Appeared in **Pure Mathematics and Applications**, 11(2000), 209-217

Scalar and vector variational inequalities

András Domokos
Babeş-Bolyai University
Department of Applied Mathematics
str. M. Kogalniceanu 1
3400 Cluj-Napoca, Romania
e-mail: domokos@math.ubbcluj.ro

Abstract

The aim of this paper is to show connections between the sensitivity of the solutions of variational inequalitites and properties of the solution sets of vector variational inequalities in the infinite dimensional setting. We will show that, in some cases, the solution set of a vector variational inequality is a bijective and continuous image of a connected subset of the surface of the unit ball.

Mathematics Subject Classification (2000): 47J20

1 Introduction

In this paper we will study the properties of the solution sets of vector variational inequalities, using existence and data dependence results for the solutions of scalar variational inequalities. The purpose of this paper is to prove the connectedness of the solution sets of vector variational inequalities under weaker assumptions and in a more general setting as in [6]. First we will recall a result regarding the sensitivity of variational inequalities [4]. Similar results appeared in [1, 3, 9] under stronger monotonicity and continuity assumptions in finite dimensional or Hilbert spaces. If we read carefully the proof from [4] we realize that the joint-continuity property of the mapping

f can be weakened. We will write Theorem 1.1 using this weaker assumption and in a special case, namely the set-valued mapping C is constant. This theorem will be used to prove our results regarding the solution sets of vector variational inequalities, which constitutes important tools in the study of the solvability of vector optimization problems, since they represent necessary conditions for a point to be a weakly efficient solution [5, 6, 7, 10].

Let X be a reflexive Banach space, (Ω, d) be a metric space. Let $x_0 \in X$, $\omega_0 \in \Omega$, and their neighborhoods X_0 of x_0 and Ω_0 of ω_0. Let $f : X_0 \times \Omega_0 \to X^*$ be a single-valued mapping and $K \subset X$ be nonempty, closed and convex.
We denote by $N_K(x)$ the normal cone to K at x, i.e.

$$N_K(x) \;=\; \{x^* \in X^* \;:\; \langle x^* \,, y - x\rangle \;\leq\; 0 \,, \,\forall\, y \in K\} \;,$$

by $B(x,r)$ a closed ball and by $\operatorname{int} B(x,r)$ its interior.

Theorem 1.1 [4] *Let us suppose that:*
(a) $0 \in f(x_0, \omega_0) + N_K(x_0)$*;*
(b) $f(\cdot, \omega)$ *is hemicontinous on* X_0 *for all* $\omega \in \Omega_0$*, and* $f(x, \cdot)$ *is continuous on* Ω_0 *for all* $x \in X_0$*.*
(c) $f(\cdot, \omega)$ *is strictly-monotone for all* $\omega \in \Omega_0$*.*
(d) For all $\omega \in \Omega_0$*, for all sequences* (x_n) *from* X_0 *and* $x \in X_0$

$$\langle f(x_n, \omega) - f(x, \omega) \,, \, x_n - x\rangle \;\to\; 0 \;\;\Longrightarrow\;\; x_n \to x \,. \tag{1}$$

Then there exist a neighborhood Ω_1 *of* ω_0 *and a unique continuos mapping* $x : \Omega_1 \to X_0$*, such that* $x(\omega_0) = x_0$ *and* $x(\omega)$ *is the solution of the variational inequality*

$$0 \;\in\; f(x, \omega) \;+\; N_K(x) \;,$$

for all $\omega \in \Omega_1$*.*

2 Main results

Let X be a reflexive Banach space, Ξ be a Banach space, $K \subset X$ be nonempty, closed and convex, $C \subset \Xi$ be a nonempty closed, convex cone with nonempty interior such that $C \cap (-C) = \{0\}$. We denote by $L(X, \Xi)$ the space of all bounded, linear mappings from X into Ξ. Moreover, let

$$S_+^* \;=\; \{\xi^* \in \Xi^* \;:\; \|\xi^*\| = 1\,,\;\; \langle \xi^*, c\rangle \;\geq\; 0,\;\; \forall\;\; c \in C\;\}\;\;,$$

and

$$\operatorname{int} S_+^* \;=\; \{\xi^* \in \Xi^* \;:\; \|\xi^*\| = 1\,,\;\; \langle \xi^*, c\rangle \;>\; 0,\;\; \forall\;\; c \in C \setminus \{0\}\;\}\;\;.$$

Using a mapping $F : K \to L(X, \Xi)$ we consider the following vector variational inequalities:

$$\left\{ \begin{array}{c} \text{Find } \;\overline{x} \in K \;\text{ such that} \\ F(\overline{x})(x - \overline{x}) \;\notin -C \setminus \{0\}\,,\;\; \forall\, x \in K \end{array} \right. \tag{VVI}$$

and

$$\left\{ \begin{array}{c} \text{Find } \;\overline{x} \in K \;\text{ such that} \\ F(\overline{x})(x - \overline{x}) \;\notin -\operatorname{int} C\,,\;\; \forall\, x \in K \end{array} \right. \tag{VVI_w}$$

For $\xi^* \in S_+^*$ arbitrarily chosen, we can consider the following scalar variational inequality:

$$\left\{ \begin{array}{c} \text{Find } \;\overline{x} \in K \;\text{ such that} \\ \langle \xi^* \circ F(\overline{x})\,,\, x - \overline{x}\rangle \;\geq 0\,,\;\; \forall\, x \in K \end{array} \right. \tag{VI_{ξ^*}}$$

The solution sets of the above problems we will denote by: $\operatorname{sol}(VVI)$, $\operatorname{sol}(VVI_w)$ and $\operatorname{sol}(VI_{\xi^*})$.
Basic properties of these solution sets are presented in the following proposition. A similar result is stated in the finite dimensional case in [6].

Proposition 2.1
(a)

$$\begin{aligned} \Sigma = \bigcup_{\xi^* \in \operatorname{int} S_+^*} &\operatorname{sol}(VI_{\xi^*}) \subseteq \operatorname{sol}(VVI) \subseteq \\ &\subseteq \operatorname{sol}(VVI_w) = \bigcup_{\xi^* \in S_+^*} \operatorname{sol}(VI_{\xi^*}) \subseteq \overline{\Sigma}\,. \end{aligned} \tag{2.1}$$

(b) If F is norm to norm continuous, then $\operatorname{sol}(VVI_w)$ *is closed.*

Proof. Let us prove the first inclusion. For this, let $\xi^* \in \operatorname{int} S_+^*$ and $\overline{x} \in \operatorname{sol}(VI_{\xi^*})$. Hence,

$$\langle \xi^* \circ F(\overline{x}) \, , \, x - \overline{x} \rangle \; = \; \langle \xi^* \, , \, F(\overline{x}) \, (x - \overline{x}) \rangle \; \geq \; 0 \, , \;\; \forall \, x \in K \; . \qquad (2.2)$$

Let us suppose that there exists $x \in K$ such that $F(\overline{x})\,(x - \overline{x}) \in -C \setminus \{0\}$. In this case $\langle \xi^* \, , \, F(\overline{x})\,(x-\overline{x}) \rangle \; < \; 0$, which is a contradiction with (2.2).
Consequently, we have proved that $\overline{x} \in \operatorname{sol}(VVI)$.
The second inclusion is obvious.
In order to prove the equality, let $\xi^* \in S_+^*$ and $\overline{x} \in \operatorname{sol}(VI_{\xi^*})$.
Then for each $x \in K$ we have $F(\overline{x})\,(x-\overline{x}) \notin -\operatorname{int} C$, because otherwise $\langle \xi^* \, , \, F(\overline{x})\,(x - \overline{x}) \rangle \; < \; 0$, which is not possible because of $\xi^* \in S_+^*$.
So $\overline{x} \in \operatorname{sol}(VVI_w)$.
For the reverse inclusion, let $\overline{x} \in \operatorname{sol}(VVI_w)$. Then

$$\{F(\overline{x})\,(x - \overline{x}) \; , \; x \in K\} \; \bigcap \; \{-\operatorname{int} C\} \; = \; \emptyset \; ,$$

and using the separation theorem for the above two convex sets, we can find $\xi^* \in \Xi^* \setminus \{0\}$, $\|\xi^*\| = 1$, such that

$$\langle \xi^* \, , \, F(\overline{x})\,(x - \overline{x}) \rangle \; \geq \; 0 \; , \;\; \forall x \in K \; ,$$

and

$$\langle \xi^* \, , \, -c \rangle \; \leq \; 0 \; , \;\; \forall c \in \operatorname{int} C \; .$$

Hence $\xi^* \in S_+^*$.
(b) Since the mapping $G_x\,(z) \; = \; F(z)(x - z)$ is continuous in z, it follows that the set $K(x) \; = \; G_x^{-1}\,(\Xi \setminus (-\operatorname{int} C))$ is closed, and so is $\operatorname{sol}(VVI_w) \; = \; \bigcap_{x \in K} \, K(x)$.

The following theorem is our first result regarding the properties of $\operatorname{sol}(VVI)$ and $\operatorname{sol}(VVI_w)$. In contrast with the Theorems 4.1 and 4.2 of [6], we use weaker monotonicity and continuity assumptions. We will strengthen then the assumptions in order to get better and better properties for the solution sets.

Theorem 2.1 *Let us suppose that:*
(a) *There exists* $\xi_0^* \in S_+^*$ *such that the mapping*

$$f(\cdot \, , \, \xi_0^*) \; = \; \xi_0^* \circ F \; : \; K \rightarrow X^*$$

is φ-monotone ([2]), i.e. there exists a continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $\varphi(r) > 0$ for $r > 0$, such that

$$\langle f(x,\xi_0^*) - f(y,\xi_0^*)\,,\, x-y\rangle \;\geq\; \varphi\left(\|x-y\|\right)\;,\;\; \forall\, x\,,\, y \in K\;.$$

(b) There exists a continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\psi(0) = 0$, such that

$$\|F(x) - F(y)\| \;\leq\; \psi\left(\|x-y\|\right)\;,\;\; \forall\, x\,,\, y \in K\;.$$

(c) There exists $\varepsilon > 0$ such that

$$\varphi(r) - \varepsilon\, r\, \psi(r) \;>\; 0\,,\; \forall\, r > 0\;.$$

(d) There exists $x_0 \in K$ such that $\langle f(x_0,\xi_0^*)\,,\, x - x_0\rangle \;\geq\; 0\,,\; \forall\, x \in K$. Then $sol(VVI) \neq \emptyset$, $sol(VVI_w) \neq \emptyset$ and both contain a continuous image of a connected subset of S_+^*.

Proof. Let us define the mapping $f : K \times S_+^* \to X^*$ by

$$f(x,\xi^*) \;=\; \xi^* \circ F(x)\;.$$

For all $x \in K$ the mapping $f(x,\cdot)$ is Lipschitz-continuous, because

$$\|f(x,\xi_1^*) - f(x,\xi_2^*)\| \;=\; \|(\xi_1^* - \xi_2^*) \circ F(x)\| \;\leq\; \|\xi_1^* - \xi_2^*\| \cdot \|F(x)\|\;.$$

For all $\xi^* \in S_+^*$, the mapping $f(\cdot,\xi^*)$ is uniformly continuous, because

$$\|f(x,\xi^*) - f(y,\xi^*)\| \;=\; \|\xi^* \circ F(x) - \xi^* \circ F(y)\| \;=\; \|\xi^* \circ (F(x) - F(y))\| \;\leq$$

$$\|\xi^*\| \cdot \|F(x) - F(y)\| \;\leq\; \psi\left(\|x-y\|\right)\;.$$

Let $\xi^* \in S_+^*$ such that $\|\xi^* - \xi_0^*\| \leq \varepsilon$. Then

$$\langle f(x,\xi^*) - f(y,\xi^*)\,,\, x-y\rangle \;=\; \langle f(x,\xi_0^*) - f(y,\xi_0^*)\,,\, x-y\rangle +$$

$$\langle f(x,\xi^*) - f(y,\xi^*) - f(x,\xi_0^*) + f(y,\xi_0^*)\,,\, x-y\rangle \;\geq$$

$$\varphi\left(\|x-y\|\right) \,+\, \langle(\xi^* - \xi_0^*)\;,\, (F(x) - F(y))\;(x-y)\rangle \;\geq$$

$$\varphi\left(\|x-y\|\right) \,-\, \|\xi^* - \xi_0^*\| \cdot \|F(x) - F(y)\| \cdot \|x-y\| \;\geq$$

$$\varphi\left(\|x-y\|\right) \,-\, \varepsilon\, \psi\left(\|x-y\|\right)\, \|x-y\|\;,$$

which shows that the mappings $f(\cdot,\xi^*)$ are $(\varphi - \varepsilon\,\psi I)$-monotone for all $\xi^* \in S_+^*$ with $\|\xi^* - \xi_0^*\| \leq \varepsilon$. This kind of monotonicity implies the

assumptions (c) and (d) of Theorem 1.1. Assumption (a) of Theorem 1.1 follows from assumption (d) (supposing $\omega_0 = \xi_0^*$).
Using Theorem 1.1 we deduce that there exists $0 < \tilde{\varepsilon} \le \varepsilon$ such that for all $\xi^* \in S_+^* \cap B(\xi_0^*, \tilde{\varepsilon})$ there exists a unique solution $x(\xi^*)$ of (VI_{ξ^*}), and $x(\xi^*)$ depends continuously on ξ^*. Using (a) of Proposition 2.1 we deduce that $\mathrm{sol}\,(VVI)$ and $\mathrm{sol}\,(VVI_w)$ are nonempty. Moreover $\mathrm{sol}\,(VVI_w)$ contains a continuous image of $S_+^* \cap B(\xi_0^*, \tilde{\varepsilon})$ and $\mathrm{sol}\,(VVI)$ contains a continuous image of $\mathrm{int}\, S_+^* \cap B(\xi_0^*, \tilde{\varepsilon})$.

Corollary 2.1 *If the function φ from assumption (a) of Theorem 2.1 is of the form $\varphi(r) = \varphi_1(r) \cdot r$, where φ_1 is continuous, strictly increasing function with $\varphi_1(0) = 0$ and $\varphi_1(r) \to +\infty$ as $r \to +\infty$ (see the definition of uniform-monotonicity* [11]*), and if in the assumption (c) of Theorem 2.1 the constant $\varepsilon > 0$ can be taken greater than the diameter of S_+^*, then sol(VVI) and sol(VVI_w) are arcwise connected. Moreover, sol(VVI_w) is closed, and if Ξ is finite dimensional then it is also compact.*

Proof. Since $\varepsilon > 0$ is large enough and the mappings $f(\cdot, \xi^*)$ are coercive in this case, it follows that (VI_{ξ^*}) has a unique solution for all $\xi^* \in S_+^*$. Theorem 1.1 can be used to prove that $x(\cdot)$ is continuous in each $\xi^* \in S_+^*$, so by Proposition 2.1 $\mathrm{sol}\,(VVI_w)$ is closed and it is a continuous image of S_+^* and hence arcwise connected.
$\mathrm{sol}\,(VVI)$ is also arcwise connected because

$$\Sigma \subseteq \mathrm{sol}\,(VVI) \subseteq \mathrm{sol}\,(VVI_w) \subseteq \overline{\Sigma}\,. \tag{2.3}$$

Indeed, let $x(\xi_1^*), x(\xi_2^*) \in \mathrm{sol}\,(VVI)$ with $\xi_1^*, \xi_2^* \in S_+^*$. Then there exists a continuous mapping $\gamma : [0,1] \to S_+^*$, such that $\gamma(0) = \xi_1^*$, $\gamma(1) = \xi_2^*$ and $\gamma((0,1)) \subset \mathrm{int}\, S_+^*$.
Hence $x \circ \gamma((0,1)) \subset \Sigma \subset \mathrm{sol}\,(VVI)$, which proves the arcwise connectedness of $\mathrm{sol}\,(VVI)$.
If Ξ is finite dimensional, then S_+^* is a compact set and $\mathrm{sol}\,(VVI_w)$, as its continuous image, it is also compact.

Remark 2.1 If, as in [6], $f(\cdot, \xi_0^*)$ is strongly-monotone, i.e. there exists $a > 0$ such that

$$\langle f(x, \xi_0^*) - f(y, \xi_0^*)\,,\, x - y \rangle \;\ge\; a\,\|x - y\|^2\,, \;\; \forall\, x, y \in X_0\,, \tag{2.4}$$

and F is Lipschitz-continuous, i.e. there exists $l \geq 0$ such that

$$\|F(x) - F(y)\| \ \leq \ l \, \|x - y\| \, , \ \ \forall \, x, y \in K \ ,$$

then assumptions (c) and (d) of Theorem 2.1 are satisfied.
Indeed, an $\varepsilon > 0$ exists such that $a > \varepsilon \, l$ and the strong-monotonicity of $f(\cdot, \xi_0^*)$ being also a coercivity condition, the solution x_0 exists.

In the following corollary we do not suppose the uniform norm-to-norm continuity of F as we did in the assumption (b) of Theorem 2.1, in order to exploit the possibility that $f(\cdot, \xi^*)$ can be only hemicontinuous.

Corollary 2.2 *Let us suppose that the function φ has the same properties as in Corollary 2.1 and:*
(a) *For all $\xi^* \in S_+^*$ the mappings $f(\cdot, \xi^*) = \xi^* \circ F(\cdot)$ are φ-monotone.*
(b) *F is continuous from the norm topology of X to the topology of pointwise convergence of $L(X, \Xi)$.*
Then $sol(VI_{\xi^*}) \neq \emptyset$ for all $\xi^* \in S_+^*$, and $sol(VVI)$ and $sol(VVI_w)$ are arcwise connected.

Proof. As in the proof Theorem 2.1, the mappings $f(x, \cdot)$ are Lipschitz-continuous for all $x \in K$.
Assumption (b) implies that for all $\xi^* \in S_+$ the mappings $f(\cdot, \xi^*)$ are demicontinuous (hence hemicontinuous [8]) on K. Indeed, let $(x_n) \subset K$ and $x \in K$ such that $x_n \to x$. Using the pointwise convergence we get that

$$F(x_n)(u) \to F(x)(u) \ , \ \ \forall \ u \in K \ ,$$

and hence

$$\xi^* \circ F(x_n)(u) \to \xi^* \circ F(x)(u) \ , \ \forall \ u \in X \ ,$$

which means that $f(x_n, \xi^*) \rightharpoonup f(x, \xi^*)$. Now we can use Theorem 1.1 and follow the proof of Corollary 2.1 to get the desired properties.

We will now prove the injectivity of the mapping $x : S_+^* \to \mathrm{sol}\,(VVI_w)$. Togheter with the surjectivity proved in Corollary 2.1 and 2.2, this means that $x : S_+^* \to \mathrm{sol}(VVI)_w$ is bijective and continuous.
Let X be a reflexive, strictly-convex Banach space with strictly-convex

dual. Let $K \subset X$ be a closed, convex set with nonempty interior and with a sufficiently smooth boundary in the sense that the normal cone $N_K(x)$ to each point x of the boundary, is a half line. Such an example can be given in the case of $K = B(0, r)$, when

$$N_K(x) \;=\; \begin{cases} \{0\}\,, & \|x\| < r \\ \{\lambda J(x)\,,\, \lambda \geq 0\}\,, & \|x\| = r \\ \emptyset\,, & \|x\| > r\,, \end{cases}$$

where $J : X \to X^*$ is the normalized duality mapping.

Corollary 2.3 *Let us suppose that the assuptions of Corollary 2.2 hold, $K \subset X$ has the above properties and the linear mappings $F(x) : X \to \Xi$ are surjective for all $x \in K$. Then the mapping $x : S_+^* \to sol(VVI_w)$ is injective.*

Proof. Let $\xi_1^*, \xi_2^* \in S_+^*$ such that $\xi_1^* \neq \xi_2^*$. Then ξ_1^* and ξ_2^* are linearly independent and for $x \in K$ arbitrarily chosen, the linear mappings $\xi_1^*(F(x))$ and $\xi_2^*(F(x))$ are also linearly independent. Indeed, because otherwise there exists $\lambda \neq 0$ sucht that $\xi_1^*(F(x))(u) = \lambda\xi_2^*(F(x))(u)$ for all $u \in X$.The surjectivity of $F(x)$ implies that $\xi_1^* = \lambda\xi_2^*$ which is impossible.
If $x(\xi_1^*)$ is the solution of the problem $(VI_{\xi_1^*})$, then

$$-\xi_1^*(F(x(\xi_1^*)) \in N_K(x(\xi_1^*))\;.$$

$\xi_2^*(F(x(\xi_1^*))$ being linearly independent from $\xi_1^*(F(x(\xi_1^*))$ follows that $x(\xi_1^*)$ is not the solution of the problem $(VI_{\xi_2^*})$. Hence the mapping $x(\cdot)$ is injective.

3 The finite dimensional case

In this section we want to show that in the case of a finite dimensional space Ξ our results from the section 2 generalize the already known scalarization methods for finite dimensional vector variational inequalities. Let $\Xi = \mathbb{R}^n$ and $C = \mathbb{R}_+^n$. Then

$$S_+^* \;=\; S_+ \;=\; \left\{\xi \in \mathbb{R}_+^n \;:\; \|\xi\| = 1\right\}$$

and

$$\operatorname{int} S_+^* \;=\; \operatorname{int} S_+ \;=\; \left\{\xi \in \operatorname{int}\mathbb{R}_+^n \;:\; \|\xi\| = 1\right\}\;.$$

The mapping F will be $F = (F_1, ..., F_n)$, where $F_i : X \to X^*$ and

$$F(x)(v) \,=\, (\langle F_1(x)\,,\,v\rangle, ..., \langle F_n(x)\,,\,v\rangle)\ .$$

The vector variational inequalities are:

$$\left\{\begin{array}{c} \text{Find } \ \overline{x} \in K \ \text{ such that} \\ F(\overline{x})(x - \overline{x}) \,\notin -\mathbb{R}^n_+ \setminus \{0\}\,, \ \ \forall\, x \in K \end{array}\right. \tag{$FVVI$}$$

and

$$\left\{\begin{array}{c} \text{Find } \ \overline{x} \in K \ \text{ such that} \\ F(\overline{x})(x - \overline{x}) \,\notin -\mathrm{int}\,\mathbb{R}^n_+\,, \ \ \forall\, x \in K\ . \end{array}\right. \tag{$FVVI_w$}$$

For $\xi = (\xi_1, ..., \xi_n) \in S_+$ and the mapping

$$f(x,\xi) \,=\, \sum_{i=1}^{n} \xi_i\, F_i(x)$$

we can consider the following scalar variational inequality:

$$\left\{\begin{array}{c} \text{Find } \ \overline{x} \in K \ \text{ such that} \\ \langle \sum_{i=1}^n \xi_i\, F_i(\overline{x})\,,\, x - \overline{x}\rangle \,\geq 0\,, \ \ \forall\, x \in K \end{array}\right. \tag{FVI_ξ}$$

Proposition 1.1 and Theorem 2.1 are valid in this special case. They constitute generalizations of the results from [6]. Using the advantages of the finite dimensionality of Ξ, we will now state another version of Theorem 2.1. In this case we can weaken the monotonocity and continuity assumptions.

Theorem 3.1 *Let us suppose that:*
(a) *There exists* $i_0 \in \{1, ..., n\}$ *such that* F_{i_0} *is strictly-monotone and*

$$\langle F_{i_0}(x_n) - F_{i_0}(x)\,,\, x_n - x\rangle \,\to\, 0 \ \ \Rightarrow \ \ x_n \to x$$

for all sequences $(x_n) \subset K$ *and* $x \in K$*.*
(b) *The mappings* F_i *are monotone and hemicontinuous.*
(c) *The mapping* F_{i_0} *is coercive (in the case of unbounded* K*).*
Then for all $\xi \in int\,S_+$ *there exists a solution* $x(\xi) \in K$ *of the problem* (FVI_ξ)*. Moreover,* $sol(FVVI)$ *contains the continuous image of* $int\,S_+$*.*

Proof. Let $\overline{\xi} = (\overline{\xi}_1, ..., \overline{\xi}_n) \in \operatorname{int} S_+$. Then $\overline{\xi}_i > 0$ for all $i \in \{1, ..., n\}$, particularily $\overline{\xi}_{i_0} > 0$.
Let $0 < \varepsilon < \overline{\xi}_{i_0}$. Let Ξ_0 be an intersection of a neighborhood of $\overline{\xi}$ with S_+ such that for all $\xi \in \Xi_0$, $\xi_{i_0} \geq \varepsilon > 0$ holds.
Then for all $\xi \in \Xi_0$, $x_n, x \in K$ we have

$$\langle f(x_n, \xi) - f(x, \xi)\, ,\, x_n - x \rangle \ \geq \ \varepsilon \,\langle F_{i_0}(x_n) - F_{i_0}(x)\, ,\, x_n - x \rangle \geq 0\ .$$

Using (a) and (b), we conclude that f satisfies assumption (c) and (d) of Theorem 1.1.
The hemicontinuity of F_i's imply that $f(\cdot, \xi)$ is hemicontinuous for all $\xi \in S_+$. Assumptions (b), (c) imply that $f(\cdot, \overline{\xi})$ is strictly-monotone, hemicontinuous and coercive (in the case of unbounded K), so there exists a unique $\overline{x} \in \operatorname{sol}(FVI_{\overline{\xi}})$.
Using Theorem 1.1 we get that $\overline{\xi}$ has a neighborhood Ξ_1 in S_+ and there exists a unique continuous mapping $x : \Xi_1 \to K$ such that $x(\overline{\xi}) = \overline{x}$ and $x(\xi) \in \operatorname{sol}(FVI_\xi)$ for all $\xi \in \Xi_1$.
$\overline{\xi}$ being chosen arbitrarily, follows that $\operatorname{sol}(FVI_\xi) \neq \emptyset$ for all $\xi \in \operatorname{int} S_+$.

Corollary 3.1 *If the assumption (b) of Theorem 3.1 is valid and assumptions (a) and (c) hold for each $i \in \{1, ..., n\}$, then sol$(FVI_\xi) \neq \emptyset$ for all $\xi \in S_+$ and, as in Corollary 2.1, sol$(FVVI)$ and sol$(FVVI_w)$ are nonempty and arcwise connected. Moreover, sol$(FVVI)$ is compact.*

Proof. It is enough to observe that this stronger form of assumption (a) implies that for all $\xi \in S_+$, $f(\cdot, \xi)$ is monotone, hemicontinuous and coercive, so $\operatorname{sol}(FVI_\xi) \neq \emptyset$. Then we can follow the proof of Corollary 2.1.

Acknowledgment: The author is grateful to the referee for helpful comments.

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