}dr \\ &\quad + \int_t^T 2(U_1 - U_2)(\Lambda_1 - \Lambda_2)dW_r, \end{aligned}$$

for $0 < t < T$. Then it holds that

$$\begin{aligned} E[|U_1(t) - U_2(t)|^2 + \int_t^T |\Lambda_1 - \Lambda_2|^2\,dr] &\leq 2E\int_t^T \{|U_1 - U_2|\,|U_1 \log U_1 - U_2 \log U_2| \\ &\quad -x^2(U_1 - U_2)^2 + x\,|U_1 - U_2|\,|\Lambda_1 - \Lambda_2|\}dr. \end{aligned}$$

Now, the second term on the right of this equation is negative. The first term $2\,|U_1 - U_2|\,|U_1 \log U_1 - U_2 \log U_2|$ is dominated by $|U_1 - U_2|^2 + |U_1 \log U_1 - U_2 \log U_2|^2$

which is also bounded by $|U_1 - U_2|^2 + \kappa(|U_1 - U_2|^2)$ using the same κ as in the proof of Proposition 16. Applying the inequality $2ab \leq \alpha a^2 + b^2/\alpha$ with $\alpha = 2$ to the third term, we get

$$2x\,|U_1 - U_2|\,|\Lambda_1 - \Lambda_2| \leq 2x^2(U_1 - U_2)^2 + |\Lambda_1 - \Lambda_2|^2/2.$$

Combining these inequalities,

$$E[|U_1(t) - U_2(t)|^2 + \frac{1}{2}\int_t^T |\Lambda_1 - \Lambda_2|^2\,dr] \leq E\int_t^T \{|U_1 - U_2|^2 + \kappa(|U_1 - U_2|^2\}dr$$

is obtained.

Define $\xi(x) \triangleq x^2 + \kappa(x^2)$. Then,

$$E[|U_1(t) - U_2(t)|^2] \leq E\int_t^T \xi(|U_1 - U_2|^2)dr$$

implies that $|U_1 - U_2|^2 = 0$, a.s., by Bihari's inequality. So $U_1 = U_2$, a.s. Then $|\Lambda_1 - \Lambda_2|^2 = 0$, a.s., too, so that $Z_1 = Z_2$, a.s. ■

Due to the monotonicity of the transformations $\psi(t,x) \triangleq \exp(-2H(t).x)$, $U_t \triangleq \psi(t, Y_t)$ and $\Lambda_t = -2H(t)U_tZ_t$, we get the following results as a Corollary:

Corollary 20 *If Condition 12 and Condition 13 hold, then the BSDE (45) where the driver f is given by (54) has a unique solution (Y, Z) in $L^2_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$*

Corollary 21 *The one-dimensional version of HJB PDE (29) has a unique viscosity solution under the assumptions of theorems above. Moreover, this solution is smooth and is given by the value function (42).*

Now, suppose that the diffusion coefficient $\sigma(t,x)$ is a uniformly Lipshitz continuous function which is uniformly positive-definite and independent of the control variable. Then, consider the following version of *(22)*:

$$\begin{aligned} dX_t &= (A(t)X_t + B(t)u_t)dt + \sigma(t, X_t)dW_t, 0 \leq s < t \leq T \\ X_0 &= x \end{aligned} \tag{63}$$

with the cost functional

$$J^u(s,x) = E^{s,x}[\int_s^T (X_t'M(t)X_t + 2X_t'L(t)u_t + u_t'N(t)u_t)dt + X_T'RX_T]. \tag{64}$$

We may use the FBSDE representation

$$\begin{aligned}
\hat{X}_t^{0,x} &= x + \int_0^t A(r, \hat{X}_r)dr + \int_0^t \sigma(r, \hat{X}_r)dW_r \\
Y_t^{s,x} &= g(\hat{X}_T) + \int_t^T f(r, \hat{X}_r, Z_r^{s,x})dr - \int_t^T Z_r^{s,x}dW_r
\end{aligned}$$

where

$$\begin{aligned}
f(t,x,z) &= x'Mx - \frac{1}{4}(\sigma'^{-1}z)'BN^{-1}B\sigma'^{-1}z \\
&= x'Mx - z'H(t,x)z \\
g(t,x) &= x'Rx
\end{aligned}$$

for the control problem *(63)-(64)*.

For simplicity, consider the one dimensional case $d = 1$, write $\hat{X} = X$ and define a process $U \triangleq \{U_t : 0 \le t \le T\}$ the following exponential tranformation

$$\begin{aligned}
U_t &= e^{-2H(t,X_t)Y_t} \\
U_T &= e^{-2H(T,X_T)Y_T}
\end{aligned}$$

where $H(t,x) \in C^{1,2}([0,T] \times R)$. The dynamics of U_t is:

$$\begin{aligned}
dU &= \{\frac{H_t}{H} + \frac{H_x}{H}AX)U\ln U + 2HMX^2U + \frac{1}{2}[\frac{H_{xx}}{H} + 2(\frac{H_x}{H})^2\ln U]\sigma^2U\ln U \\
&\quad +\sigma\frac{H_x}{H}(1+\ln U)\Lambda\} + (\sigma\frac{H_x}{H}U\ln U + \Lambda)dW \qquad (65)
\end{aligned}$$

with $\Lambda = -2HUZ$. In general H_t, H_x, H_{xx} are complicated functions of t and x. For the linear case $\sigma(t,x) = C(t)x$ with $C \ge \epsilon > 0$, some straightforward computations show that

$$\begin{aligned}
\frac{H_t}{H} &= \frac{2\dot{B}N}{B} - \dot{N} + 2\dot{N}\frac{\dot{C}}{C} \\
\frac{H_x}{H} &= -2/x \\
\frac{H_{xx}}{H} &= 6/x^2
\end{aligned}$$

so that all the terms in the BSDE *(65)* above becomes state-invariant. Note that the results of LQR problem applies to this case, i.e. the value function is known to be a smooth quadratic function of x and to be a maximal solution of the corresponding HJB PDE. However, we are not able to prove the uniqueness of the solution to the corresponding BSDE. The main difficulty is coming from the term $U(\ln U)^2$ which grows very fast in a neighborhood of zero. The arguments of Proposition 16 doesn't apply directly to such a function.

2.3. Multidimensional Extension

In this section, we will consider the case $\tilde{\sigma}(t,x) = \sigma(t) \in \Re^{d\times m}$, $d, m \geq 1$. We will also assume that $L \equiv 0$ although all of the following results will remain valid in the general case $L \not\equiv 0$ provided that the matrix $\tilde{M} = M - LN^{-1}L'$ is nonnegative definite: $\tilde{M} \geq 0$.

With $L \equiv 0$, we have $\tilde{M} = M$ and $\tilde{A} = A$. Therefore, when $d = m$, the driver $f(t,x,z)$ simplifies to

$$f(t,x,z) = x'M(t)x + x'\tilde{A}'\sigma'^{-1}(t)z - z'H(t)z$$

with

$$H(t) = \frac{1}{4}\sigma^{-1}BN^{-1}B'\sigma'^{-1}(t). \tag{66}$$

Otherwise if $d \neq m$, replace the matrices σ^{-1} and σ'^{-1} with $\sigma'(\sigma\sigma')^{-1}$ and $(\sigma\sigma')^{-1}\sigma$, respectively. In multidimensional case, the exponential transformation method is much more complicated and we will only consider some special cases.

Now, let (Y, Z) be the maximal solution considered in section 2.1 Then applying Ito's rule to a transformation of the form $\varphi(t,y) \triangleq e^{-2\lambda(t)y}$, the dynamics of the process $U_t \triangleq \varphi(t, Y_t)$ is governed by the BSDE

$$\begin{aligned} dU_t &= d\varphi(t,Y_t) = -h(t,X_t,U_t,\Lambda_t)dt + \Lambda_t dW_t \\ U_T &= \varphi(T,Y_T) = \exp(-2\lambda(T)g(X_T)) \end{aligned} \tag{67}$$

with

$$h(t,x,u,z) = -\frac{\dot{\lambda}}{\lambda}u\log u - 2\lambda u x'Mx + x'A(\sigma')^{-1'}z + \frac{1}{2u}z'(I - \frac{H}{\lambda})z \tag{68}$$

where I is the $d \times d$ identity matrix.

When $H(t)$ is of the form $\lambda(t)I$, where $\lambda(t)$ is a real-valued continuous function which doesn't vanish on $[0,T]$, the results in the previous section can be generalized to the multidimensional case, as stated by the following theorem:

Theorem 22 *Let the function $H(.)$ be given by (66) and be of the form $\lambda(t)I$ where $\lambda : [0,T] \to (0,\infty)$ is a continuous function satisfying Condition 12. Define a* *transformation* $\varphi(t,y) \triangleq e^{-2\lambda(t)y}$*, for* $(t,y) \in [0,T] \times [0,\infty)$*. Then, the dynamics of the process* $U_t \triangleq \varphi(t,Y_t)$ *is given by the BSDE*

$$\begin{aligned} dU_t &= d\varphi(t,Y_t) = -h(t,X_t,U_t,\Lambda_t)dt + \Lambda_t dW_t \\ U_T &= \varphi(T,Y_T) = \exp(-2\lambda(T)g(X_T)) \end{aligned} \tag{69}$$

with

$$h(t,x,u,z) = -\frac{\dot{\lambda}}{\lambda}u\log u - 2\lambda u x' M x + x' A (\sigma')^{-1'} z \tag{70}$$

$$\Lambda_t = -2\lambda(t)U_t Z_t. \tag{71}$$

This system has a unique solution (U_t,Λ_t) *in* $L^\infty_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$*.*

Proof. The first part follows from Ito's lemma. Since the quadratic dependence on z disappears in *(70)* and the time dependent matrix valued functions are bounded on $[0,T]$, the existence and uniqueness arguments are similar to the proof for the one dimensional case. ∎

Corollary 23 *Under the assumptions of Theorem 22, the FBSDE system (45) has a unique solution* (Y,Z) *in* $L^2_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$*, where*

$$\begin{aligned} Y_t &= \frac{-1}{2\lambda(t)}\ln(U_t) \\ Z_t &= \frac{-1}{2\lambda(t)U_t}\Lambda_t \end{aligned}$$

with (U_t,Λ_t) *being the unique solution to the system (69)-(71). Moreover, the value function given by (42) is the unique viscosity solution to the quasilinear PDE (29) and this solution is smooth.*

The results above suggest that we can identify certain systems of FBSDE's as LQR problems through the representations *(45)*-*(47)* and solve them by means of Riccati type equations:

Corollary 24 *Consider the following FBSDE system*

$$\begin{aligned} dX_t &= \sigma(t)dW_t, \;\; s \le t \le T \\ dY_t &= -f(t,X_t,Z_t)dt + Z_t dW_t, \;\; s \le t \le T \\ X_s &= x; Y_T = X_T' R X_T \end{aligned} \tag{72}$$

with

$$f(t,x,z) = x'M_1(t)x + x'M_2(t)z + z'M_3(t)z$$

and proper conditions on the matrix valued functions $M_j(t)$, for $j = 1, 2, 3$. Then the pair of processes (Y, Z) which are given by

$$Y_t = V(t, X_t), \;\; Z_t = \sigma'(t)V_x(t, X_t)$$

solve the system (72) with $V(t, x) = x'Sx + a(t)$ where $S(t)$ and $a(t)$ satisfy

$$\begin{aligned} \dot{S} + M_2\sigma'S + S\sigma M_2' + 4S\sigma M_3\sigma'S + M_1 &= 0 \\ \dot{a} + tr(\sigma\sigma'S) &= 0 \\ S(T) &= R \\ a(T) &= 0. \end{aligned}$$

Corollary 25 *Consider the following FBSDE system:*

$$\begin{aligned} dX_t &= A(t)X_t dt + \sigma(t)dW_t, \;\; s \le t \le T \\ dY_t &= -f(t, X_t, Z_t)dt + Z_t dW_t, \;\; s \le t \le T \\ X_s &= x; Y_T = X_T'RX_T \end{aligned} \tag{73}$$

with

$$f(t, x, z) = x'M(t)x + z'H(t)z$$

Then the pair of processes (Y, Z) which are given by

$$\begin{aligned} Y_t &= V(t, X_t), \\ Z_t &= \sigma'(t)V_x(t, X_t) \end{aligned}$$

solve the system (72) with $V(t, x) = x'Sx + a(t)$ where $S(t)$ and $a(t)$ satisfy

$$\begin{aligned} \dot{S} + A'S + SA + 4S\sigma H\sigma'S + M &= 0 \\ \dot{a} + tr(\sigma\sigma'S) &= 0 \\ S(T) &= R \\ a(T) &= 0. \end{aligned}$$

2.4. Some Comparison Results

Note that the dynamics of the Riccati equation *(38)* depends on the expression $BN^{-1}B'$ rather than on the matrices B and N individually. By taking advantage of this property as well as the symmetry properties of the matrices involved, we can obtain the following comparison results for the solutions to the stochastic LQR problems.

Theorem 26 *Assume that the diffusion matrix $\sigma(.)$ has full column rank and consider the following LQR problems with state dynamics*

$$\begin{aligned} dX_t^i &= (A(t)X_t^i + B_i(t)u_t)dt + \sigma(t)dW_t, \ 0 < t \leq T \\ X_0^i &= x, \end{aligned} \tag{74}$$

for $i = 1, 2$ and cost functionals

$$J_i^u(s, x) = E^{s,x}[\int_s^T (X_t^{i\prime} M(t)X_t^i + u_t' N_i(t)u_t)dt + X_T^{i\prime} R X_T^i], \ 0 \leq s \leq T. \tag{75}$$

Define the value functions $V^i(s, x) = \inf_u J_i^u(s, x)$, $i = 1, 2$ and set

$$H_i(t) \triangleq \frac{1}{4}[\sigma'(\sigma\sigma')^{-1} B_i N_i^{-1} B_i' (\sigma\sigma')^{-1}\sigma](t), \ i = 1, 2.$$

If $H_i(.) > 0$ on $[0, T]$, then the followings hold:
(i) If the matrix inequality $H_1(t) \leq H_2(t)$ holds on $[s, T]$, then $V^1(s, x) \geq V^2(s, x)$.
(ii) If, in particular, $H_i(t) = \lambda_i(t)I$ for $i = 1, 2$ with $0 < \lambda_1(t) \leq \lambda_2(t) < c$, for some $c > 0$ on $[0, T]$, then the value functions $V^i(s, x)$ are the unique solutions to the HJB equations

$$\begin{aligned} v_s^i + \frac{1}{2} tr(\sigma\sigma' v_{xx}^i(s, x)) + f(s, x, \sigma' v_x^i(s, x)) &= 0 \\ v^i(T, x) &= x'Rx \end{aligned} \tag{76}$$

with

$$\begin{aligned} f(s, x, z) &= x'Mx + x'A'(\sigma\sigma')^{-1}\sigma z - \frac{1}{4}((\sigma\sigma')^{-1}\sigma z)' B N^{-1} B' (\sigma\sigma')^{-1}\sigma z \\ v^i(T, x) &= x' Rx \end{aligned}$$

for $i = 1, 2$ and satisfy $V^1(s, x) \geq V^2(s, x)$.
(iii) Let $H_1(.) \geq \lambda I$ for some $\lambda > 0$ on $[0, T]$, and $V(s, x)$ be the value function for a LQR problem (and unique solution to the corresponding HJB equation) of the form (74)-(75) with $H(.) = \lambda I$. Then any classical solution $v_1(s, x)$ of the PDE (76) with $i = 1$ satisfies $v_1(s, x) \leq V^1(s, x) \leq V(s, x)$.

Proof. Note that the matrix inequality $S_1 \geq S_2$ implies

$$V_1(s, x) = x'S_1(s)x + a_1(s) \geq x'S_2(s)x + a_2(s) = V_2(s, x)$$

Then the claims (i)-(ii) follow from the FBSDE representation of (74)-(75) and Lemma B.2 noting that

(i) $B_1 N_1^{-1} B_1'(t) = 4\sigma\sigma'(t)\lambda_1(t) \le 4\sigma\sigma'(t)\lambda_2(t) = B_2 N_2^{-1} B_2'(t)$, $\forall\ t \in [s,T]$ and

(ii) $x'(B_2 N_2^{-1} B_2'(t) - B_1 N_1^{-1} B_1'(t))x = x'\sigma(H_2 - H_1)\sigma'(t)x \ge 0$, for any $t \in [s,T]$ and any nonzero x since σ has full column rank and $(H_2 - H_1)(t) \ge 0$.

The proof of (iii) is a result of (i)-(ii) and Remark *8.* ■

Corollary 27 *Consider the following decoupled FBSDE system:*

$$
\begin{aligned}
X_t^i &= x + \int_0^t \sigma(s)dW_s \\
Y_t^i &= g(X_T^i) + \int_t^T f^i(s, X_s^i, Z_s^i)ds - \int_t^T Z_s^{i\prime} dW_s
\end{aligned} \tag{77}
$$

where

$$
\begin{aligned}
f^i(s,x,z) &= x'M(s)x - z'H_i(s)z, \\
g(y) &= y'Ry
\end{aligned} \tag{78}
$$

and $H_i(.) > 0$ on $[0,T]$, for $i = 1,2$. Let $(\bar{Y}^i, \bar{Z}^i)$, $i = 1,2$ be the maximal solutions to the system (77)-(78) as in Lemma 5. Then

(i) If $H_1(t) \le H_2(t)$ holds for $0 \le t \le T$, then $\bar{Y}_t^1 \ge \bar{Y}_t^2$, $\forall\ t \in [0,T]$, a.s.

(ii) If $H_i(t) = \lambda_i(t)I$ for $i = 1,2$ with $0 < \lambda_1(t) \le \lambda_2(t) < c$, for some $c > 0$ on $[0,T]$, then $(\bar{Y}^i, \bar{Z}^i)_{i=1,2}$ are the unique solutions to the systems (77)-(78) for $i = 1,2$ such that $\bar{Y}_t^1 \ge \bar{Y}_t^2$, $\forall\ t \in [0,T]$, a.s.

(iii) Let $H_1(.) \ge \lambda I$ for some $\lambda > 0$ on $[0,T]$, and $V(s,x)$ be the value function for a LQR problem (74)-(75) with $H(.) = \lambda I$. If $(Y_t^1, Z_t^1) \in L_{\mathcal{F}}^2(\Re) \times L_{\mathcal{F}}^2(\Re^m)$ is any solution of (77)-(78) for $i = 1$, then

$$
P\{Y_t^1 \le \bar{Y}_t^1 \le \bar{Y}_t, \forall t \in [0,T]\} = 1
$$

where $(\bar{Y}_t, \bar{Z}_t) = (V(t, X_t), \sigma'(t)V_x(t, X_t))$ is the unique (maximal) solution to a BSDE of the form (77)-(78) with $H(.) = \lambda I$.

Remark 28 *In section 3.1, we will implement a numerical procedure under which an approximating sequence V^n converges to the solution V given by (42). Note that* *the process Y with $Y_t = V(t, X_t)$ satisfies the identity*

$$\begin{aligned} E_t[Y_T] &= E_t[(X_t + \int_t^T \tilde{\sigma}(r, X_r)dW_r)'R(X_t + \int_t^T \tilde{\sigma}(r, X_r)dW_r)] \\ &= X_t'RX_t + \int_t^T tr(\tilde{\sigma}\tilde{\sigma}'R)dr,\ t < T. \end{aligned} \tag{79}$$

which will be used in Chapter 3.

In the next section, we introduce a more general class of LQR problems which have numerous applications in financial economics.

2.5 Control Dependent Diffusion Terms

In this section, we review some known results about solvability of a class of LQR problems that a linear control is present in the diffusion of the state dynamics as in *(22)* and the cost functional as in *(24)* but with $L \equiv 0$:

$$J^u(s,x) = E^{s,x}[\int_s^T (X_t'M(t)X_t + u_t'N(t)u_t)dt + X_T'RX_T],\ s \leq T. \tag{80}$$

The optimization problems of this form have various applications in mathematical finance where decision making is also related to the scale of the uncertanty. When the parameters of the equation *(22)* are allowed to be random, the LQR problem is related to the stochastic Riccati BSDE's which are nonlinear BSDE's with quadratic growth. The general theory of these equations are reviewed in subsection 2.6.3 and an application to mean-variance portfolio optimization problem is provided in section 4.2.

Note that the FBSDE representation given in the previous section doesn't work when $D(t) \neq 0$. Moreover, the optimal control process turns out to be different than the one given in section 2.1. The stochastic LQR problems with state dynamics *(22)* are extensively studied under the condition that the control weighting matrix $N > 0$, see e.g. Wonham (1968), Bizmut (1976). However, this condition has been recently shown to be relaxed (see Remark *29* below).

Now, associated with *(22)* and *(80),* we have the following generalized Riccati equation:

$$
\begin{aligned}
0 &= \dot{P} + (PA + A'P + \sum_{j=1}^{m} C_j' P C_j + M) \\
&\quad -(PB + \sum_{j=1}^{m} C_j' P D_j)(N + \sum_{j=1}^{m} D_j' P D_j)^{-1}(PB + \sum_{j=1}^{m} C_j' P D_j) \quad (81) \\
P(T) &= R,\ (N + \sum_{j=1}^{m} D_j' P D_j)(t) > 0, \quad t \leq T. \quad (82)
\end{aligned}
$$

Remark 29 *Note that the assumption (82)*

$$(N + \sum_{j=1}^{m} D_j' P D_j)(t) > 0, t \leq T$$

is part of the solution and is a generalization of the condition $N > 0$ which is a standing assumption in the standard LQR problems. To the best of our knowledge, the extension (82) is first introduced by Chen et al (1998) with $m = 1$ (one dimensional Brownian motion), assuming $C_j \equiv 0$, for all j. Later, Chen and Zhou (2000) discussed some special cases with non-zero C_j's. See also Ait Rami et. al. (2001) for the general case $L \not\equiv 0$. The results for $m > 1$ are not essentially different. See, for instance, Kohlmann and Tang (2000), Hu and Zhou (2003).

When all the coefficients are deterministic, the following general result connecting the generalized Riccati equation to the LQR problem (81)-(82) is well known. Our formal argument to prove it is based on DPP in one-dimensional homogenous case in the next subsection.

Proposition 30 *If the Riccati equation (81)-(82) has a solution, then the stochastic LQR problem with state dynamics (22) is well posed: $V(s, x) > -\infty$. Moreover, the feedback control*

$$u^* = -(N + \sum_{j=1}^{m} D_j' P D_j)^{-1}(PB + \sum_{j=1}^{m} C_j' P D_j)' X_t \tag{83}$$

is optimal for the LQR problem.

2.5.1 Homogenous Case with Deterministic Coefficients

Let $\alpha \equiv 0$, $\sigma \equiv 0$ in *(22)* and consider the case $m = 1$ with cost functional *(80).* Then *(22)* reduces to the homogenous equation

$$\begin{aligned} dX_t &= (A(t)X_t + B(t)u_t)dt + (C(t)X_t + D(t)u_t)dW_t, 0 \leq s < t \leq T \\ X_0 &= x_0, \end{aligned} \tag{84}$$

the control *(83)* becomes

$$u^* = -(N + D'PD)^{-1}(PB + C'PD)'X_t$$

and the corresponding Riccati equation is given by

$$\begin{aligned} 0 &= \dot{P} + (PA + A'P + C'PC + M) \\ &\quad -(PB + C'PD)(N + D'PD)^{-1}(PB + C'PD) \\ P(T) &= R, \ (N + D'PD)(t) > 0, \ t \leq T. \end{aligned} \tag{85}$$

A proof of Proposition *(30)* under the assumptions of this subsection is given by Chen et al (1998) using a square completion technique which also works in random coefficient case. For a Maximum Principle approach, see Yong and Zhou (1999). We prefer to use DPP to also show the connections with the methods used in section 2.1. Notice that the HJB PDE is now fully nonlinear and corresponds to a fully coupled FBSDE system which is far more complicated than the systems considered in the Chapter 2. For simplicity, a (formal) proof will be given only in one-dimensional case.

Proof of Propositon *(30)*. Define $\sigma^u(t,x) \triangleq C(t)x + D(t)u$. Then the HJB equation satisfies

$$\begin{aligned} V_s + M(s)x^2 + A(s)xV_x + \inf_u g(s,x,u) &= 0 \\ V(T,x) &= Rx^2 \end{aligned} \tag{86}$$

with

$$g(s,x,u) = B(s)uV_x + N(s)u^2 + \frac{1}{2}(\sigma^u(s,x))^2 V_{xx}.$$

Assuming a solution of the form $V(s,x) = x^2 P(s)$, one gets

$$\inf_u g(s,x,u) = g(s,x,u^*) = \frac{-(B + CD)^2 P^2}{N + D^2 P}(s)x^2$$

corresponding to

$$u^* = \frac{-(B+CD)P}{N+D^2P}(s)x.$$

So, the equation *(86)* becomes

$$\begin{aligned} x^2(\dot{P} + 2AP + C^2P - \frac{(B+CD)^2P^2}{N+D^2P} + M) &= 0 \\ N + D^2P &> 0, P(T) = R \end{aligned} \tag{87}$$

If the equation *(87)* has a solution, this solution should be unique, symmetric and continuous on $[0,T]$, which is also positive semidefinite (Lemma B.4 in Appendix B). Now, the system equation *(84)* with the optimal control candidate $u^*(t, X_t)$ takes the following form:

$$\begin{aligned} dX &= (A - B\tilde{N}^{-1}\tilde{P})Xdt + (C - D\tilde{N}^{-1}\tilde{P})dW_t \\ X_s &= x \end{aligned} \tag{88}$$

with $\tilde{N} = N + D^2P > 0$, and $\tilde{P} = PB + CPD$. Since P is bounded, $\tilde{N} > 0$, and coefficients are time-continuous (uniformly), the equation *(88)* has a unique adapted solution. By the Verification Theorem, $u^*(t, X_t)$ is the unique (feedback) optimal control for the optimization problem which is finite. ■

Corollary 31 *Suppose that $D'D > 0$ or $N > 0$ on $[0,T]$. Then the Riccati equation (85) has a solution P, and the value function for the stochastic LQR problem with state dynamics (84) is $V(s,x) = x'P(s)x$ Moreover, the optimal control is $u^* = -(N + D'PD)^{-1}(PB + C'PD)'X_t$.*

Proof. Follows from Proposition B.5 and Proposition *30.* ■

Remark 32 *In general, the Riccati equation (85) is only locally solvable around T, by the general theory of the first order ODE's. When C doesn't vanish, there is no general result about the global existence of the solutions although results similar to the ones in Appendix B are available for special cases.*

2.5.2 Random Nonhomogenous Terms

It is easy to see that the arguments in the previous subsection can be generalized to the nonhomogenous case (with deterministic coefficients). We will now give a more general result assuming that the nonhomogenous terms $\alpha(.)$ and $\sigma(.)$ in *(22)* are $\Re^d$ valued square integrable adapted processes (again with $m = 1$, for simplicity). Let

(Y, Z) be the solution to the following linear BSDE:

$$\begin{aligned} dY_t &= -[\bar{A}'y + \bar{C}'z + P\alpha + \bar{C}'P\sigma]dt + ZdW_t \\ Y_T &= 0, \end{aligned} \tag{89}$$

where

$$\begin{aligned} \bar{A} &= A - B\tilde{N}^{-1}\tilde{P}, \quad \bar{C} = C - D\tilde{N}^{-1}\tilde{P} \\ \tilde{P} &= B'P + D'PC, \;\; \tilde{N} = N + D'PD \end{aligned}$$

and $P(.)$ is a solution to *(85)* (if it exists). Then the following result can be shown by a square completion method as in Chen and Zhou (2000):

Theorem 33 *If $P(.)$ is a solution to (85), then the LQR problem (22) and (80) is solvable with an optimal control*

$$u_t^* = \tilde{N}^{-1}(\tilde{P}X_t + h(t))$$

where

$$h(t) = B'Y_t + D'Z + D'P\sigma$$

Moreover, the optimal value is

$$\begin{aligned} J^{u^*}(0, x_0) &= \int_0^T \{-h'\tilde{N}^{-1}h + \sigma'P\sigma + 2Y'\alpha + 2Z'\sigma\}dt \\ &\quad + x_0'P(0)x_0 + Y'(0)x_0. \end{aligned}$$

2.5.3 Random Parameters and Riccati BSDE's

When the parameters of the LQR problem *(22)* and *(80)* are bounded random functions which are adapted to $\mathcal{F}$, then the solutions to the LQR problem can be described in terms of the following matrix equation, called *backward Riccati SDE (BRSDE,* in short*)* or *Riccati BSDE*:

$$
\begin{aligned}
dP &= -\{PA + A'P + \sum_{j=1}^{m}(Z_j'C_j + C_j'Z_j + C_j'PC_j) + M \\
&\quad -[PB + \sum_{j=1}^{m}(C_j'P + Z_j)D_j]\tilde{N}^{-1}[PB + \sum_{j=1}^{m}(C_j'P + Z_j)D_j]'\}dt \qquad (90) \\
&\quad + \sum_{j=1}^{m} Z_j' dW_j(t) \\
\tilde{N} &= N + \sum_{j=1}^{m} D_j'PD_j > 0,\ P(T) = R \qquad (91)
\end{aligned}
$$

where the matrices M, N, R are symmetric. A solution $(P;Z_j, j = 1, ...m)$ to this BSDE is a set of matrix valued symmetric processes $P, Z_j \in L^2_{\mathcal{F}}([0,T]; \Re^{d\times d})$ such that the conditions *(91)* are satisfied.

Remark 34 *The Riccati BSDE is a highly nonlinear equation with a quadratic qrowth in Z and involves the reciprocal of the unknown P which makes it very hard to show the global existence of solutions. Analogous to the deterministic Riccati ODE's, some special cases can yield the uniqueness and existence of the solutions as in the following Lemma:*

Lemma 7 (Peng, 2000) *Consider the Riccati BSDE (90) with $D_j \equiv 0$, $\forall j = 1, ...m$, and assume that $M \geq 0, R \geq 0$ and $N > 0$. Then the equation (90) admits a unique solution $(P; Z_j, j = 1, ...m)$ such that $P \in L^\infty_{\mathcal{F}}([0,T]; \Re^{d\times d})$, $Z_j \in L^2_{\mathcal{F}}([0,T]; \Re^{d\times d})$, $j = 1, ..., m$ and $P \geq 0$.*

Now, for any given $(s, \eta) \in [0,T]\times\ L^2_{\mathcal{F}_s}(\Re^d)$, we consider the following SDE on $[s,T]$:

$$
\begin{aligned}
dX_t &= (A(t)X_t + B(t)u(t))dt + (\sum_{j=1}^{m} C_j(t)X_t + D_j(t)u(t))dW_j(t), \qquad (92) \\
X_s &= \eta
\end{aligned}
$$

where $A, C_j \in L^\infty_F([s,T];\Re^{d\times d})$ and $B, D_j \in L^\infty_F([s,T];\Re^{d\times m})$ for $j = 1, ..., m$. Define the cost functional $J^u(s,\eta)$ for $u(.) \in L^2_F([s,T];\Re^m)$ as

$$J^u(s,\eta) = E^{s,\eta}[\int_s^T (X_t' M(t) X_t + u_t' N(t) u_t) dt + X_T' R X_T],\ s \le T. \tag{93}$$

With the value function

$$V(s,\eta) = \min_{u \in L^2_F([s,T];\Re^m)} J^u(s,\eta), \tag{94}$$

we have the following result for the solvability of the stochastic LQR problem *(92)-(94)* via Riccati BSDE's:

Lemma 8 *Let $\tilde{N}$ be as in (91), $\eta \in L^2_{F_s}(\Re^d)$ and consider the control process u^* :*

$$u_t^* = -\tilde{N}^{-1}(PB + \sum_{j=1}^m C_j' P D_j + Z_j)' X_t \tag{95}$$

under which the forward SDE (92) becomes

$$\begin{aligned} dX_t &= [A - B\tilde{N}^{-1}(B'P + \sum_{j=1}^m D_j'(PC_j + Z_j))]X_t dt \\ &\quad + \sum_{j=1}^m [C_j - D_j(t)\tilde{N}^{-1}(B'P + \sum_{j=1}^m D_j'(PC_j + Z_j))]X_t dW_j(t), \\ X_s &= \eta. \end{aligned} \tag{96}$$

If the Riccati BSDE (90) has a solution $P \in L^\infty_F([0,T];\Re^{d\times d})$, $Z_j \in L^2_F([0,T];\Re^{d\times d})$, $j = 1, ..., m$ and the SDE has a solution $X(.) \in L^2_F([0,T];\Re^d)$, then the stochastic LQR problem (94) is solvable, (95) is the optimal feedback control and the minimal cost is

$$V(s,\eta) = J^{u^*}(s,\eta) = \eta' P(s) \eta \tag{97}$$

Proof. See Chen et. al (1998) for the deterministic η. The same argument is valid here since $\eta \in L^2_{F_s}(\Re^d)$, as also stated by Hu and Zhou (2003). ■

The one-dimensional Riccati BSDE's of the following form have unique bounded solutions and are used frequently in mean-variance hedging and portfolio selection problems (see the section 4.2)

$$
\begin{aligned}
dp &= -\{Ap + B'z - \frac{z'z}{p}\}dt + z'dW_t \\
p(T) &= M, \quad p(t) > 0, \forall\ t \in [0,T],
\end{aligned}
\tag{98}
$$

where $A \in L^\infty_{\mathcal{F}}(\Re), B \in L^\infty_{\mathcal{F}}(\Re^m), M \in L^\infty_{\mathcal{F}_T}(\Re)$. As shown in Lim and Zhou (2002), this BSDE is a monotonic transformation of the linear BSDE

$$
\begin{aligned}
dY &= \{AY - B'Z\}dt + Z'dW_t \\
Y(T) &= \frac{1}{M} \geq \delta > 0, \text{ a.s.}
\end{aligned}
\tag{99}
$$

for some finite constant δ. This linear BSDE has a unique bounded solution $(Y, Z) \in L^\infty_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$. Then a solution to the Riccati BSDE *(98)* is given by $(p, z) = (\frac{1}{Y}, \frac{-Z}{Y^2}) \in L^\infty_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$ which can be verified by Ito's rule. Moreover, this solution is unique since the function $f(x) = 1/x$ is monotonic when $x > 0$. So we have

Lemma 9 *The equation (98) has a unique solution* $(p, z) \in L^\infty_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$.

Chapter 4
Numerical Implementation

In this section, the discrete-time approximation and numerical solution of FB-SDE's are discussed by taking LQR problem as a benchmark. Recall that the value function for the LQR problem can be written as

$$\begin{aligned} V(s,x) &= x'S(s)x + K'(s)x + \int_s^T (\sum_{j=1}^m \sigma^{j'} S \sigma^j - \frac{1}{4} K' B N^{-1} B' K) dt \qquad (100) \\ V(T,x) &= x^{'} R x, \end{aligned}$$

where S, K and a solves the ODE's *(35)*-*(37)* with boundary conditions

$$S(T) = R, \;\; K(T) = 0, \;\; a(T) = 0.$$

In practice, one may only hope to get a numerical solution to these equations and hence to *(100)*. A numerical approximation of $V(s,x)$ which involves the time discretization to approximate the ODE's corresponding to S, K and a on $[0,T]$ is not very costly, even in high dimensions. See Figure 1 in Example *(38)*. However, in general, the solution of a parabolic PDE of the form *(29)*-*(30)* with a state dependent diffusion term doesn't reduce to a system of ODE's. The numerical solution of such an equation with finite difference methods requires the state discretization in each dimension which makes the procedure very costly in higher dimensions.

Now, we rewrite the FBSDE system *(45)* with $s = 0$, and $m = d$:

$$\begin{aligned} X_t &= x + \int_0^t \tilde{\sigma}(r, X_r) dW_r \qquad (101) \\ Y_t &= g(X_T) + \int_t^T f(r, X_r, Z_r) dr - \int_t^T Z_r' dW_r \end{aligned}$$

where

$$\begin{aligned} f(t,x,z) &= x'\tilde{M}\; x + x'\tilde{A}'(\tilde{\sigma}')^{-1} z - \frac{1}{4} z'(\tilde{\sigma}')^{-1} B N^{-1} B' (\tilde{\sigma}')^{-1} z \\ g(x) &= x' R x \qquad (102) \\ \tilde{\sigma}^j(t, X_t) &= C_j X_t + \sigma^j \end{aligned}$$

with $\tilde{\sigma}^j$ being the j^{th} column of $\tilde{\sigma}$.

In the next subsection, a discrete time approximation scheme for the FBSDE system *(101)-(102)* will be introduced as a numerical solution to this system. Then a Monte Carlo approach will be presented to estimate the conditional expectations (regression estimation) in the numerical scheme when no explicit representation can be obtained. However, it turns out that for the LQR problem, the discretized numerical solution of the corresponding FBSDE system reduces to a deterministic iterative procedure which doesn't require any regression estimation. The resulting deterministic sequence which is a backward Euler discretization is a numerical solution of S, K and a in (*100)* and converges to the value function (*100)* strongly for any $x \in \Re^d$. Therefore, for a linear state process X whose diffusion coefficient doesn't depend on the control variable u, the most efficient procedure for the LQR problem seems to be solving the system of deterministic equations *(35)*-*(37)* numerically. Moreover, this scheme is not much costly because only the time component is discretized.

We want to obtain error estimates for the approximation procedure (*105)* similar to the results given by Theorem 3.1 of Bouchard and Touzi (2004) or the ones given by Zhang (2001). We will also discuss the error due to the regression approximation based on Malliavin calculus with a three dimensional example. In general, in order to improve efficiency of regression approximation, one needs to find upper and lower bounds for $Y^{\pi}_{t_i}$ and $Z^{\pi}_{t_i}$ in terms of $X^{\pi}_{t_i}$. We mainly use the approximation $\hat{E}^{\pi}_i$ of E^{π}_i as in Bouchard and Touzi (2004) with slight adjustments. Since the functions $f(t,x,z)$ and $g(x)$ are neither Lipshitz nor have linear growth in state variables, the regression error is large and heavily depends on the *a priory* bounds for $Y^{\pi}_{t_i}$ and $Z^{\pi}_{t_i}$. In high dimensions, the estimation of Skorohod integrals becomes relatively harder. When the forward state process is Gaussian, we could evaluate d-iterated Skorohod integrals in terms of the discretized Brownian motion (see the subsection 3.2 and Appendix C).

3.1 Discrete Time Approximation of FBSDE

Let $T = 1$ be fixed. Then for a partition π: $0 = t_0 < ... < t_n = 1$ of $[0, 1]$, with uniform mesh size $|\pi| \triangleq \max_{1\le i\le n} \Delta t_i = 1/n$,the discretized approximation X^{π} of the forward process X in (*101)* is computed using classical Euler discretization and Monte-Carlo simulation:

$$\begin{aligned} X^{\pi}_{t_0} &= x \\ X^{\pi}_{t_i} &= X^{\pi}_{t_{i-1}} + \tilde{\sigma}(t_{i-1}, X^{\pi}_{t_{i-1}})\Delta^{\pi} W_i, \text{ for } i = 1, ...n, \end{aligned} \tag{103}$$

where $\Delta^\pi W_i = W_i - W_{i-1}$. Moreover, for $t_{i-1} \le t < t_i$, the continuous time approximation X_t^π of X is defined as

$$X_t^\pi \triangleq X_{t_{i-1}}^\pi + \tilde{\sigma}(t_{i-1}, X_{t_{i-1}}^\pi)(W_t - W_{i-1}).$$

Then it is easy to check that the following estimates hold for the processes $X_{t_i}^\pi$ and X_t^π:

$$\begin{aligned} E[\sup_{0\le t\le 1} |X_t^\pi|^4] &\le c(1+|x|^4) \\ E[\sup_{0\le t\le 1} |X_t - X_t^\pi|^2] &\le c(1+|x|^2)\,|\Delta t| \\ \max_{1\le i\le n} \sup_{t_{i-1}\le t<t_i} E\Big[\Big|X_t - X_{t_{i-1}}^\pi\Big|^2\Big] &\le c(1+|x|^2)\,|\Delta t| \end{aligned} \tag{104}$$

where $|\Delta t|$ would be replaced by $|\pi|$ for a non-uniform partition π. The error term for this scheme is well known:

$$\limsup_{|\pi|\to 0} |\pi|^{-1/2} E[\sup_{0\le t\le 1} |X_t - X_t^\pi|^p + \max_{1\le i\le n} \sup_{t_{i-1}\le t\le t_i} \Big|X_t - X_{t_{i-1}}^\pi\Big|^p] < \infty$$

for $p \ge 1$, see Kloeden and Platen (1992), for example.

For the discrete-time approximation of the processes (Y^π, Z^π), the following backward procedure is implemented:

$$\begin{aligned} Y_{t_n}^\pi &= g(X_{t_n}^\pi), \\ Z_{t_{i-1}}^\pi &= (\Delta t)^{-1} E[Y_{t_i}^\pi \Delta W_i | \mathcal{F}_{t_{i-1}}] \\ Y_{t_{i-1}}^\pi &= E[Y_{t_i}^\pi | \mathcal{F}_{t_{i-1}}] + f(t_{i-1}, X_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi).\Delta t \end{aligned} \tag{105}$$

where $\Delta W_i = W_{t_i} - W_{t_{i-1}}$, $\{\mathcal{F}_t\}$ is the (completed) filtration generated by W (in other words, generated by X) and f is given by *(102)*. This backward scheme will be relied on the Markov structure of the FBSDE system and the fact that (Y_t, Z_t) is a deterministic function of (t, X_t), $t \le 1$. Then the conditional expectations above reduce to the regression of $Y_{t_i}^\pi$ and $Y_{t_i}^\pi \Delta W_i$ on $X_{t_{i-1}}^\pi$, which in general requires further approximations. One may use the Malliavin calculus approach first introduced in Fournie et al. (2001) and then developed by Zhang (2001), Bouchard et al. (2004) and Bouchard and Touzi (2004). For other approaches (e.g., kernel estimators, quantization, basis projection methods) and references, see Gobet et.al (2004) and Bouchard and Touzi (2004).

Notice that the filtrations for the conditional expectations above can be replaced by the discrete-time filtration $\mathcal{F}_i^\pi$ which is the σ-algebra generated by $\{X_{t_j}^\pi, j \le i\}$. Then, $E_{i-1}^\pi[Y_{t_i}^\pi] = E[Y_{t_i}^\pi | X_{t_{i-1}}^\pi]$ and $E_{i-1}^\pi[Y_{t_i}^\pi \Delta^\pi W_i] = E[Y_{t_i}^\pi \Delta^\pi W_i | X_{t_{i-1}}^\pi]$ with

$E^{\pi}_{i-1}[.] \triangleq E[Y^{\pi}_{t_i}|\mathcal{F}^{\pi}_{i-1}]$ for any $i = 1, ..., N$. Conditional on $\mathcal{F}^{\pi}_{i-1}$, $E^{\pi}_{i-1}[X^{\pi}_{t_i}]$ is normally distributed with mean $X^{\pi}_{t_{i-1}}$ and variance

$$\left\|\tilde{\sigma}(t_{i-1}, X^{\pi}_{t_{i-1}})\right\|^2 /n$$

for each $i = 1, ..., n$. If $Y^{\pi}_{t_i} \in L^2_{\mathcal{F}}$, $\forall\ i = 1, ..., n$ (which indeed holds by Corollary 37), then we can define a continuous time approximation $(Y^{\pi}_t, Z^{\pi}_t) \in L^2_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$ as follows: For all $i = 1, ..., n$, by the martingale representation theorem, there exists a process $Z^{\pi} \in L^2_{\mathcal{F}}(\Re^m)$ such that

$$Y^{\pi}_{t_i} = E^{\pi}_{i-1}[Y^{\pi}_{t_i}] + \int_{t_{i-1}}^{t_i} Z^{\pi}_t dW_t. \tag{106}$$

Then, following Bouchard and Touzi (2004), we define Y^{π}_t

$$Y^{\pi}_t \triangleq Y^{\pi}_{t_i} - (t - t_i) f(t_i, X^{\pi}_{t_i}, Z^{\pi}_{t_i}) + \int_{t_i}^{t} Z^{\pi}_s dW_s,\ \ t_i \leq t < t_{i+1} \tag{107}$$

and an auxiliary process $\tilde{Z}^{\pi}_{t_i}$:

$$\tilde{Z}^{\pi}_{t_{i-1}} \triangleq \frac{1}{\Delta t_i} E_{i-1}[\int_{t_{i-1}}^{t_i} Z_t dt], 1 \leq i \leq n. \tag{108}$$

A similar approximation is introduced in Zhang (2001) by using step processes. See also Zhang (2004).

By *(106)* and the Ito isometry, $Z^{\pi}_{t_i}$ satisfies

$$\begin{aligned} Z^{\pi}_{t_{i-1}} \Delta t_i &= E_{i-1}[Y^{\pi}_{t_i} \Delta^{\pi} W_i] = E_{i-1}[\Delta^{\pi} W_i \int_{t_{i-1}}^{t_i} Z_t dW_t] \\ &= E_{i-1}[\int_{t_{i-1}}^{t_i} 1 dW_t \int_{t_{i-1}}^{t_i} Z'_t dW_t] \\ &= E_{i-1}[\int_{t_{i-1}}^{t_i} Z_t dt] \end{aligned} \tag{109}$$

The following estimate is obtained in Zhang (2001) under a uniform Lipshitz condition on both the driver f and the terminal condition g which is not satisfied in our case:

Lemma 10 (Zhang, 2001) *Assume that the functions $f(t,x,z)$ and $\tilde{\sigma}(t,x)$ in a FBSDE system of the form (101) are uniformly Lipshitz in state variables (x,z) and in x, respectively and Hölder-$\frac{1}{2}$ continuous in time variable t. Assume also that $g(x)$ is uniformly (L^∞) Lipshitz. Then, for any partition π of $[0,1]$, the following estimate holds:*

$$\max_{1\leq i\leq n} \sup_{t_{i-1}\leq t<t_i} E[\left|X_t - X_{t_{i-1}}\right|^2 + \left|Y_t - Y_{t_{i-1}}\right|^2] + \sum_{i=1}^{n} E[\int_{t_{i-1}}^{t_i} \left|Z_t - \tilde{Z}^{\pi}_{t_{i-1}}\right|^2 dt \leq \rho\left|\pi\right|, \tag{110}$$

where ρ is independent of π.

Remark 35 *For more general functions f and g, it is hard to get an L^2-type regularity result (110). For example, the functions $f(t,x,z)$ and $g(x)$ given by (102) have quadratic growth in (x,z), and in x, respectively. The maximal solution $(\bar{Y},\bar{Z})$ satisfies*

$$\bar{Z}_t = 2\tilde{\sigma}'(t,X_t)S(t)X_t + K(t)$$

which has quadratic growth in X_t, too. Because of the Markovian feature of the solution $(\bar{Y},\bar{Z})$, the driver f then has a quartic (fourth order) dependence on X_t, which usually complicates the analysis of the regularity properties of the numerical approximations. In order to eliminate such technical problems, we will again consider the standard case as we did in Lemma (6).

Lemma 11 *Suppose that $C_j \equiv 0$, $j=1,...m$, and $L\equiv 0$. Consider the Gaussian process $X_t = x + \int_0^t \sigma(r)dW_r, 0<t\leq T$, where $\sigma(.)\in\Re^{d\times m}$ is uniformly Lipshitz continuous on $[0,1]$: $\exists\ c>0$ such that $\|\sigma(t)-\sigma(s)\|\leq c\left|t-s\right|$, $\forall\ s,t\in[0,1]$. Let (Y,Z) be the maximal solution*

$$\begin{aligned} Y_t &= V(t,X_t) = X_t'S(t)X_t + a(t), \\ Z_t &= \sigma(t)'V_x(t,X_t) = 2\sigma(t)'S(t)X_t \end{aligned}$$

to the LQR problem in this standard case. Then,

(a) The triple (X,Y,Z) satisfy the estimate (110).

(b) Moreover, the following regularity result

$$\max_{1\leq i\leq n}\sup_{t_{i-1}\leq t<t_i}\{E[\left|X_t-X_{t_{i-1}}\right|^2+\left|Y_t-Y_{t_{i-1}}\right|^2]+\sum_{i=1}^{n}E[\int\limits_{t_{i-1}}^{t_i}\left|Z_t-Z_{t_{i-1}}\right|^2dt\}\leq\rho\left|\pi\right| \tag{111}$$

holds.

Proof. The part (b) follows from Lemma *(6)* with $\rho=k+1$ where the constant k is as in *(50)*.

To prove part (a), it suffices to show that

$$\sum_{i=1}^{n}E[\int\limits_{t_{i-1}}^{t_i}\left|Z_t-\tilde{Z}^{\pi}_{t_{i-1}}\right|^2dt\}\leq C\left|\pi\right|,$$

for some $C>0$. Note that

$$\begin{aligned}\tilde{Z}^{\pi}_{t_{i-1}} &= \frac{1}{\Delta t_i}E_{i-1}[\int\limits_{t_{i-1}}^{t_i}Z_tdt]\\ &= 2\frac{X_{t_{i-1}}}{\Delta t_i}\int\limits_{t_{i-1}}^{t_i}\sigma'S(s)ds, 1\leq i\leq n,\end{aligned}$$

and for any $t\in[t_{i-1},t_t)$,

$$Z_t-\tilde{Z}^{\pi}_{t_{i-1}}=2\sigma'S(t)(X_t-X_{t_{i-1}})+2(\sigma'S(t)-\frac{1}{\Delta t_i}\int\limits_{t_{i-1}}^{t_i}\sigma'S(s)ds)X_{t_{i-1}}$$

so that

$$\begin{aligned}E\left|Z_t-\tilde{Z}^{\pi}_{t_{i-1}}\right|^2 &\leq 8\{dE\left|X_t-X_{t_{i-1}}\right|^2+c^2dE\left|X_t-X_{t_{i-1}}\right|^2\}\\ &\leq 8\{d^2\Delta t_i+c^2d(1+\left|x\right|^2)(\Delta t_i)^2\}\\ &\leq C\left|\pi\right|\end{aligned}$$

with $C=8d(d+c^2(1+\left|x\right|^2))$ and c,d as in the proof of Lemma *(6)*. ■

Before going into the details of the error estimates, we show that the discrete-time approximation (*105)* reduces to the numerical solution of the value function through Riccati equation *(35)*.

Proposition 36 *The procedure given by (105) results in the following iteration for Y^π and Z^π:*

$$
\begin{aligned}
Z^\pi_{t_i} &= \eta_{t_i} X^\pi_{t_i} + \xi_{t_i} \\
Y^\pi_{t_i} &= X^{\pi\prime}_{t_i} \lambda_{t_i} X^\pi_{t_i} + \phi'_{t_i} X^\pi_{t_i} + \gamma_{t_i}
\end{aligned} \tag{112}
$$

where the sequence of coefficients $\eta_{t_i}, \lambda_{t_i} \in \Re^{d\times d}$, $\phi_{t_i}, \xi_{t_i} \in \Re^{d\times 1}$ and $\gamma_{t_i} \in \Re$ satisfy

$$
\begin{aligned}
\lambda_{t_n} &= R,\ \eta_{t_n} = 0, \phi_{t_n} = 0, \xi_{t_n} = 0 = \xi_{t_{n-1}}, \gamma_{t_n} = 0 \\
\lambda_{t_i} &= \frac{1}{n}\{\hat{M}_{t_i} + \hat{A}'_{t_i}\lambda_{t_{i+1}} + \lambda_{t_{i+1}}\hat{A}_{t_i} - \lambda'_{t_{i+1}} B_{t_i} N^{-1}_{t_i} B'_{t_i}\lambda_{t_{i+1}} \\
&\quad + \sum_{j=1}^m C'_j(t_i)\lambda_{t_{i+1}} C_j(t_i)\} + \lambda_{t_{i+1}} \\
\phi_{t_i} &= \phi_{t_{i+1}} + \frac{1}{n}\{2\sum_{j=1}^m C'_j(t_i)\lambda_{t_{i+1}}\sigma^j(t_i) + (\hat{A}_{t_i} - \lambda'_{t_{i+1}} B_{t_i} N^{-1}_{t_i} B'_{t_i})\phi_{t_{i+1}}\} \\
\gamma_{t_i} &= tr(\sigma_{t_i}\sigma'_{t_i}\lambda_{t_{i+1}})/n + \gamma_{t_{i+1}} \\
\eta_{t_i} &= 2\tilde{\sigma}'_{t_i}\lambda_{t_{i+1}};\ \xi_{t_i} = \tilde{\sigma}'_{t_i}\phi_{t_{i+1}}
\end{aligned} \tag{113}
$$

Proof. *Backward induction on i.* The case $i = n$ is clear with the convention $Z^\pi_{t_n} = 0$. For $i = n-1$, the conditional expectations $E^\pi_{i-1}[Y^\pi_{t_i}]$ and $E^\pi_{i-1}[Y^\pi_{t_i}\Delta^\pi W_i]$ could be computed explicitly: By writing

$$X^\pi_{t_n} = X^\pi_{t_{n-1}} + \tilde{\sigma}_{t_{n-1}}\Delta^\pi W_n,$$

with

$$
\begin{aligned}
\tilde{\sigma}_{t_{n-1}} &= \tilde{\sigma}(t_{n-1}, X^\pi_{t_{n-1}}), \\
\sigma_{t_{n-1}} &= \sigma(t_{n-1})
\end{aligned}
$$

and using *(79)*, the conditional expectations $E^\pi_{n-1}[Y^\pi_{t_n}]$, $E^\pi_{n-1}[Y^\pi_{t_n}\Delta^\pi W_n]$ can be computed as follows:

$$
\begin{aligned}
E^\pi_{n-1}[Y^\pi_{t_n}] &= E^\pi_{n-1}[X^{\pi'}_{t_n} R X^{\pi'}_{t_n}] \\
&= X^{\pi'}_{t_{n-1}} R X^\pi_{t_{n-1}} + tr(\tilde{\sigma}_{t_{n-1}}\tilde{\sigma}'_{t_{n-1}} R)/n
\end{aligned}
$$

and

$$E^\pi_{n-1}[Y^\pi_{t_n}\Delta^\pi W_n] = E^\pi_{n-1}[X^{\pi'}_{t_n} R X^\pi_{t_n}\Delta^\pi W_n] = 2\tilde{\sigma}'_{t_{n-1}} R X^\pi_{t_{n-1}}/n$$

so that

$$Z_{t_{n-1}} = nE^{\pi}_{i-1}[Y^{\pi}_{t_i}\Delta^{\pi}W_i] = 2\tilde{\sigma}'_{t_{n-1}}RX^{\pi}_{t_{n-1}} = \eta_{t_{n-1}}X^{\pi}_{t_{n-1}} + \xi_{t_{n-1}}$$

with

$$\eta_{t_{n-1}} = 2\tilde{\sigma}'_{t_{n-1}}R, \quad \xi_{t_{n-1}} = 0$$

and hence

$$\begin{aligned}
Y_{t_{n-1}} &= \frac{1}{n}f(n-1, X^{\pi}_{t_{n-1}}, Z^{\pi}_{t_{n-1}}) + E^{\pi}_{n-1}[X^{\pi'}_{t_n}RX^{\pi'}_{t_n}] \\
&= \frac{1}{n}\{X^{\pi'}_{t_{n-1}}\hat{M}_{t_{n-1}}X^{\pi}_{t_{n-1}} + X^{\pi'}_{t_{n-1}}\hat{A}'_{t_{n-1}}(\tilde{\sigma}'_{t_{n-1}})^{-1}2\tilde{\sigma}'_{t_{n-1}}RX^{\pi}_{t_{n-1}} \\
&\quad -(2\tilde{\sigma}'_{t_{n-1}}RX^{\pi}_{t_{n-1}})'(\frac{1}{4}\tilde{\sigma}^{-1}_{t_{n-1}}B_{t_{n-1}}N^{-1}_{t_{n-1}}B'_{t_{n-1}}(\tilde{\sigma}'_{t_{n-1}})^{-1})(2\tilde{\sigma}'_{t_{n-1}}RX^{\pi}_{t_{n-1}})\} \\
&\quad +X^{\pi'}_{t_{n-1}}RX^{\pi}_{t_{n-1}} + tr(\tilde{\sigma}'_{t_{n-1}}\tilde{\sigma}_{t_{n-1}}R)/n \\
&= X^{\pi'}_{t_{n-1}}\{\frac{1}{n}(\hat{M}_{t_{n-1}} + \hat{A}'_{t_{n-1}}R + R\hat{A}_{t_{n-1}} - RB_{t_{n-1}}N^{-1}_{t_{n-1}}B'_{t_{n-1}}R \\
&\quad +\sum_{j=1}^{m}C'_jRC_j(t_{n-1})) + R\}X^{\pi}_{t_{n-1}} \\
&\quad +\frac{2}{n}(\sum_{j=1}^{m}C'_j(t_{n-1})R\sigma^j_{t_{n-1}})'X^{\pi}_{t_{n-1}} + tr(\sigma_{t_{n-1}}\sigma'_{t_{n-1}}R)/n \\
&= X^{\pi'}_{t_{n-1}}\lambda_{n-1}X^{\pi}_{t_{n-1}} + \phi'_{t_{n-1}}X^{\pi}_{t_{n-1}} + \gamma_{t_{n-1}}
\end{aligned}$$

Now, assume that (Y_{t_i}, Z_{t_i}) together with the coefficients $\eta_{t_i}, \lambda_{t_i} \in \Re^{d\times d}$, $\phi_{t_i}, \xi_{t_i} \in \Re^{d\times 1}$ and γ_{t_i} are given by *(112)-(113)*. Then, writing $X^{\pi}_{t_i}$ in the form

$$X^{\pi}_{t_i} = X^{\pi}_{t_{i-1}} + \tilde{\sigma}_{t_{i-1}}\Delta^{\pi}W_i$$

and following similar arguments as in the computations for $t = t_{n-1}$ above, we get the

$$\begin{aligned}
Z_{t_{i-1}} &= n(E^{\pi}_{i-1}[(X^{\pi\prime}_{t_i}\lambda_{t_i}X^{\pi}_{t_i})\Delta^{\pi}W_i] + E^{\pi}_{i-1}[\phi'_{t_{n-1}}X^{\pi}_{t_{n-1}}\Delta^{\pi}W_i] + E^{\pi}_{i-1}[\gamma_{t_i}\Delta^{\pi}W_i]) \\
&= n(2\frac{1}{n}\tilde{\sigma}'_{t_{i-1}}\lambda_{t_i}X^{\pi}_{t_{i-1}} + \tilde{\sigma}'_{t_{i-1}}\phi_{t_i}/n) \\
&= 2\tilde{\sigma}'_{t_{i-1}}\lambda_{t_i}X^{\pi}_{t_{i-1}} + \tilde{\sigma}'_{t_{i-1}}\phi_{t_i} = \eta_{t_{i-1}}X^{\pi}_{t_{i-1}} + \xi_{X^{\pi}_{t_{i-1}}}
\end{aligned}$$

and similarly for $Y_{t_{i-1}}$:

$$\begin{aligned}
Y_{t_{i-1}} &= \frac{1}{n}f(i-1, X^{\pi}_{t_{i-1}}, Z^{\pi}_{t_{i-1}}) + E^{\pi}_{i-1}[X^{\pi'}_{t_i}\lambda_{t_i}X^{\pi'}_{t_i} + \phi'_{t_{n-1}}X^{\pi}_{t_{n-1}} + \gamma_{t_i}] \\
&= X^{\pi'}_{t_{i-1}}\{\frac{1}{n}(\tilde{M}_{t_{i-1}} + \tilde{A}'_{t_{i-1}}\lambda_{t_i} + \lambda_{t_i}\tilde{A}_{t_{i-1}} - \lambda_{t_i}B_{t_{n-1}}N^{-1}_{t_{n-1}}B'_{t_{n-1}}\lambda_{t_i} \\
&\quad + \sum_{j=1}^{m} C'_j(t_{i-1})\lambda_{t_i}C_j(t_{i-1})) + \lambda_{t_i}\}X^{\pi}_{t_{i-1}} \\
&\quad + \frac{2}{n}(\sum_{j=1}^{m} C'_j(t_{i-1})\lambda_{t_i}\sigma^{j}_{t_{i-1}})'X^{\pi}_{t_{n-1}} + tr(\sigma_{t_{i-1}}\sigma'_{t_{i-1}}\lambda_{t_i})/n + \gamma_{t_i} \\
&= X^{\pi'}_{t_{i-1}}\lambda_{t_{i-1}}X^{\pi}_{t_{i-1}} + \phi'_{t_{i-1}}X^{\pi}_{t_{i-1}} + \gamma_{t_{i-1}},
\end{aligned}$$

with the coefficients

$$\begin{aligned}
\lambda_{t_{i-1}} &= \frac{1}{n}(\tilde{M}_{t_{i-1}} + \tilde{A}'_{t_{i-1}}\lambda_{t_i} + \lambda_{t_i}\tilde{A}_{t_{i-1}} - \lambda_{t_i}B_{t_{i-1}}N^{-1}_{t_{i-1}}B'_{t_{i-1}}\lambda_{t_i} \\
&\quad + \sum_{j=1}^{m} C'_j(t_{i-1})\lambda_{t_i}C_j(t_{i-1})) + \lambda_{t_i} \\
\phi_{t_{i-1}} &= \phi_{t_i} + \frac{1}{n}\{2\sum_{j=1}^{m} C'_j(t_{i-1})\lambda_{t_i}\sigma^{j}_{t_{i-1}} \\
&\quad + (\tilde{A}_{t_{i-1}} - \lambda'_{t_i}B_{t_{i-1}}N^{-1}_{t_{i-1}}B'_{t_{i-1}})\phi_{t_i}\}
\end{aligned}$$

and

$$\gamma_{t_{i-1}} = tr(\sigma_{t_{i-1}}\sigma'_{t_{i-1}}\lambda_{t_i})/n + \gamma_{t_i}.$$

■

Corollary 37 *Let* $(Y^{\pi}_{t_i}, Z^{\pi}_{t_i})_{i=0,\ldots n}$ *be the discrete time approximation given by procedure (105). Then*

$$Y^{\pi}_{t_0} = x'\lambda_{t_0}x + \phi'_{t_0}x + \gamma_{t_0}, \tag{114}$$

where the deterministic coefficients $\lambda_{t_0}, \phi_{t_0}$ *and* γ_{t_0} *can be computed recursively from (113). Moreover,* $Y^{\pi}_{t_0}$ *converges to the value function (42) of the LQR problem, for any* $x \in \Re^d$*:* $\|\lambda_{t_0} - S(0)\| \to 0$*,* $\left|\phi_{t_0} - K(0)\right| \to 0$*, and* $\left|\gamma_{t_0} - a(0)\right| \to 0$ *as* $n \to \infty$*.*

Proof. It's easy to see that the scheme given by *(112)-(113)* is a backward discretization of *(35)-(37)* where some of the time-dependent coefficients are evaluated at lower bounds of the integration. Because of the continuity of these matrix-valued functions, the result follows. ■

Now, we provide a three dimensional example with $L = 0$ and $C_j = 0$, for all $j = 1, ...m$, which corresponds to the standard Riccati equation (38):

$$\dot{S} + M - SBN^{-1}B'S + A'S + SA = 0$$

Example 38 *Consider the following matrix valued functions for the standard LQR problem on* $[0,1]$ *with* $x = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}'$*:*

$$A(t) = -\begin{bmatrix} 1/2 & t/2 & 1/2 \\ 0 & 1/2 & -1/2 \\ 1 & 1-\frac{t}{2} & 1/3 \end{bmatrix} /5$$

$$B(t) = \begin{bmatrix} t+\frac{2}{3} & t/2 & 0 \\ 1/3 & t+\frac{2}{3} & 1 \\ 0 & 1/2 & 1+t \end{bmatrix} /6$$

$$\sigma(t) = \begin{bmatrix} 1+t/2 & 0 & 0 \\ .9 & 1.2 & 0 \\ 0 & .5 & 1+\frac{t}{4} \end{bmatrix} /4$$

$$M(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2-t & 0 \\ 0 & 0 & 1+t^2/2 \end{bmatrix} /3$$

$$N(t) = \begin{bmatrix} 1.5 & 1 & 0 \\ 1 & 2.5 & 1 \\ 0 & 1 & 1.5 \end{bmatrix} /4$$

Let $S(1) = R = I_{3\times 3}$*. Then the value function is*

$$V(0,x) = x\prime S(0)x + a(0)$$

where S *and* a *solve the ODE system*

$$\begin{aligned} \dot{S} + M - SBN^{-1}B'S + A'S + SA &= 0 \\ \dot{a} + tr(\sigma\sigma' R) &= 0 \end{aligned}$$

with $S(1) = I_{3\times 3}$ *and* $a(1) = 0$*. For any partition* π*, the approximating sequence (114) is now given by*

$$Y^{\pi}_{t_0} = x'\lambda_{t_0}x + \gamma_{t_0}, \tag{115}$$

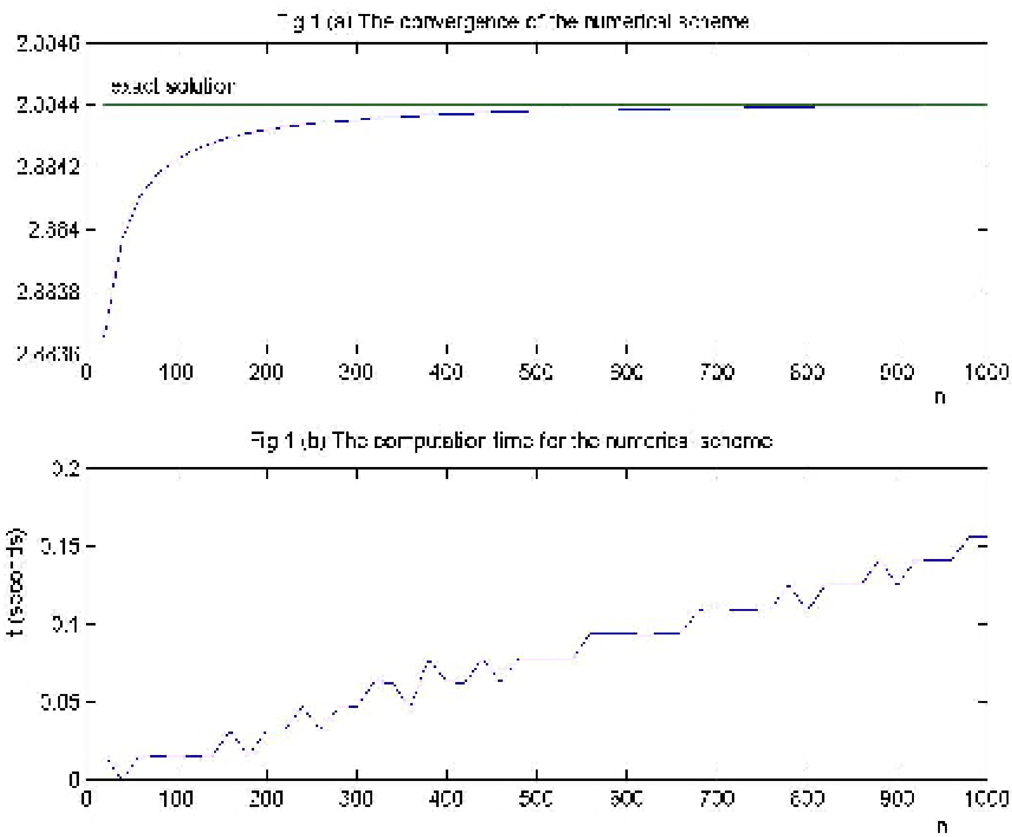
where λ_{t_0} *and* γ_{t_0} *can be computed recursively through (113). Now, using this algorithm with* $n = 100000$ *on a uniform partition of* $[0,1]$ *results in the following numerical solution for the system*

$$S(0) \sim \lambda_{t_0} = \begin{bmatrix} 1.0577 & -0.0091 & -0.2710 \\ 0.0394 & 1.2055 & -0.1174 \\ -0.2710 & -0.1174 & 1.0755 \end{bmatrix}$$

and $a(0) \sim \gamma_{t_0} = 0.3407$ *so that*

$$V(0, x) = x'S(0)x + a(0) = 2.8844.$$

We consider this result as exact when comparing with other computations. Although the numerical scheme (which is some type of Euler discretization) for this example is very fast, the convergence is rather slow since the accuracy of $a(0)$ *also depends on the mesh size of the partition* π *and usually reqiures a large number of discretization points. See the Figure 1 (a)-(b) below:*



Convergence of Riccati equation

Remark 39 *The Corollary 37 and the Example 38 above discuss only the con-*

vergence of $Y^{\pi}_{t_0}$ to the value function $V(0,x) = Y^{0,x}_0$ of the associated LQR problem. In general, the L^2 type convergence of the continuos time process (Y^{π}_t, Z^{π}_t) to (Y_t,Z_t)=$(\bar{Y}_t,\bar{Z}_t)$ is not very easy to prove since, in our case, it may require strong regularity of Z^{π}_t which is only known to have L^2-regularity. The following error estimate is obtained both by Zhang (2001) and Bouchard and Touzi (2004) assuming the driver f is Lipshitz in state variables:

$$\sup_{0\leq t\leq 1} E[|Y_t - Y^{\pi}_t|^2] + E[\int_0^1 |Z_t - Z^{\pi}_t|^2\, dt\} \leq c\,|\pi|$$

One may obtain similar estimates in our setup by assuming $Z^{\pi} \in L^4_F(\Re^d)$, for example. Since $Z_t = 2\sigma'(t)S(t)X_t$ has the same regularity as X_t and $X \in L^4_F(\Re^d)$, this assumption is not too strong. Now, leaving this discussion aside, thanks to the fine properties of the solutions to the standard Riccati equation, we have the following error estimate as a corollary to Proposition 36:

Corollary 40 $\max\limits_{1\leq i\leq n} \sup\limits_{t_{i-1}\leq t\leq t_i} E[\left|Y_{t_i} - Y^{\pi}_{t_i}\right|^2] + \sum\limits_{i=1}^{n} E[\int\limits_{t_{i-1}}^{t_i} \left|Z_{t_i} - Z^{\pi}_{t_i}\right|^2 dt\} \leq c\,|\pi|.$

3.2 Regression Approximation

Thanks to Proposition 37, the discretized pair $(Y^{\pi}_{t_i}, Z^{\pi}_{t_i})$ given by *(112)* can be described as deterministic functions of $X^{\pi}_{t_i}$, $i = 1, ..., n$. By using this explicit representation and the estimated value $Y^{\pi}_{t_0} \sim V(0,x)$, we can now test the Malliavin calculus based regression approximation on LQR problem through the Example 38. Due to the involved technical steps and the notation, we are not going to explain the details of the algorithm. The reader can refer to the paper by Bouchard and Touzi (2004) for a deeper understanding of the method. We also put some of the computations and technical steps in the Appendix C to simplify the presentation in this section.

The approximation of the forward process $\{X^{\pi}_{t_i},\, i = 1, ..., n\}$ is based on Monte-Carlo simulation and the classical Euler discretization given by *(103).* The terminal condition $Y^{\pi}_{t_n} = g(X^{\pi}_{t_n})$ with $t_n = 1$ is computed from the simulated random variables $X^{\pi}_{t_n}$ for each $j = 1, ...N$, accordingly. Then the pair of processes $(Y^{\pi}_{t_i}, Z^{\pi}_{t_i})$ are computed by approximating the conditional expectation operator E^{π}_i in *(103)* by an approppriate Malliavin regression estimator $\tilde{E}^{\pi}_i$ as follows: If $\{X^{\pi^{(j)}},\, j = 1, ..., N\}$ are N independent copies of $\{X^{\pi}_{t_i},\, i = 1, ..., n\}$ and a smooth localizing function

$\phi : \Re^d \to \Re$ is given, then

$$\tilde{E}_i^\pi[\phi(X_{t_{i+1}}^\pi)] \triangleq \frac{\sum_{j=1}^{N} \phi(X_{t_{i+1}}^{\pi(j)}) H_{X_{t_i}^\pi}(X_{t_i}^{\pi(j)}) \xi^{(j)}}{\sum_{j=1}^{N} H_{X_{t_i}^\pi}(X_{t_i}^{\pi(j)}) \xi^{(j)}} \tag{116}$$

where

$$H_X(y) \triangleq \prod_{i=1}^{d} 1(X_i \leq y_i)$$

is the Heaviside unit function and, $\xi^{(j)}$ are independent copies of some random variable ξ which has some Skorohod integral representation of the form

$$S_i^{h_{i-1}}[F] \triangleq \int_0^\infty (F h_t^i)' dW_t, i = 1, ..., d \tag{117}$$

where F is a random variable (usually another Skorohod integral) and h^i is the i^{th} column of a matrix valued process h satisfying

$$\begin{aligned} \int_0^\infty D_t X_{t_i}^\pi h_t^i dt &= I_{d \times d} \\ \int_0^\infty D_t X_{t_{i+1}}^\pi h_t^i dt &= 0_{d \times d}. \end{aligned} \tag{118}$$

For the exact form of the matrix h_t and the computation of the Malliavin derivatives and the iterated Skorohod integrals corresponding to the Example *38*, see Appendix C.

Since this scheme requires n independent copies of N discrete-time process X on a partition π, it is computationally costly. Moreover, the estimator *(116)* is not necessarily integrable. In order to improve efficiency of the algorithm and to eliminate problems related to the possible non-integrability of the estimators, it is essential to use some *a priory bounds* on $Y_{t_i}^\pi$ and $Z_{t_i}^\pi$ during the implementation. Since the function $f(t, x, z)$ depends quadratically on z, any error in the estimation of the process Z^π is magnified by f. Then the accuracy in the computation of Y^π is affected by both regression estimation $\tilde{E}_{i-1}^\pi[Y_{t_i}^\pi]$ of $E_{i-1}^\pi[Y_{t_i}^\pi]$ and the error in the approximation $f(t_{i-1}, X_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi)$ which has quadratic growth in both state variables. For this reason, during the simulation, we first apply the bounds on the regression estimation $\tilde{E}_i^\pi[Y_{t_i}^\pi]$ if such bounds are available and then compute

$$Y_{t_{i-1}}^\pi = E[Y_{t_i}^\pi | \mathcal{F}_{t_{i-1}}] + f(t_{i-1}, X_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi).\Delta t_i.$$

The random variables ξ of the form *(117)* involves the evaluation of 2^d terms which are iterated Skorohod integral and might pose additional problems for high dimensions and finer partitions. However, the three dimensional case discussed in Example *38* reduces to the simulation of Brownian motion and to the computation of discretized values of h_t due to the choice of a constant localizing function and the *nice* conditional distribution of the state process X.

3.2.1 A Computational Example

We now apply this regression estimation to the LQR problem of Example *38* by first choosing $\phi(x) \equiv 1$ which reduces the number of terms in the Skorohod integral ξ that appear in the formula (*116)*.

Remark 41 *We implemented the procedure in Matlab and because of some sequential programming, the computation time is very large which could be reduced significantly by using C++ or Fortran, for example.*

Example 42 *We use the same model and setup as in Example 38. Noting that the diffusion matrix depends only on t, the deterministic function*

$$h_{i,t} = \frac{1}{|\pi|}(\sigma(t_{i-1})^{-1}1_{[t_{i-1},t_i)}(t) - \sigma(t_i)^{-1}1_{[t_i,t_{i+1})}(t))$$

satisfies the identities (118). By using the localizing function $\phi(x) \equiv 1$ in the estimation of both conditional expectations $E_{i-1}^\pi[Y_i^\pi]$ and $E_{i-1}^\pi[Y_i^\pi \triangle W_i]$, the term $\xi^h[\phi]$ reduces to the iterated Skorohod integrals $S_{\{1,2,3\}}^{h_{i-1}}[1]$ and $S_{\{1,2,3\}}^{h_{i-1}}[\triangle W_i^u]$ which can be computed in terms of the single Skorohod integrals

$$S_k^{h_{i-1}}[1] = (h_{i-1,t_{i-2}}^k \cdot \triangle W_{i-1} + h_{i-1,t_{i-1}}^k \cdot \triangle W_i)$$

and

$$S_k^{h_{i-1}}[\triangle W_i^u] = (\triangle W_i^u)(h_{i-1,t_{i-2}}^k \cdot \triangle W_{i-1} + h_{i-1,t_{i-1}}^k \cdot \triangle W_i) - h_{i-1,t_{i-1}}(u,k)/n \quad (119)$$

where W_i^u denotes the u^{th} component of W_i^u, for $1 \leq u, k \leq d$. For the details, see the Appendix C.
The following simulation results are obtained from one application of the regression estimation where $N \sim n^{3.5}$ paths are simulated at each step corresponding to the number of intervals $n = 8, 10$ and 12:

	Exact ($n = 10^5$)	$n = 8$ $N = 1450$	$n = 10$ $N = 3150$	$n = 12$ $N = 6000$
Y_0^π	2.8844	3.1324	2.5478	3.0893
computation time	14.7	522	3114	14767
absolute error	0	.2480	.3075	.2049

Table 1. Computations for One Set of Simulations

For the significance of $N \sim n^{3.5}$ *as an optimal number of simulations, one may refer to Bouchard and Touzi (2004). Since the computation results are subject to change in each implementation, we repeated the scheme for each* N *and computed the empirical mean, standard deviation and computation time (in units of seconds) to have a more objective idea about the performance of the method. We fixed* $n = 10$ *(recall that* $10^{3.5} \sim 3162$*) and had* 20 *sets of results for* $N = 2000$, $N = 3000$ *and* $N = 4000$ *and* 12 *sets for* $N = 5000$, *(we also tried* $N > 5000$ *but observed no significant improvement with increased computational cost). The following table compares the simulation results for various* N *where "emp." and comp" are abbreviations for "empirical" and "computation", respectively:*

$n = 10$	*Exact*	$N = 2000$	$N = 3000$	$N = 4000$	$N = 5000$
emp. mean	2.8844	2.7518	2.9140	3.0308	3.0051
emp. std.	0	0.7010	0.7202	0.7268	0.7752
comp. time	14.8	1220.1	2707.3	4727.8	7727.9
abs. error	0	.1296	.0326	.1494	.1237

Table 2. Results for Repeated Simulations

The FBSDE approximation scheme by (pure) simulation using the regression estimation doesn't seem to be performing well for this problem. Moreover, increasing the number of simulated paths are not necessarily helping to get better estimates with a reduced variance. One possible reason would be that the number of data sets for large N *(10 sets) is less than that of small* N *(20 sets). However, comparing* $N = 2000$, $N = 3000$ *and* $N = 4000$, *we see that the case* $N = 2000$ *(although slightly) performs better with the same (20 sets) number of data. Another reason for this would be the poor estimation of Skorohod integrals when* N *is large. One significant observation is that the minimum error is obtained when* $N = 3000$ *which is closer to* $10^{3.5} \sim 3162$. *Since the computation time increases quadratically with*

N, it is not desirable to use large values of N. To address this question empirically, we will mention briefly the applicability of the variance reduction techniques.

3.2.2 Variance Reduction

The error due to the regression estimation could be reduced by using some variance reduction techniques. Introducing a *control variate* method would require some additional computations to estimate the optimal estimator by simulation. As reported by Bouchard et. al (2004), the gain from this method is not significant (even in one dimensional case). We can also consider the separable localizing functions of the form $\phi(x) = e^{\eta' x}$, for $x \in \Re^d$. Bouchard et al. (2004) characterized such a function as the unique integrated mean-square-error minimizer among the class of separable localizing functions. Although the nonlinear system of equations that they provided could be solved by an iterative method, it involves expected values of some random variables which could be quite complicated and would usually require extra simulation to be estimated. In order to control the convergence rate when n is large (the case of finer discretization), it is also essential to normalize $\phi(x)$ as $\phi_n(x) = \phi(\sqrt{n}x)$.

By choosing a localizing function of the form $\phi(x) = e^{\eta' x}$ for the estimation of $E^{\pi}_{i-1}[Y^{\pi}_i]$ and $\phi(x) = 1$ for $E^{\pi}_{i-1}[Y^{\pi}_i \bigtriangleup W_i]$, we apply a partial variance reduction. The reason is that the Skorohod integrals in the computation of $E^{\pi}_{i-1}[Y^{\pi}_i]$ have a simpler form than that of $E^{\pi}_{i-1}[Y^{\pi}_i \bigtriangleup W_i]$, and it is less costly from computational point of view. The following table compares the results of the Table 2 with the results using this partial variance reduction for $N = 2000$ and 3000.

$n = 10$	From	Table 2	Variance	Reduction
(20 sets)	N=2000	N=3000	N=2000	N=3000
emp. mean	2.7518	2.9140	2.7258	2.7908
empirical std.	0.7010	0.7202	0.6795	0.6723
comp. time	1220.1	2707.3	1227.0	2741.9
abs. error	. 1296	.0326	.1556	.0 906

Table 3. Partial Variance Reduction

We notice that a localizing function of the form $\phi(x) = e^{\eta' x}$ helps in reducing the empirical standard deviation. However, it doesn't contribute significantly to minimize the absolute mean error. The increase in the computation time due to using a localizing function is negligible (around 1%). Although $n = 10$ doesn't correspond to a small mesh size, we also tried the localizing function $\phi_n(x) = \phi(\sqrt{n}x)$ for the computation of $E^{\pi}_{i-1}[Y^{\pi}_i]$. We haven't observed any significant difference between $\phi(x)$ and $\phi(\sqrt{n}x)$. A full variance reduction includes estimating also the term $E^{\pi}_{i-1}[Y^{\pi}_i \triangle W_i]$ with separable localizing functions of the exponential form and might perform better by reducing both the empirial standard deviation and the mean absolute error.

Chapter 4
Applications to Economics and Finance

4.1 An Application to Economics

The following classical-impulse control application is based on the paper by Cadenillas and Zapatero (2000) where they solve the problem of a Central Bank that wants to control the exchange rates by using both interest rates and reserves. We first summarize their results on the infinite time domain and then discuss a modified terminal value problem which has an FBSDE representation.

4.1.1 The Case of Infinite Time Domain

Let r and $\bar{r}$ denote domestic and the target interest rates, respectively and define the dynamics for the controlled exchange rate (the domestic currency units per unit of foreign currency) process as

$$X_t = \int_0^t (\mu X_s + K u_s) ds + \int_0^t \sigma X_s dW_s + \sum_{i=1}^{\infty} 1(\tau_i < t)\xi_i$$

where $u(t) = \log \frac{r(t)}{\bar{r}}$; the parameters $\mu \in \Re$, $K \leq 0$, $\sigma > 0$ are constant; τ_i and ξ_i are the time and the intensity of the i^{th} intervention, respectively. For the economical significance and a detailed interpretation of these parameters, one can refer to Cadenillas and Zapatero (2000) and the references there.

Then the problem is to choose a triple (u, τ, ξ) where $u \in L^2_{\mathcal{F}}([0, \infty) \times \Re)$ is a classical control and (τ, ξ) is an impulse control with $P\{\forall t \geq 0 : X(t) \geq 0\} = 1$ to minimize the following cost functional J:

$$J(x; u, \tau, \xi) = E[\int_0^{\infty} e^{-\lambda t} f(X_t, u_t) dt + \sum_{i=1}^{\infty} e^{-\lambda \tau_i} 1(\tau_i < \infty) g(\xi_i)]$$

where

$$f(x, u) = (x - \rho)^2 + k u^2, \tag{120}$$

$$g(z) = \left\{ \begin{array}{cc} C + cz, & z > 0 \\ \min(C, D) & z = 0 \\ D - dz & z < 0 \end{array} \right\}, \tag{121}$$

with the nonnegative constants $\lambda, \rho, C, c, D, d$ and $k \in [0, \infty)$.

The value function is defined for all $x > 0$ as follows

$$V \triangleq \inf J(x; u, \tau, \xi : (u, \tau, \xi) \in \mathcal{A}(x)$$

where $\mathcal{A}(x)$ is the set of admissible controls which satisfy the following identities:

$$\begin{aligned} E[\int_0^\infty e^{-\lambda t} X_t^2 dt] &< \infty; \quad E[\int_0^\infty e^{-\lambda t} u_t^2 dt] < \infty \\ \lim_{n\to\infty} e^{-\lambda t} X(T+) &= 0; \quad P\{\lim_{n\to\infty} \tau_n \leq T\} = 0, \forall T > 0. \end{aligned}$$

By developing the DPP for the mixed classical-impulse control problem above, they conjecture an optimal solution $(\hat{u}, \hat{\tau}, \hat{\xi})$ characterized by four parameters a, α, β, b with $0 < a < \alpha \leq \beta < b < \infty$ such that the optimal strategy is to stay in the (continuation) region $[a, b]$ and jump to α (respectively, to β) when reaching a (respectively, b). This conjecture implies that the process

$$\hat{u} = -\frac{K}{k}\frac{dV}{dx}(\hat{X}_t)$$

is the optimal control in the continuation region, which result in the following non-linear ODE for the value function V:

$$\frac{1}{2}\sigma^2 x^2 \frac{d^2 V(x)}{dx^2} - \frac{K^2}{k}(\frac{dV}{dx})^2 + \mu x \frac{dV}{dx} - \lambda V + (x - \rho)^2 = 0. \tag{122}$$

Remark 43 *For the details of the following remarks and some examples, see the original paper.*

(i) The values for the constants a, α, β, b are determined by a system of equations numerically.

(ii) The quasilinear ODE (122) doesn't have an explicit solution and should be solved numerically.

This application can be extended to the higher dimensions, to terminal value problems including deterministic and possibly random (bounded) parameters. However, in the next subsection, we will consider the one dimensional case of a corresponding terminal value problem and its FBSDE interpretation assuming that the model parameters are constant.

4.1.2 Terminal Value Problem and FBSDE Formulation

Let $T > 0$ be given. Then, a natural modification of the cost functional on $[0, T]$ is

$$J(x; u, \tau, \xi) = E[\int_0^T e^{-\lambda t} f(X_t, u_t)dt + e^{-\lambda T} |x-\rho|^2 + \sum_{i\geq 1} e^{-\lambda \tau_i} 1(\tau_i < T) g(\xi_i)] \quad (123)$$

where f and g are as in *(120)-(121)*. Since the diffusion coefficient in doesn't depend on the control variable, an analog of the DPP for this classical-impulse control problem would hold and the optimal control is given by

$$\hat{u} = -\frac{K}{k} V_x(\hat{X}_t).$$

Hence, in the continuation region (a, b), the value function V satisfies the quasilinear PDE

$$\begin{aligned} V_t + \frac{1}{2}\sigma^2 x^2 V_{xx} - \frac{K^2}{k}(V_x)^2 + \mu x V_x - \lambda V + (x-\rho)^2 &= 0 \\ V(T, x) &= e^{-\lambda T}(x-\rho)^2 \end{aligned} \quad (124)$$

which can be represented by the following (decoupled) FBSDE system:

$$\begin{aligned} X_s &= x \in (a, b) \\ dX_t &= \mu X_t dt + \sigma X_t dW_t; s < t \leq T \\ dY_t &= -F(X_t, Y_t, Z_t)dt + Z_t dW_t; Y_{T=} e^{-\lambda T}(X_T - \rho)^2 \end{aligned} \quad (125)$$

where

$$s = \inf\{t : X_t \in (a, b), 0 \leq t \leq T\}$$

and the driver

$$F(x, y, z) = |x-\rho|^2 - \lambda y - \frac{K^2}{k\sigma^2 x^2} z^2$$

is a quadratic function in z.

Note that X is an *exponential martingale* with an explicit unique solution

$$X_t = x e^{(\mu - \frac{1}{2}\sigma^2)(t-s) + \sigma(W(t) - W(s))}.$$

Although the generator $F(x, y, z)$ involves x^2 and the reciprocal of x which are not desirable in general, both $|x-\rho|^2$ and $1/x^2$ are bounded on the continuation region and the following estimate holds:

$$|F(x,y,z)| \leq M(1+|y|+z^2)$$

for some positive constant M. Therefore, the results of Kobilansky imply that the BSDE (*125)* has a unique minimal solution. One can then attempt to solve this system numerically by probabilistic methods discussed in Chapter 3.

4.2 Mean-Variance Portfolio Optimization

In this section, we first consider the application of stochastic LQR problem (hence, of Riccati BSDE's) to the continuous time mean-variance portfolio selection problem with random coefficients. Then we discuss the problem using martingale (or convex) duality method. An LQR formulation for the mean-variance type problems is first introduced by Zhou and Li (2000) in the deterministic coefficients case which can also be handled by DPP approach through the solution of the corresponding HJB equation. In the general case of random coefficients, HJB equation is not a deterministic PDE and DPP approach is not easy to apply. However, the LQR and martingale duality methods work quite well in complete market situations where the number of the risky assets is equal to the dimension of the Brownian motion. Some of the results and methods that are used in subsections 4.2.1 and 4.2.2 are adapted from Lim and Zhou (2002). Throughout the section, we ignore the transaction costs, taxes and consumption and assume the completeness of the market.

4.2.1 Problem Formulation

We consider a complete financial securities market consisting of a locally riskless bond with dynamics

$$\begin{aligned} dS_0(t) &= r(t)S_0(t)dt, 0 < t \leq T \\ S_0(0) &= s_0 > 0 \end{aligned} \tag{126}$$

and d risky assets whose prices $S_i(t)$, $i = 1, ..., d$, with $S = (S_1, S_2 ..., S_d)'$, satisfy the following linear SDE

$$\begin{aligned} dS(t) &= diag(S(t))[\mu(t)dt + \sigma(t)dW(t)], 0 < t \leq T \\ S(0) &= s \in \Re^d, \end{aligned} \tag{127}$$

where $s = (s_1, s_2 ..., s_d)'$ with $s_i > 0$, $i = 1, ..., d$; the interest rate $r(.)$, the appreciation rate $\mu(.)$ and the volatility matrix $\sigma(.)$ are $\mathcal{F}$-adapted, essentially bounded processes. We also assume that $\sigma(.)$ is uniformly non degenerate and define the process

$$\theta(t) \triangleq \sigma^{-1}(t)(\mu - r1),$$

known as the *risk premium* or the *market price of risk.*

Now, let $x_0 > 0$ be the initial wealth of a small investor (in the sense that he cannot change the dynamics of *(126)-(127))* and let $\pi = (\pi_1, \pi_2, ..., \pi_d)'$ be the portfolio strategy of the investor where π_i denotes the market value of the wealth in the i^{th} asser at time t. Then the investor's wealth process X has the following dynamics:

$$\begin{aligned} dX(t) &= [rX + \pi'(\mu - r1)]dt + \pi'\sigma dW(t) \\ X(0) &= x_0 > 0, \end{aligned} \tag{128}$$

where 1 stands for the d dimensional column vector composed of ones. Sometimes, we write $X^\pi(t)$ to emphasize the dependence of X on π. However, we usually skip this superscript for notational simplicity. We denote the set of admissible portfolio processes by $\mathcal{U} = \{\pi \in L^2_{\mathcal{F}}(\Re^d)$:*(128) has unique solution*$\}$.

We assume that the investor has mean-variance preferences: For a given expected terminal wealth $d > 0$ (which is assumed to be reachable), his aim is to minimize the variance

$$J(\pi(.)) = E[(X^\pi(T) - d)^2]$$

over $\pi \in \mathcal{U}$ subject to $E[X(T)] = d$. Then the value function is

$$\begin{cases} V(s,x) = \inf\limits_{\pi \in \mathcal{U}} J(\pi) \text{ subject to} \\ \qquad\qquad E[X(T)] = d. \end{cases} \tag{129}$$

We will assume in the remaining of this section that for $d > 0$ given, the optimization problem *(129)* is feasible: $V(s,x) < \infty$. A solution $(var(X_T), d)$ to this problem is called *the efficient point*, and the set of all efficient points corresponding to feasible d values is called an *efficient frontier.*

4.2.2 A Modified Problem and Riccati BSDE

By introducing a Lagrange multiplier $\lambda \in \Re$, the original optimization problem *(129)* can be written as

$$\begin{aligned} \Gamma(\lambda) &= \inf_{\pi \in \mathcal{U}} E[(X^\pi(T) - d)^2] - 2\lambda(E[X^\pi(T)] - d) \\ &= \inf_{\pi \in \mathcal{U}} E[(X^\pi(T) - (d + \lambda)^2] - \lambda^2 \end{aligned} \tag{130}$$

Our aim is to first solve the unconstrained problem $\Gamma(\lambda)$ for any $\lambda > 0$ and then to optimize $\Gamma(\lambda)$ over λ solve the original problem thanks to the convexity of the cost functional J. To this end, consider the following Riccati BSDE:

$$\begin{aligned} dp(t) &= -\{(2r - |\theta|^2)p - 2\theta' Z - \frac{|Z|^2}{p}\}dt + Z' dW_t \\ p(T) &= 1,\ p(t) > 0,\ \forall\, t \in [0,T] \end{aligned} \tag{131}$$

together with the linear BSDE

$$\begin{aligned} dh(t) &= (rh + \theta' \rho)dt + \rho' dW_t \\ h(T) &= -(\lambda + d). \end{aligned} \tag{132}$$

Note that when $\sigma(.)$ is uniformly non-degenerate on $[0,T]$, $|\theta|^2(.)$ is a bounded process. Therefore, by Lemma *(9)*, the BSDE *(131)* has a unique solution $(p, Z) \in L^{\infty}_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$. The linear equation *(132)* is a special case of the BSDE *(99)* and has unique solution $(h, \rho) \in L^{\infty}_{\mathcal{F}}(\Re) \times L^2_{\mathcal{F}}(\Re^m)$ with $h(.) \geq \delta > 0$, for some constant δ. Now, for any admissible control $\pi \in \mathcal{U}$, applying the Ito's rule to the expressions $(X^{\pi}_t + \pi_t)^2$ and $p_t((X^{\pi}_t + h_t)^2$, and noting that

$$p_T((X^{\pi}_T + h_T)^2 = (X^{\pi}_T - (\lambda + d))^2,$$

a square completion method, as in Chen et. al (1998) and Lim and Zhou (2002), would yield that the cost functional $E[(X^{\pi}(T) - (d+\lambda)^2]$ can be written as

$$E[(X^{\pi}(T) - (d+\lambda)^2] = p_0(x_0 + h_0)^2 + E[\int_0^T p_t(\pi_t + \pi^*_t)' \sigma\sigma'(t)(\pi_t + \pi^*_t)]$$

where

$$\begin{aligned} \pi^*_t &= -(\sigma\sigma'(t))^{-1}[((\mu - r1)(t) + \sigma(t)\frac{Z_t}{p_t})(X_t + h_t) + \sigma(t)\rho_t] \\ &= -\sigma'(t)^{-1}[(\theta(t) + \frac{Z_t}{p_t})(X_t + h_t) + \rho_t]. \end{aligned} \tag{133}$$

Noting that the equation *(128)* with $\pi = \pi^* \in L^2_{\mathcal{F}}(\Re^m)$ is a linear non-homogenous SDE, we have $\pi^* \in \mathcal{U}$ and the following result holds:

Proposition 44 *For any $\lambda > 0$, the LQR problem (130) is solvable with the unique optimal feedback control π^*_t given by (133) and with the optimal cost*

$$\Gamma(\lambda) = p_0(x_0 + h_0)^2. \tag{134}$$

Remark 45 *The optimal cost (134) depends on λ through the initial value h_0 of the process h in (132). However, it is easy to see that*

$$(h_t, \rho_t) = -(\lambda + d)(g_t, \kappa_t)$$

where (g_t, κ_t) solves the linear BSDE

$$\begin{aligned} dg_t &= (rg + \theta'\kappa)dt + \kappa' dW_t \\ g_T &= 1,\ g(t) > 0 \text{ on } [0, T]. \end{aligned} \tag{135}$$

Corollary 46 *For any $\lambda > 0$, the optimal cost for the LQR problem (130) is given by*

$$\Gamma(\lambda) = p_0(x_0 - (\lambda + d)g_0)^2, \tag{136}$$

where g solves (135).

4.2.3 Solution to Mean Variance Optimization Problem

Since the wealth process X has a linear dynamics, the cost functional

$$J(\pi(.)) = E[(X^\pi(T) - d)^2]$$

is strictly convex. Moreover, *(129)* is finite and can be written as

$$\begin{aligned} V(s,x) &= \sup_{\lambda>0}(\inf_{\pi\in\mathcal{U}} E[(X^\pi(T) - (d+\lambda)^2] - \lambda^2) \\ &= \sup_{\lambda>0}(\Gamma(\lambda) - \lambda^2) = \sup_{\lambda>0}[p_0(x_0 - (\lambda + d)g_0)^2 - \lambda^2] \end{aligned} \tag{137}$$

by Lagrange duality theorem (see for example Luenberger (1968)). Clearly, the optimal λ is

$$\lambda^* = \frac{p_0 g_0(x_0 - dg_0)}{g_0^2 - 1}$$

with $g_0 > 0$, and the minimum variance is given by

$$V(0, x_0) = \frac{p_0 g_0^2}{1 - p_0 g_0^2}(d - \frac{x_0}{g_0})^2.$$

Hence the efficient frontier is characterized by the pairs $(\frac{p_0 g_0^2}{1-p_0 g_0^2}(d - \frac{x_0}{g_0})^2, d)$ for all feasible $d > 0$.

Remark 47 *A sufficient condition for the feasibility of the problem is given by Lim and Zhou (2002) under which $1 - g_0^2 p_0 > 0$ holds. They also proved that $0 < g_0 \leq 1$,* *if $r(t) \geq 0$, a.s. on $[0,T]$. Note that indeterministic case, g simply becomes the discount factor $g(t) = e^{-\int_t^T r(s)ds} > 0$ and p is given by $p(t) = e^{\int_t^T (2r - |\theta|^2)(s)ds}$ so that*

$$0 < p_0 g_0^2 = e^{-\int_0^T |\theta|^2(s)ds} < 1$$

and

$$V(0, x_0) = \frac{e^{-\int_0^T |\theta|^2(s)ds}}{1 - e^{-\int_0^T |\theta|^2(s)ds}} (d - x_0 e^{\int_0^T r(s)ds})^2. \tag{138}$$

In this case, it is natural to consider the values of d such that

$$d - x_0 e^{\int_0^T r(s)ds} > 0,$$

otherwise the investor would just invest in the riskless bond (so with zero risk, zero variance) and come up a wealth amount of at least d at time T. Under these assumptions, we have $\lambda^* > 0$ and $0 < V(s,x) < \infty$.

4.2.4 The Martingale Duality Approach

The solution of the mean variance portfolio selection problem using the LQR approach depends on the solvability of the corresponding Riccati equation. In the previous subsection, the Riccati BSDE could be written as a simple transformation of another linear BSDE which has a unique bounded solution and the optimal portfolio could be obtained via a square completion technique. However, in mathematical finance, when there are some constraints on the admissible portfolio processes, these methods tricks may not apply. See for example Li et. al (2002) for short selling constraints; Hugonnier and Kaniel (2004) and Cetin (2005) for mutual fund portfolio optimization problems with constraints on the portfolios of the investor. In this subsection, without going into the details, we will introduce the martingale duality method which is a powerful tool for the optimization problems in complete market settings. For the general theory and applications of the duality methods to utility maximization problems, one can refer to the Karatzas and Shreve (1998) and Cvitanic and Karatzas (1992).

We begin with introducing the exponential martingale $Z_0(t)$:

$$\begin{aligned} dZ_t &= -Z_0(t)\theta(t)dW_t, \\ Z_t &= 1 \end{aligned} \tag{139}$$

which is a strictly positive and uniformly integrable martingale (under the conditions of the previous subsections). Then the state price density process $H = \{H_t : 0 \leq t \leq T\}$ is defined as

$$H_t \triangleq e^{-\int\limits_0^t r(s)ds} Z_t$$

and has dynamics

$$dH = -H(rdt + \theta' dW(t).$$

By Ito's rule, it is easy to see that the process HX is a martingale with

$$\begin{aligned} d(H_t X_t) &= H_t(\pi'\sigma - \theta' X_t)dW_t \\ H_t X_t &= x > 0. \end{aligned} \tag{140}$$

Now, by martingale representation theorem, there exists an adapted square integrable process $\varphi \in L^2_F(\Re^d)$ such that

$$\begin{aligned} E[H_T X_T] &= H_t X_t = E[H_T X_T] + \int\limits_0^T \varphi'(s)dW_s \\ &= x + \int\limits_0^T \varphi'(s)dW_s. \end{aligned} \tag{141}$$

Moreover, by Clark-Ocone formula (see section 1.2.4) and comparing the equations *(140)-(141)*,

$$H_t(\sigma'\pi - \theta X_t) = \varphi(t) = E_t[D_t(H_T X_T)]. \tag{142}$$

Now, we turn back to the optimization problem *(129)* which is, by the arguments of the previous subsection, equivalent to

$$\begin{aligned} V(0,x) &= \sup_{\lambda > 0}(\inf_{\pi \in \mathcal{U}} E[(X^\pi(T) - (d+\lambda)^2] - \lambda^2) \\ &= \sup_{\lambda>0}(\Gamma(\lambda) - \lambda^2) \end{aligned} \tag{143}$$

where $\Gamma(\lambda)$ is as in *(130)*. We now define the following quadratic utility function:

$$U(x) \triangleq -(x-k)^2, \tag{144}$$

for some $k > 0$. Clearly, $U(.)$ is a smooth and strictly concave function of x with the derivative

$$\frac{dU}{dx} = -2(x-k).$$

We then introduce the following functions:

$$\begin{aligned} I(y) &\triangleq (\frac{dU}{dx})^{-1}(y) \\ \tilde{U}(y) &\triangleq \max_{x>0}\{U(x) - xy\},\ y \in \Re. \end{aligned}$$

which takes the following form in our case:

$$\begin{aligned} I(y) &= k - y/2 \\ \tilde{U}(y) &= U(I(y)) - yI(y) \\ &= \frac{y^2}{4} - yk. \end{aligned}$$

Remark 48 *The most frequently used utility functions (e.g. exponential, logarithmic, power utilities) in the literature are strictly increasing (functions of the consumption or wealth). Although (130) doesn't satisfy this condition, the martingale duality methods applies to our problem since U is strictly concave and the functions $I(y)$ and $\tilde{U}(y)$ are well defined. The function $\tilde{U}(y)$ is related to the Legendre-Fenchel transform (of a convex function) as mentioned in Karatzas and Shreve (1998). The following properties of $\tilde{U}(.)$ are straightforward.*

Lemma 12 *The function*

$$\tilde{U}(y) = \frac{y^2}{4} - yk$$

satisfy the following properties:
(i) $\tilde{U}(.) \in C^2(\Re)$ and $\tilde{U}(.)$ is convex
(ii) $\tilde{U}(.)$ is decreasing on $(-\infty, 2k)$, and increasing on $(2k, \infty)$
(iii) $\frac{d\tilde{U}}{dx}(y) = -I(y)$
(iv) $\forall x \in \Re$, $U(x) = \inf_{y\in\Re}\{\tilde{U}(y) + xy\} = \tilde{U}(\frac{dU}{dx}(x)) + x\frac{dU}{dx}(x)$

Now, we rewrite the problem $\Gamma(\lambda)$ in the equivalent form as follows: Maximize the expected terminal utility $E[U(X_T)]$ with $k = (\lambda + d)$, over all admissible portfolio processes such that the static budget constraint

$$E[H_T X_T] = x$$

is satisfied. By introducing another Lagrange multiplier $\alpha > 0$, this can be described as

$$\sup_{X(T),\alpha}\ E[U(X_T)] - \alpha(x - E[H_T X_T]) \tag{145}$$

which characterizes the optimal terminal wealth first. Therefore, the original problem transforms to a static one in *(130)*. By a simple application of the arguments above and the Lemma *12*, the necessary and sufficient conditions for the optimality of (X_T, α, π) are given by the following identities:

$$\begin{aligned} \hat{X}_T^{\pi} &= I(\hat{\alpha} H_T) && (146) \\ x &= E[H_T \hat{X}_T^{\hat{\pi}}] = E[I(\hat{\alpha} H_T) H_T]. && (147) \end{aligned}$$

From *(146),* we get

$$\hat{X}_T^{\pi} = k - \frac{\hat{\alpha} H_T}{2}$$

which together with *(147)* implies that

$$x = kE[H_T] - \frac{\hat{\alpha}}{2} E[H_T^2]$$

so that the optimal values $(\hat{X}_T, \hat{\alpha})$ are given by

$$\begin{aligned} \hat{\alpha} &= 2\frac{kE[H_T] - x}{E[H_T^2]} \\ \hat{X}_T^{\pi} &= k - \frac{kE[H_T] - x}{E[H_T^2]} H_T && (148) \end{aligned}$$

if there exists a portfolio such that *(148)* is attained for some admissible $\pi \in L_{\mathcal{F}}^2(\Re^d)$ and $\hat{\alpha} > 0$. Thanks to the completeness of the market, such a portfolio (called a *replicating portfolio*) always exists, and indeed, is given by *(142).* For the technical details and the financial interpretation of market completeness, one can refer to Karatzas and Shreve (1998).

The application of the identities in section 1.2.4 and using the fact that XH is a martingale, we obtain

$$D_t(X_T H_T) = k\theta_t H_T - 2\theta_t X_T H_T,$$

and hence the optimal values satisfy

$$\begin{aligned} H_t(\sigma_t' \pi_t^* - \theta_t \hat{X}_t^{\pi^*}) &= \varphi_t = E_t[D_t(H_T \hat{X}_T^{\pi^*})] \\ &= k\theta_t E_t[H_T] - 2\theta_t H_t \hat{X}_t^{\pi^*} \end{aligned}$$

from which we get the optimal feedback control

$$\pi^* = -\sigma'(t)^{-1}[\theta(t)(\hat{X}_t^{\pi^*} - kE_t[e^{-\int_t^T (r + \frac{1}{2}|\theta|^2)(s) ds - \int_t^T \theta(s)' dW_s}]) \qquad (149)$$

Remark 49 *By uniqueness of* π^**, the expressions (133) and (149) with* $k = d + \lambda$ *are equal a.s. However, the representation (149) is more intuitive from financial point* *of view (thinking H as a discount factor) and simpler computationally (expressed in terms of model parameters). In the deterministic parameters case, both expressions reduce to the following form:*

$$\begin{aligned}\pi^* &= -\sigma'(t)^{-1}\theta(t)(X_t - (d+\lambda)e^{-\int_t^T r(s)ds}) \\ &= -(\sigma\sigma')^{-1}(\mu - r1)[X_t - (d+\lambda)e^{-\int_t^T r(s)ds}]\end{aligned}$$

which still depends on λ. We now complete the solution of the mean variance problem.

Theorem 50 *The value function $V(0,x)$ for the mean-variance problem (143) is given by*

$$V(0,x) = var(\hat{X}_T) = \frac{E[(dH_T - x)^2]}{var(H_T)}.$$

with optimal λ

$$\lambda^* = \frac{d(E[H_T])^2 - xE[H_T]}{var(H_T)}$$

Proof. For any $\lambda > 0$, the optimal terminal wealth is

$$\hat{X}_T = \lambda + d - \frac{(\lambda+d)E[H_T] - x}{E[H_T^2]}H_T.$$

Now, instead of maximizing the expression $\Gamma(\lambda) - \lambda^2$ in *(143), we find* λ such that the static terminal condition $E[\hat{X}_T] = d$ is satisfied. This means

$$d = \lambda + d - \frac{(\lambda+d)E[H_T] - x}{E[H_T^2]}E[H_T]$$

so that

$$\lambda^* = \frac{d(E[H_T])^2 - xE[H_T]}{var(H_T)}$$

which also maximizes $\Gamma(\lambda) - \lambda^2$. Then by *(148),* the corresponding $\hat{\alpha}$ and $\hat{X}_T$ are

$$\begin{aligned}\hat{\alpha} &= 2\frac{dE[H_T] - x}{var(H_T)} \\ \hat{X}_T &= \frac{dE[H_T^2] - xE[H_T]}{var(H_T)} - \frac{dE[H_T] - x}{var(H_T)}H_T.\end{aligned}$$

Then, some straightforward computations show that the minimum variance is

$$var(\hat{X}_T) = \frac{E[(dH_T - x)^2]}{var(H_T)}.$$

■

Corollary 51 *The efficient frontier*

$$(var(X_T), E[X_T])$$

for the the mean variance problem (129) is given by

$$(var(X_T), E[X_T]) = (V(0, x_0), d) = (\frac{E[(dH_T - x_0)^2]}{var(H_T)}, d)$$

When all the coefficients are deterministic, $V(0, x_0)$ coincides with (138).

4.3 Other Applications and Concluding Remarks

In the previous section, the power of the martingale duality method comes from the fact that when an optimal terminal wealth is obtained, there is always an admissible portfolio strategy to replicate this payoff in a complete market setting. Another advantage of this approach is that it could be applied even under some portfolio constraints (Cvitanic and Karatzas, 1992). A recent application is the mutual fund portfolio optimization problem that the investor is not allowed to short sell the fund. Hugonnier and Kaniel (2004) considered log utility for the investor and an increasing utility for the manager. They obtained the optimal strategies for both agents as well as the optimal fund proportions using convex duality technique and backward BSDE's. When the agents have mean-variance preferences, the optimal strategies differ significantly and need to be computed numerically, even in deterministic coefficients case (Cetin, 2005). One important drawback of the martingale duality is that when the financial market is incomplete (which means that some claims or payoffs may not be replicated) it is not so easy to apply.

In a similar way, the LQR approach works well if the corresponding Riccati equation is solvable. When there are some constraints on the portfolios, the square completion technique is not guaranteed to work. Moreover, the resulting Riccati BSDE may not be always obtained from a simple transformation of a linear BSDE. For the recent applications of LQR approach to the quadratic hedging problems under incomplete market settings, one may refer to Lim (2003).

We can also mention the applications of quadratic BSDE's to the problems of stochastic differential utility by Schroder and Skiadas (1999, 2003).

References

[1] Ait Rami, M., Chen, S., Moore, J.B. (2001). Solvability and asypmtotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls. *IEEE Transactions on Automatic Control*, 46, 3.

[2] Antonelli, F. (1993). Backward-forward stochastic differential equations. *Ann. Appl. Probab.* 3, 777-793.

[3] Bihari, I. (1956). A generalization of a lemma of Bellman and its application to uniqueness problem ofdifferential equations. *Acta Math. Acad. Sci. Hungar.* 7, 71-94

[4] Bizmut, J.M. ((1976)). Linear quadratic optimal stochastic control with random coefficients, *SIAM J. Control Optim.*, 14, 419-444

[5] Bouchard, B., Ekeland, I. and Touzi (2002). On the Malliavin calculus approach to Monte Carlo approximation of conditional expectations, *Finance Stochastics* 8, 45-71

[6] Bouchard, B. and Touzi (2004). Discrete-time approximation and Monte Carlo simulation of backward stochastic differential equations, preprint, *Stoch. Process. Appl.* **111**, 175-206

[7] Cadenillas, A. and Zapatero, F. (2000). Classical and impulse stochastic control of the exchange rate using interest rates and reserves, *Mathematical Finance*, Vol. 10, 2, 141-156

[8] Cetin, C. (2005). Investor-manager portfolio optimization under mean-variance preferences, Working paper

[9] Chen, S., Li, X. and Zhou, X.Y. (1998). Stochastic linear quadratic regulators with indefinite control weight costs, *SIAM J. Control Optim*, 36, 1685-1702

[10] Chen, S., and Zhou, X.Y. (2000). Stochastic linear quadratic regulators with indefinite control weight costs, *SIAM J. Control Optim*, 39, 1065-1081

[11] Cvitanic, J. and Karatzas, I. (1992). Convex duality in constrained portfolio optimization, *Annals of Probability*, 2, 767-818

[12] Cvitanic, J., Lazrak, A., Quenez, M.C., Zapatero, F. (2001). Incomplete information with recursive preferences, *International J. of Theoretical and Appl. Finance*, 4, 245-261

[13] Cvitanic, J, Wan, Xuhu and Zhang, J. (2004). First-best contracts for continuous-time principal-agent problems, submitted to *Mathematics of Operations Research*

[14] Cvitanic, J. and Ma, J. (1996). Hedging options for a large investor and forward-backward SDE's. *Ann. Appl. Probab.* 6, 370-398

[15] Davis, M.H.A. (1977). Linear Estimation and Stochastic Control. Chapman and Hall.

[16] Douglas, J., Ma, J. and Protter, P. (1999). Numerical methods for forward-backward stochastic differential equations. *The Annals of Probab.*, **6**, 940-968

[17] Duffie, D., Ma, J., Yong, J. (1995). Black's console rate conjecture, *Ann. Appl. Probab.*, **5**, 356-382

[18] El Karoui, N., Hamadene, S. (2003). BSDE's and risk-sensitive control, zero-sum and non-zero-sum game problems of stochastic functional differential equations. *Stoch. Process. Appl.* **107** 145-169

[19] El Karoui, N., Peng, S., Quenez, M. C. (1997). Backward stochastic differential equations in finance. *Math. Finance* **7** 1-71

[20] Fleming, W. H., and Rishel, R. W. (1975). Deterministic and Stochastic Optimal Control, Springer-Verlag

[21] Fleming, W.H. and Soner, H.M. (1993). Controlled Markov Processes and Viscosity Solutions, Springer-Verlag.

[22] Fournie, E., Lasry, J.M., Lebuchoux, J., Lions, P.L. (2001). Applications of Malliavin Calculus to Monte Carlo methods in finance, *Finance and Stochastics*, **3**, 391-412

[23] Gobet, E., Lemor, J-P., Warin, X. (2004). A regression-based Monte-Carlo method to solve backward stochastic differential equations, working Paper, Ecole Polytechnique

[24] Hu, Y. and Zhou, X.Y. (2003). Indefinite stochastic Riccati equations, *SIAM J. Control Optim.*, 42, 1, 123-137

[25] Hugonnier, J. , Kaniel, R. (2004), Mutual Fund Portfolio Choice in the Presence of Dynamic Flows, submitted

[26] Karatzas, I and Shreve, S.E. (1999). Brownian Motion and Stochastic Calculus, Springer-Verlag

[27] Karatzas, I and Shreve, S.E. (1998). Methods of Mathematical Finance, Springer-Verlag

[28] Kobylanski, M (2000). Backward stochastic differential equations and partial differential equations with quadratic growth. *The Annals of Prob.* **28** 558-602

[29] Kohlmann, M. and Tang, S. (2000). Recent advances in backward stochastic Riccati equations and their applications, working paper

[30] Lepeltier, J.P. and San Martin, J. (1997). Backward stochastic differential equations with continuous coefficients. *Statist Probab. Lett.* **32** 425-430

[31] Lepeltier, J.P. and San Martin, J. (1998). Existence for BSDE with superlinear-quadratic coefficient. *Stochastics Stochastic Rep.* **63** 227-240

[32] Lim, A. (2003). Quadratic hedging and mean-variance portfolio selection with random parameters in an incomplete market setting, Working paper

[33] Lim, A., Zhou, X.Y. (2001), Mean-variance portfolio selection with random parameters in a complete market. *Math. Oper. Res.* 27:1, 101-120

[34] Luenberger, D.G. (1968). Optimization by Vector Space Methods, John Wiley

[35] Ma, J., Protter, P. and Yong, J. (1994). Solving forward-backward stochastic differential equations explicitly-a four step scheme. *Prob. Theory Related Fields* **98**, 339-359

[36] Ma, J., and Yong, J. (1999). Forward-Backward Stochastic Differential Equations and Their Applications, Lecture Notes in Math, 1702, Springer

[37] Ma, J. and Zhang, J. (2002) Representation Theorems for BSDE's, *The Annals of Probab., Vol.12, 4, 1390-1418*

[38] Mao, X. (1995). Adapted solution of backward stochastic differential equations with non-Lipshitz coefficients. *Stoch. Process. Appl.* **58** 281-292

[39] Nualart, D. (1995). The Malliavin Calculus and Related Topics, Springer

[40] Oksendal, B. (2000). Stochastic Differential Equations, Springer.

[41] Pardoux, E. and Peng, S. (1990). Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14** 55-61

[42] Pardoux, E. and Peng, S. (1992). Backward stochastic differential equations and quasilinear partial differential equations, Lecture Notes in CIS, vol. **176** Springer, 200-217

[43] Pardoux, E. and Peng, S. (1994). Some backward stochastic differential equations with non-Lipshitz coefficients, Prépublication LATP, 94-03

[44] Peng, S. (1992). Stochastic Hamilton-Bellman equations, *SIAM J. Control Optim.*, **30**, 284-304

[45] Schroder, M. and Skiadas, C. (1999). Optimal Consumption and Portfolio Selection with Stochastic Differential Utility, *J. of Economic Theory* **89**, 68-126

[46] Schroder, M. and Skiadas, C. (2003). Optimal Lifetime Consumption-Portfolio Strategies under Trading Constraints and Generalized Recursive Preferences, *Stoch. Process. Appl,* **108**, 155-202

[47] Wonham, W.M (1968). On a matrix Riccati equation of stochastic control, *SIAM J. Control Optim.*, **6**, 312-326

[48] Yong, J. and Zhou, X.Y. (1999). Stochastic Controls: Hamiltonian Systems and HJB Equations, Springer-Verlag, New York

[49] Zhang, J. (2001). Some Fine Properties of Backward Stochastic Differential Equations, PhD Thesis, Purdue University

[50] Zhang, J. (2004). A numerical scheme for BSDE's, *The Annals of Probab.*, **14**, 459-488

Appendices

Appendix A. Some Useful Inequalities

Lemma A.1. (**Bihari's Inequality**) *Let $T > 0$, $u_0 \geq 0$, $u(t)$ and $v(t)$ be continuous functions on $[0, T]$.Let $\xi : \Re_+ \to \Re_+$ be continuous and nondecreasing such that $\xi(r) > 0$ for $r > 0$. If*

$$u(t) \leq u_0 + \int_0^t v(s)\xi(u(s))ds$$

for all $0 \leq t \leq T$, then

$$u(t) \leq G^{-1}(G(u_0) + \int_0^t v(s)ds)$$

for all such $t \in [0, T]$ that $G(u_0) + \int_0^t v(s)ds \in Dom(G^{-1})$, where $G(r) = \int_1^r \frac{1}{\xi(s)}ds$, for $r \geq 0$. In particular,when $u_0 = 0$ and $\int_{0^+} \frac{1}{\xi(s)}ds = \infty$, then $u(t) = 0$ for all $0 \leq t \leq T$.

(Bihari, 1956; Mao, 1995)

Lemma A.2. **The Burkholder-Davis-Gundy Inequality.** *Let M be a continuous martingale which, along with its quadratic variation process $\langle M \rangle_t$, is bounded and define $M_t^* = \max_{0 \leq s \leq t} |M_s|$. Then for any $m > 0$ there exist universal positive constant c_m, d_m (depending only on m) such that*

$$c_m E[\langle M \rangle_\tau^m] \leq E[|M_\tau^*|^{2m}] \leq d_m E[\langle M \rangle_\tau^m]$$

holds for any stopping time τ

Appendix B. Riccati ODE's

The most of the following results are well known in the literature and can be found in Yong and Zhou (1999), Chen at al (1998) and Chen and Zhou (2000).

Lemma B.1. *Let S be the unique $d \times d$ symmetric matrix valued solution to the standard Riccati equation (38):*

$$\begin{aligned} \dot{S} + M - SBN^{-1}B'S + A'S + SA &= 0, \ 0 \leq t < T \\ S(T) &= R \end{aligned}$$

where A, B, M, N, R are as in (22)-(24). Then, $S \geq 0$. Moreover, if $M > 0$, or $R > 0$ on $[0, T]$, then $S > 0$ on $[0, T]$.

Lemma B.2. *Let S_i, for $i = 1, 2$, be $d \times d$ symmetric matrix valued functions on $[0, T]$, satisfying the Riccati equation*

$$\begin{aligned} \dot{S}_i + M - S_i B_i N_i^{-1} B_i' S_i + A' S_i + S_i A &= 0, \ 0 \leq t < T \\ S_i(T) &= R_i \geq 0, \ i = 1, 2 \end{aligned} \tag{B.1}$$

where A, M, R, B_i, N_i are as in (22)-(24), $i = 1, 2$.
(i) If

$$B_1 N_1^{-1} B_1' \leq B_2 N_2^{-1} B_2'$$

on $[0, T]$ and $R_1 \geq R_2$ then, $S_1 \geq S_2$, on $[0, T]$

(ii) If $N_1 \geq N_2 > 0$ on $[0, T]$ and $R_1 \geq R_2$ then, $S_1 \geq S_2$, on $[0, T]$

Proof. (i) Set $S = S_1 - S_2$. Then it is easy to see that S satisfies the following standard Riccati equation:

$$\begin{aligned} \dot{S} + \tilde{A}' S + S \tilde{A} - S B N^{-1} B' S &= 0, \ 0 \leq t < T \\ S(T) &= R_1 - R_2 \geq 0 \end{aligned}$$

with $\tilde{A} = A - B S_2$. By the previous Lemma, S is the unique symmetric non-negative definite solution and therefore $S_1 \geq S_2$ on $[0, T]$.(ii) Now, $S = S_1 - S_2$ satisfies

$$\begin{aligned} \dot{S} + \tilde{A}' S + S \tilde{A} - S B N^{-1} B' S + \tilde{M} &= 0, \ 0 \leq t < T \\ S(T) &= R_1 - R_2 \geq 0 \end{aligned}$$

with $\tilde{A} = A - B N_2^{-1} B' S_2$ and $\tilde{M} = S_1 B (N_2^{-1} - N_1^{-1}) B' S_1$. Since $N_1 \geq N_2$ and $S_1 \geq 0$, we have $N_2^{-1} - N_1^{-1} \geq 0$ and $\tilde{M} \geq 0$. So the equation above is a standard Riccati ODE which satisfies $S \geq 0$. ■

Lemma B.3. *Consider the following linear ODE:*

$$\begin{aligned} \dot{P} + M + A' P + P A + \sum_{j=1}^{m} C_j' P C_j &= 0, \ 0 \leq t < T \\ P(T) &= R \end{aligned} \tag{B.2}$$

Then there exist a unique symmetric solution $P \in C^1(\Re^d)$ with $P \geq 0$ on $[0, T]$. Moreover, $P > 0$ if $M > 0$ or $R > 0$ on $[0, T]$.

Proof. The case $m = 1$ follows from a Lemma 4.1 in Chen and Zhou (2000). Since (B.2) is a linear equation with bounded coefficients, it is not difficult to extend the proof to the general case. ■

Lemma B.4. *Consider the following generalized Riccati ODE with continuous coefficients where N, M and R are symmetric:*

$$\begin{aligned} \dot{S} &= -(M + A'S + SA + C'SC) + (SB + C'SD)(N + D'SD)^{-1}(SB + C'SD)' \\ S(T) &= R,\ N + D\prime SD > 0 \end{aligned} \tag{B3}$$

*If $S \in C([0, T]; \Re^d)$ is a solution to (**??**), then this solution is unique and symmetric.*

Proposition B.5 *Consider the Riccati ODE (**??**) with $M \geq 0, R \geq 0, N \geq 0$ (uniformly on $[0, T]$). Then, the equation (**??**) has a continuous symmetric solution if either of the followings hold:*

(a) $N > 0$ on $[0, T]$.

(b) N is singular and $D'D > 0$ on $[0, T]$.

Appendix C. Regression Approximation

Let $\sigma(t)$ be uniformly parabolic, π be a partition of $[0, 1]$ with uniform mesh size: $|\pi| = \triangle t = 1/n$. Using the notation of Bouchard and Touzi (2004), the process

$$h_{i,t} = \frac{1}{|\pi|}(\sigma(t_{i-1})^{-1} 1_{[t_{i-1}, t_i)}(t) - \sigma(t_i)^{-1} 1_{[t_i, t_{i+1})}(t))$$

satisfies $\int\limits_0^\infty D_t X_{t_i}^\pi h_{i,t} dt = I_d$ and $\int\limits_0^\infty D_t X_{t_{i+1}}^\pi h_{i,t} dt = 0$ since $D_t X_{t_i}^\pi = \sigma(t_{i-1})$ for $t \in (t_{i-1}, t_i)$, $1 \leq i \leq n$. Let a be a generic notation for the affine functions a_0 and $a_k : \Re^d \to \Re, k = 1, ..., d$, defined by $a_0(x) = 1$; $a_k(x) = x_k$ (k^{th} component of x). Then

$$S_{\{k\}}^{h_{i-1}}[a(\triangle W_i)] = a(\triangle W_i)(h_{i-1,t_{i-2}}^k \cdot \triangle W_{i-1} + h_{i-1,t_{i-1}}^k \cdot \triangle W_i) - \nabla a(\triangle W_i) \cdot h_{i-1,t_{i-1}}^k / n$$

where h^k denotes the k^{th} column of the matrix h and the dots $(\cdot)$ denote the inner product in $\Re^d$ to simplify the notation.

Then, for $1 \leq k, u \leq d$,

$$S_{\{k\}}^{h_{i-1}}[a_0(\triangle W_i)] = S_{\{k\}}^{h_{i-1}}[1] = (h_{i-1,t_{i-2}}^k \cdot \triangle W_{i-1} + h_{i-1,t_{i-1}}^k \cdot \triangle W_i)$$

and

$$S_{\{k\}}^{h_{i-1}}[a_u(\triangle W_i)] = (\triangle W_i^u)(h_{i-1,t_{i-2}}^k \cdot \triangle W_{i-1} + h_{i-1,t_{i-1}}^k \cdot \triangle W_i) - h_{i-1,t_{i-1}}(u, k)/n$$

By choosing the localizing function φ as the constant function $\varphi \equiv 1$, the representation for $S^{h_{i-1}}[a(\triangle W_i)]$ can be simply written as $S^{h_{i-1}}_{\{1,\ldots,d\}}[a(\triangle W_i)]$ (see, for example, Bouchard and Touzi (2004)). Our aim is to evaluate these expressions in terms of the discretized processes h_i and W_i for a given dimension d.

For $d = 3$,

$$\begin{aligned}
S^{h_{i-1}}[1] &= S^{h_{i-1}}_{\{1,2,3\}}[1] = \int S^{h_{i-1}}_{\{2,3\}} h^1_{i-1} \cdot dW_t \\
&= S^{h_{i-1}}_{\{2,3\}}(h^1_{i-1,t_{i-2}} \cdot \triangle W_{i-1} + h^1_{i-1,t_{i-1}} \cdot \triangle W_i) - \int D_t S^{h_{i-1}}_{\{2,3\}} \cdot h^1_{i-1} dt \\
&= S^{h_{i-1}}_{\{2,3\}} S^{h_{i-1}}_{\{1\}} - \int D_t S^{h_{i-1}}_{\{2,3\}} \cdot h^1_{i-1} dt
\end{aligned}$$

with

$$\begin{aligned}
S^{h_{i-1}}_{\{2,3\}} &= \int S^{h_{i-1}}_{\{3\}} h^2_{i-1} \cdot dW_t \\
&= S^{h_{i-1}}_{\{3\}} S^{h_{i-1}}_{\{2\}} - \int D_t S^{h_{i-1}}_{\{3\}} \cdot h^2_{i-1} dt
\end{aligned}$$

where $S^{h_{i-1}}_{\{k\}} = S^{h_{i-1}}_{\{k\}}[1]$,

$$D_t S^{h_{i-1}}_{\{k\}} = h^k_{i-1,t_{i-2}} 1_{[t_{i-2},t_{i-1})}(t) + h^k_{i-1,t_{i-1}} 1_{[t_{i-1},t_i)}(t)$$

and

$$D_t S^{h_{i-1}}_{\{2,3\}} = S^{h_{i-1}}_{\{2\}} D_t S^{h_{i-1}}_{\{3\}} + S^{h_{i-1}}_{\{3\}} D_t S^{h_{i-1}}_{\{2\}}$$

Therefore,

$$\begin{aligned}
S^{h_{i-1}}[1] &= S^{h_{i-1}}_{\{2,3\}} S^{h_{i-1}}_{\{1\}} - \int_{t_{i-2}}^{t_{i-1}} [S^{h_{i-1}}_{\{2\}} h^3_{i-1,t_{i-2}} \cdot h^1_{i-1,t_{i-2}} + S^{h_{i-1}}_{\{3\}} h^2_{i-1,t_{i-2}} \cdot h^1_{i-1,t_{i-2}}] dt \\
&\quad - \int_{t_{i-1}}^{t_i} [S^{h_{i-1}}_{\{2\}} h^3_{i-1,t_{i-1}} \cdot h^1_{i-1,t_{i-1}} + S^{h_{i-1}}_{\{3\}} h^2_{i-1,t_{i-1}} \cdot h^1_{i-1,t_{i-1}}] dt
\end{aligned}$$

The computation for $S^{h_{i-1}}[\triangle W_i^u]$ is much more complicated:

$$\begin{aligned} S^{h_{i-1}}[\triangle W_i^u] &= S^{h_{i-1}}_{\{1,2,3\}}[\triangle W_i^u] = \int S^{h_{i-1}}_{\{2,3\}} h^1_{i-1} \cdot dW_t \\ &= S^{h_{i-1}}_{\{2,3\}} S^{h_{i-1}}_{\{1\}}[1] - \int_{t_{i-2}}^{t_i} D_t S^{h_{i-1}}_{\{2,3\}} \cdot h^1_{i-1} dt \end{aligned}$$

with

$$\begin{aligned} S^{h_{i-1}}_{\{2,3\}} &= \int S^{h_{i-1}}_{\{3\}} h^2_{i-1} \cdot dW_t \\ &= S^{h_{i-1}}_{\{3\}} S^{h_{i-1}}_{\{2\}}[1] - \int D_t S^{h_{i-1}}_{\{3\}} \cdot h^2_{i-1} dt \end{aligned}$$

where

$$\begin{aligned} S^{h_{i-1}}_{\{k\}} &= S^{h_{i-1}}_{\{k\}}[\triangle W_i^u], \\ D_t S^{h_{i-1}}_{\{3\}} &= \triangle W_i^u (h^3_{i-1,t_{i-2}} 1_{[t_{i-2},t_{i-1})}(t) + h^3_{i-1,t_{i-1}} 1_{[t_{i-1},t_i)}(t)) \\ &+ \begin{bmatrix} \delta_{1u} \\ \delta_{2u} \\ \delta_{3u} \end{bmatrix} (h^3_{i-1,t_{i-2}} \cdot \triangle W_{i-1} + h^3_{i-1,t_{i-1}} \cdot \triangle W_i) 1_{[t_{i-1},t_i)}(t) \end{aligned}$$

$$\begin{aligned} & D_t S^{h_{i-1}}_{\{2,3\}}[\triangle W_i^u] = \{(h^3_{i-1,t_{i-2}} \triangle W_i^u 1_{[t_{i-2},t_{i-1})}(t) + [(h^3_{i-1,t_{i-2}} \cdot \triangle W^u_{i-1} \\ & + h^3_{i-1,t_{i-1}} \cdot \triangle W_i) 1_{[t_{i-1},t_i)}(t) \begin{bmatrix} \delta_{1u} \\ \delta_{2u} \\ \delta_{3u} \end{bmatrix} + \triangle W_i^u h^3_{i-1,t_{i-1}}]\}((h^2_{i-1,t_{i-2}} \cdot \triangle W_{i-1} + h^2_{i-1,t_{i-1}} \triangle W_i) \\ & + \triangle W_i^u (h^3_{i-1,t_{i-2}} \cdot \triangle W_{i-1} + \triangle W_i \cdot h^3_{i-1,t_{i-1}}) \{(h^2_{i-1,t_{i-2}} 1_{[t_{i-2},t_{i-1})}(t) + h^2_{i-1,t_{i-1}} 1_{[t_{i-1},t_i)}(t))\} \\ & - \int_{t_{i-2}}^{t_{i-1}} (h^3_{i-1,t_{i-2}} \cdot h^2_{i-1,t_{i-2}})(v) dv \begin{bmatrix} \delta_{1u} \\ \delta_{2u} \\ \delta_{3u} \end{bmatrix} 1_{[t_{i-1},t_i)}(t) \\ & \quad - \int_{t_{i-1}}^{t_i} (h^3_{i-1,t_{i-1}} \cdot h^2_{i-1,t_{i-1}})(v) dv \begin{bmatrix} \delta_{1u} \\ \delta_{2u} \\ \delta_{3u} \end{bmatrix} 1_{[t_{i-1},t_i)}(t) \\ & - \int_{t_{i-1}}^{t_i} (h^3_{i-1,t_{i-2}} 1_{[t_{i-2},t_{i-1})}(t) + h^3_{i-1,t_{i-1}} 1_{[t_{i-1},t_i)}(t)) h_{i-1,t_{i-1}}(u,2) dv \\ & - (\int_{t_{i-1}}^{t_i} h_{i-1,t_{i-1}}(u,3) dv)(h^2_{i-1,t_{i-2}} 1_{[t_{i-2},t_{i-1})}(t) + h^2_{i-1,t_{i-1}} 1_{[t_{i-1},t_i)}(t) \end{aligned}$$