THE SMALL GROWTH INVARIANTS OF GOURSAT DISTRIBUTIONS

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Abstract. This is the second of a pair of papers devoted to the local invariants of Goursat distributions. The study of these distributions naturally leads to a tower of spaces over an arbitrary surface, called the monster tower, and thence to connections with the topic of singularities of curves on surfaces. In the prior paper we studied those invariants of Goursat distributions akin to those of curves on surfaces, which we call structural invariants. In this paper we study invariants arising from the small growth sequence of a Goursat distribution, and relate them to the the structural invariants.

1. Introduction

A *distribution* D on a manifold M is a subbundle of the tangent bundle TM. It is called *Goursat* if the *Lie square sequence*

$$D = D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$$

(as defined in Section 3) is a sequence of vector bundles for which

$$\operatorname{rank} D_{i+1} = 1 + \operatorname{rank} D_i$$

until one reaches the full tangent bundle. This is the second of a pair of papers devoted to the local invariants of Goursat distributions. In the first paper [2] we gave an account of those invariants that are akin to well-known invariants in the theory of curves on surfaces. Here we turn our attention to invariants stemming more directly from the definition of Goursat distributions, which have no existing counterpart in the theory of curves on surfaces; we will call them *small growth invariants*. We continue, however, to use the connection with that other theory in order to analyze them and work out recursive schemes for computing them. The invariants of the earlier paper, when applied in the context of curves on surfaces, are all what singularity theorists call *complete topological invariants*. The small growth invariants are somewhat coarser: this is because curves that are not of the same topological type may define the same Goursat distribution. Nevertheless, they seem to be more challenging to calculate.

Figure 1 shows most of the invariants that are analyzed in the two papers. (In the body of this paper we will introduce several other invariants, including the table of values for the quantities e_{hi} of Section 8, its associated *b vector*, and the *proximity diagram*.) Figure 2 presents an example, using the same layout.

As explained in Section 2, we give our explanations and arguments simultaneously in three settings: smooth manifolds, complex manifolds, and nonsingular algebraic varieties. We begin in Section 3 by reviewing the definitions of the Lie square sequence, Goursat

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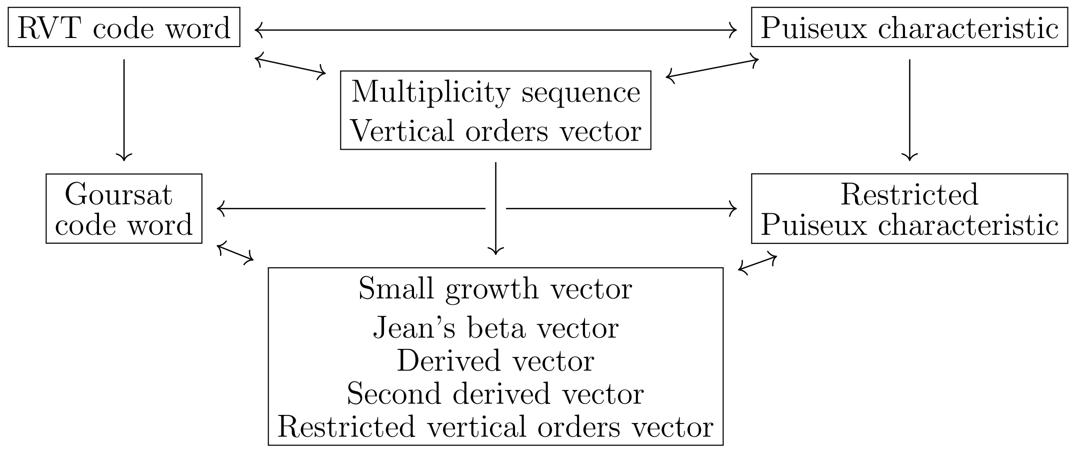

FIGURE 1. The invariants listed in the top five boxes were studied in our prior paper [2]; those listed in the bottom box are invariants of the small growth sequence of a Goursat germ. See Figure 2 for an example.

distributions, the small growth sequence and small growth vector, Jean's beta vector and its derived vectors, and the degree of nonholonomy. In Section 4 we briefly recall the content of our earlier paper [2]. We merely name the most essential vocabulary from that paper; to fully understand what we are doing here, one will need to read many of its details. Since we make extensive use of the coordinate systems on the standard charts of the monster spaces, in Section 5 we recall their construction; we also introduce two sequences of standard vector fields. In the rather technical Section 6, we work out formulas for the Lie brackets of these vector fields, and find a particular basis in which these formulas look relatively simple. Section 7 explicates the notions of focal order and vertical order for functions on the monster spaces. Section 8 presents Theorem 19, our main technical result; it provides detailed information about the sheaves in the small growth sequence, leading to inequalities (in Corollary 22) that compare structural invariants and small growth invariants. We want to show that in fact we have equalities, and to do so we need to exhibit specific sections of the sheaves. We do exactly that in Section 9; the basic idea is that we want to avoid certain cancellations of leading terms. The remaining sections are the payoff. Section 10 briefly explains the simple relationships between three structural invariants and their three small growth counterparts. Section 11 discusses how one can calculate these six invariants as functions of the RVT or Goursat code words via recursion. Our line of argument naturally leads to front-end recursions, but we also remark on back-end recursions, most notably that of Jean for his beta vector [3]. We conclude in Section 12 by showing that the degree of nonholonomy for a Goursat distribution coincides with the last entry in its associated Puiseux characteristic; it seems that this was never noticed previously.

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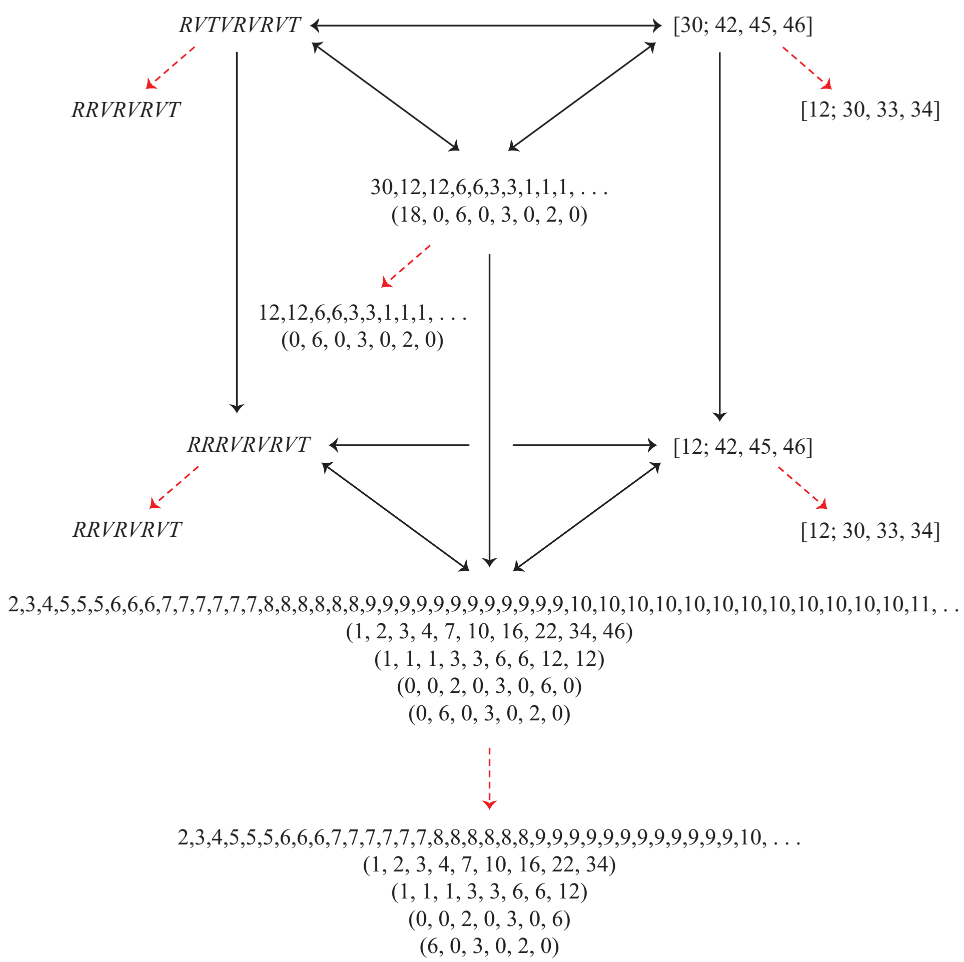


FIGURE 2. An example of the invariants of Figure 1. The dashed arrows indicate compatible front-end recursions.

2. THE THREE SETTINGS

We begin with a space M of dimension $m \geq 2$, by which we mean either

(1) a smooth manifold, or
(2) a complex manifold, or
(3) a nonsingular algebraic variety over an algebraically closed field of characteristic 0.

All the constructions of this paper are understood to be in the chosen setting, e.g., in setting 2 we work with holomorphic sections of holomorphic bundles, with m denoting the complex dimension of M, whereas in setting 3 we work with algebraic bundles and sections. In a similar way, we use the word "surface" to mean either a smooth manifold of dimension 2, a complex manifold of dimension 2, or a nonsingular algebraic surface over an algebraically closed field of characteristic 0, with a similar convention for the word "curve." We remark that although in our prior paper [2] we assumed the first setting, all of the results obtained there are equally valid in all three settings.

3. Small growth

We repeat here selected definitions from [2]. Let D be a distribution on M, i.e., a subbundle of its tangent bundle TM. Let $\mathcal{E}$ be its sheaf of sections, which is a subsheaf of the sheaf Θ_M of sections of TM. In other words, let Θ_M be the sheaf of vector fields on M, and let $\mathcal{E}$ be the subsheaf of vector fields tangent to D.

The *Lie square* of $\mathcal{E}$ is

$$\mathcal{E}_2 = [\mathcal{E}, \mathcal{E}],$$

meaning the subsheaf of Θ_M whose sections are generated by Lie brackets of sections of $\mathcal{E}$ and the sections of $\mathcal{E}$ itself. Note that in general the rank of $\mathcal{E}_2$ may vary from point to point. If, however, $\mathcal{E}_2$ is the sheaf of sections of a distribution D_2, then this rank is constant. Beginning with $\mathcal{E}_1 = \mathcal{E}$, recursively we define $\mathcal{E}_{i+1}$ to be the Lie square of $\mathcal{E}_i$, and we call

$$\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{E}_3 \subseteq \cdots \tag{3.1}$$

the *Lie square sequence*.

Let d be the rank of D. We say that D is a *Goursat distribution* if each sheaf $\mathcal{E}_i$ is the sheaf of sections of a distribution D_i and if

$$\operatorname{rank} D_{i+1} = 1 + \operatorname{rank} D_i$$

for $i = 1, \ldots, m-d$. In particular D_{m-d+1} is the tangent bundle TM. For a Goursat distribution, the sequence

$$D = D_1 \subset D_2 \subset D_3 \subset \cdots \subset D_{m-d} \subset D_{m-d+1} = TM$$

is also called the Lie square sequence. Since a distribution of rank 1 is always integrable, a Goursat distribution necessarily has rank at least 2. As explained at the end of Section 6 of [2], the essential case for understanding all Goursat distributions is the case of rank 2. For the remainder of the paper we assume that D is a Goursat distribution with $d = 2$.

Again denoting the sheaf of sections of D by $\mathcal{E}$, we consider the *small growth sequence*:

$$\mathcal{E} = \mathcal{E}^1 \subseteq \mathcal{E}^2 \subseteq \mathcal{E}^3 \subseteq \cdots$$

defined by

$$\mathcal{E}^j = [\mathcal{E}, \mathcal{E}^{j-1}],$$

meaning the subsheaf of Θ_M whose sections are generated by Lie brackets of sections of $\mathcal{E}$ with sections of $\mathcal{E}^{j-1}$ and by the sections of $\mathcal{E}^{j-1}$. Note how this differs from the Lie square sequence in (3.1): at each step we form Lie brackets with vector fields from the beginning distribution. For each point $p \in M$ we let SG_i denote the rank of $\mathcal{E}^i$ at p; we call

$$\mathrm{SG}(D, p) = \mathrm{SG}_1, \mathrm{SG}_2, \ldots$$

the *small growth vector*. At each step the rank grows at most by one, and eventually the entries of SG stabilize at the value m. Simple examples show that this vector may differ from point to point of M.

Jean's *beta vector* [3]

$$\beta(D, p) = (\beta_2, \beta_3, \ldots, \beta_m)$$

records the positions in the small growth vector where the rank increases:

$$\beta_i = \min\{j : \mathrm{SG}_j = i\}.$$

The last entry β_m, which records how many steps it takes to reach the full tangent bundle, is called the *degree of nonholonomy*.

From the definitions we immediately see that $\mathcal{E}^1 = \mathcal{E}_1$, $\mathcal{E}^2 = \mathcal{E}_2$, and that $\mathcal{E}^3$ is a subsheaf of $\mathcal{E}_3$. In fact $\mathcal{E}^3 = \mathcal{E}_3$. To establish the opposite containment, we consider a generator of $\mathcal{E}_3$ and write it, using the Jacobi identity, as a sum of two sections of $\mathcal{E}^3$:

$$[[a, b], [c, d]] = -[c, [d, [a, b]]] + [d, [c, [a, b]]].$$

Thus the small growth vector of a rank 2 Goursat distribution always begins with $2, 3, 4$ and the initial components of the beta vector are $1, 2, 3$.

The differences of successive terms

$$\mathrm{der}_i = \beta_i - \beta_{i-1}$$

form the *derived vector*

$$\mathrm{der}(D, p) = (\mathrm{der}_3, \mathrm{der}_4, \ldots, \mathrm{der}_m),$$

and the second differences

$$\mathrm{der}^2{}_i = \mathrm{der}_i - \mathrm{der}_{i-1}$$

form the *second derived vector*

$$\mathrm{der}^2(D, p) = (\mathrm{der}^2{}_4, \mathrm{der}^2{}_5, \ldots, \mathrm{der}^2{}_m).$$

We note that the derived vector must begin with a pair of 1's and the second derived vector begins with 0.

Example 1. If the small growth vector of D at p is

$$\mathrm{SG}(D, p) = 2, 3, 4, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 8, 8, 9, 9, 9, 9, 9, 9, 9, 9, 9, 10, 10, \ldots$$

then

$$\beta(D, p) = (1, 2, 3, 4, 5, 8, 11, 17, 26)$$
$$\mathrm{der}(D, p) = (1, 1, 1, 1, 3, 3, 6, 9)$$
$$\mathrm{der}^2(D, p) = (0, 0, 0, 2, 0, 3, 3).$$

4. Recollection of the first paper

This paper relies heavily on our previous paper [2]. In the following breezy recollection, we italicize vocabulary whose definitions can be found there. In that paper we recalled, following [5] and [6], how one assigns a *Goursat code word* to the germ of a Goursat distribution at a point. Following [5], we explained the notion of *prolongation* of Goursat distributions, which leads to a construction of the *monster tower*, a tower of *monster spaces* $S(k)$ over a surface S; each $S(k)$ is a space of dimension $k + 2$. These spaces are universal for Goursat distributions: given a germ of Goursat distribution of rank 2 and corank k, one can find a point on $S(k)$ for which the germ of the *focal distribution* $\Delta(k)$ at that point is *equivalent* to the specified germ. The spaces $S(k)$ are naturally stratified by their *divisors at infinity* and the prolongation of these divisors; using them we introduce *RVT code words*, which agree with the Goursat code words if one avoids the divisor at infinity I_2. Working with *focal curve germs*, we defined various *structural invariants* — those shown in the top five boxes of Figure 1 — and explicated recursive schemes for calculating them. We introduced the notion of *Goursat invariants*; the present paper is devoted to further study of these invariants.

5. Charts, coordinates, focal vector fields

We will use the standard charts on the monster space $S(k)$ explained in Section 8 of [2]. We repeat here, mostly verbatim, our descriptions of those charts.

We begin with a specified chart U on S with coordinates $\mathbf{r}_0$ and $\mathbf{n}_0$. On $S(k)$ there are 2^k standard charts over U, each of which is a copy of $U \times \mathbf{A}^k$, and on each such chart there are $k+2$ coordinate functions, namely $\mathbf{r}_0$ and $\mathbf{n}_0$ together with k affine coordinates. At each level j, by a recursive procedure, two of these coordinates are designated as *active coordinates*. One is the *new coordinate* $\mathbf{n}_j$ and the other is the *retained coordinate* $\mathbf{r}_j$. In addition, for $j > 0$, a third coordinate is designated as the *deactivated coordinate* $\mathbf{d}_j$.

To describe the recursive procedure, we begin with a standard chart on $U(j)$ with coordinates $\mathbf{n}_j$, $\mathbf{r}_j$, and $\mathbf{d}_j$ together with $j-1$ unnamed coordinates. At each point of the chart, the fiber of $\Delta(j)$ (except for the zero vector) consists of tangent vectors for which either the restriction of the differential $d\mathbf{n}_j$ or that of $d\mathbf{r}_j$ is nonzero. Create a standard chart at the next level by choosing one of the following two options:

- Assuming the restriction of $d\mathbf{r}_j$ is nonzero, let $\mathbf{n}_{j+1} = d\mathbf{n}_j/d\mathbf{r}_j$; then set $\mathbf{r}_{j+1} = \mathbf{r}_j$ and $\mathbf{d}_{j+1} = \mathbf{n}_j$. We call this the *ordinary choice*.
- Assuming the restriction of $d\mathbf{n}_j$ is nonzero, let $\mathbf{n}_{j+1} = d\mathbf{r}_j/d\mathbf{n}_j$; then set $\mathbf{r}_{j+1} = \mathbf{n}_j$ and $\mathbf{d}_{j+1} = \mathbf{r}_j$. We call this the *inverted choice*.

In every standard chart the names of the coordinates are $\mathbf{r}_0$, $\mathbf{n}_0$, $\mathbf{n}_1$, $\ldots$, $\mathbf{n}_j$, but their meaning depends on the chart. The charts are given names such as $\mathcal{C}(oiiooi)$, where each symbol o or i records which choice has been made, either ordinary or inverted.

In an alternative procedure for naming coordinates, we begin with coordinates $x^{(0)} = x$ and $y^{(0)} = y$. At each level, the two active coordinates will be $x^{(i)}$ and $y^{(j)}$, for some nonnegative integers i and j. If we create our chart at the next level by designating $x^{(i)}$ as the retained coordinate, then the new active coordinate is $y^{(j+1)} = dy^{(j)}/dx^{(i)}$; if we designate $y^{(j)}$ as the retained coordinate, then the new active coordinate is $x^{(i+1)} = dx^{(i)}/dy^{(j)}$.

As remarked in the earlier paper, the focal sheaf $\Delta(k)$ consists of those vector fields annihilated by the differential form

$$d\mathbf{d}_i - \mathbf{n}_i \, d\mathbf{r}_i \tag{5.1}$$

for $i = 1, \ldots, k$; we call them *focal vector fields*. Since k is fixed, we abbreviate to Δ. Similarly, the other sheaves Δ_i in the Lie square sequence, which (by equation (5.3) of [2]) are extensions of focal sheaves from spaces lower in the tower, consist of vector fields annihilated by (5.1) for $i = 1, \ldots, k-i+1$.

On each standard chart we now define two sequences of *standard vector fields*. The sequence v_0, v_1, $\ldots$, v_k is defined by

$$v_i = \frac{\partial}{\partial \mathbf{n}_i}. \tag{5.2}$$

The last vector field v_k is vertical, i.e., its projection to $S(k-1)$ is the zero vector field. For $1 \le i \le k-1$, each v_i projects to a vertical vector field at level i (and hence to the zero vector field at level $i-1$). The sequence f_0, f_1, $\ldots$, f_k is defined by recursion. To begin, set

$$f_0 = \frac{\partial}{\partial \mathbf{r}_0}\,. \tag{5.3}$$

If at level i we make the ordinary choice, then we define

$$f_i = f_{i-1} + \mathbf{n}_i \, v_{i-1} \, . \tag{5.4}$$

If at level i we make the inverted choice, then we define

$$f_i = \mathbf{n}_i f_{i-1} + v_{i-1} \, . \tag{5.5}$$

The last vector field f_k is focal, and each f_i projects to a focal vector field at level i.

Example 2. In chart $\mathcal{C}(ooioii)$ we have

$$\begin{aligned}
f_0 &= f_0 \\
f_1 &= f_0 + \mathbf{n}_1 v_0 \\
f_2 &= f_0 + \mathbf{n}_1 v_0 + \mathbf{n}_2 v_1 \\
f_3 &= \mathbf{n}_3 f_0 + \mathbf{n}_1\mathbf{n}_3 v_0 + \mathbf{n}_2\mathbf{n}_3 v_1 + v_2 \\
f_4 &= \mathbf{n}_3 f_0 + \mathbf{n}_1\mathbf{n}_3 v_0 + \mathbf{n}_2\mathbf{n}_3 v_1 + v_2 + \mathbf{n}_4 v_3 \\
f_5 &= \mathbf{n}_3\mathbf{n}_5 f_0 + \mathbf{n}_1\mathbf{n}_3\mathbf{n}_5 v_0 + \mathbf{n}_2\mathbf{n}_3\mathbf{n}_5 v_1 + \mathbf{n}_5 v_2 + \mathbf{n}_4\mathbf{n}_5 v_3 + v_4 \\
f_6 &= \mathbf{n}_3\mathbf{n}_5\mathbf{n}_6 f_0 + \mathbf{n}_1\mathbf{n}_3\mathbf{n}_5\mathbf{n}_6 v_0 + \mathbf{n}_2\mathbf{n}_3\mathbf{n}_5\mathbf{n}_6 v_1 + \mathbf{n}_5\mathbf{n}_6 v_2 + \mathbf{n}_4\mathbf{n}_5\mathbf{n}_6 v_3 + \mathbf{n}_6 v_4 + v_5 \, .
\end{aligned}$$

If we prefer the alternative coordinate names, we use x, y, and

$$\begin{aligned}
y' &= dy/dx \\
y'' &= dy'/dx \\
x' &= dx/dy'' \\
x'' &= dx'/dy'' \\
y^{(3)} &= dy''/dx'' \\
x^{(3)} &= dx''/dy^{(3)}.
\end{aligned}$$

In these coordinates we have

$$\begin{aligned}
f_0 &= \partial/\partial x \\
v_0 &= \partial/\partial y \\
v_1 &= \partial/\partial y' \\
v_2 &= \partial/\partial y'' \\
v_3 &= \partial/\partial x' \\
v_4 &= \partial/\partial x'' \\
v_5 &= \partial/\partial y^{(3)} \\
v_6 &= \partial/\partial x^{(3)}
\end{aligned}$$

and

$$
\begin{aligned}
f_1 &= f_0 + y' v_0 \\
f_2 &= f_0 + y' v_0 + y'' v_1 \\
f_3 &= x' f_0 + y' x' v_0 + y'' x' v_1 + v_2 \\
f_4 &= x' f_0 + y' x' v_0 + y'' x' v_1 + v_2 + x'' v_3 \\
f_5 &= x' y^{(3)} f_0 + y' x' y^{(3)} v_0 + y'' x' y^{(3)} v_1 + y^{(3)} v_2 + x'' y^{(3)} v_3 + v_4 \\
f_6 &= x' y^{(3)} x^{(3)} f_0 + y' x' y^{(3)} x^{(3)} v_0 + y'' x' y^{(3)} x^{(3)} v_1 + y^{(3)} x^{(3)} v_2 + x'' y^{(3)} x^{(3)} v_3 + x^{(3)} v_4 + v_5\,.
\end{aligned}
$$

The recursive formulas (5.4) and (5.5) lead to an explicit formula for f_i in terms of this basis. To state it, we let IP denote the set of positions at which we have made the inverted choice, and then define (for $1 \le i \le j \le k$)

$$a_{ij} = \prod_{\substack{h \in \mathrm{IP} \\ i+1 \le h \le j}} \mathbf{n}_h \tag{5.6}$$

and (for $0 \le i < j \le k$)

$$b_{ij} = \begin{cases} a_{i+1,\,j} & \text{if } i+1 \in \mathrm{IP} \\ \mathbf{n}_{i+1} a_{i+1,\,j} & \text{if } i+1 \notin \mathrm{IP}. \end{cases} \tag{5.7}$$

By an easy induction on j we have

$$f_j = a_{1j} f_0 + \sum_{i=0}^{j-1} b_{ij} v_i. \tag{5.8}$$

The vector fields

$$f_{k-i+1}, v_{k-i+1}, v_{k-i+2}, \ldots, v_k \tag{5.9}$$

form a basis of sections of Δ_i. One extreme case is the focal bundle Δ, whose sections are spanned by f_k and v_k. At the other extreme is the full tangent bundle $TS(k)$, whose sections are spanned by $f_0, v_0, v_1, \ldots, v_k$. For future reference, we note an alternative basis for Δ_i.

Lemma 3. *For each i, there is an alternative basis for Δ_i:*

(a) If $k-i+2 \notin \mathrm{IP}$, then $f_{k-i+2}, v_{k-i+1}, v_{k-i+2}, \ldots, v_k$ is an alternative basis.
(b) If $k-i+2 \in \mathrm{IP}$, then $f_{k-i+2}, f_{k-i+1}, v_{k-i+2}, \ldots, v_k$ is an alternative basis.

Proof. In case (a), formula (5.4) tells us that we may replace the basis element f_{k-i+1} by f_{k-i+2}. Similarly in case (b), formula (5.5) tells us that we may replace the basis element v_{k-i+1} by f_{k-i+2}. □

6. Lie derivatives and Lie brackets

In this section we work out some needed facts about Lie derivatives and Lie brackets, using the standard coordinate functions and standard vector fields of Section 5. As we will see, the formulas for Lie brackets are somewhat irregular. In Lemma 9, however, we will identify a particular basis mixing the standard vertical and focal vector fields (the precise mixture being determined by the choice of chart); for this basis there is a uniform formula, as expressed in property (5) of the lemma. This basis will be used in Theorem 19 to characterize the sections of the small growth sheaves.

If z is a vector field on $S(k)$ and a is a smooth function, we use the usual notation $z(a)$ for the Lie derivative. Looking at a specified standard chart on $S(k)$, we now present two lemmas about specific Lie derivatives.

Lemma 4. *For $i < j$ we have that*

$$f_j(\mathbf{n}_i) = b_{ij}. \tag{6.1}$$

Proof. This is an immediate consequence of formula (5.8). □

Lemma 5. *Suppose that the function a is a monomial in the standard coordinates. Then $v_k(a)$ and $f_k(a)$ are linear combination of monomials with positive coefficients.*

Proof. Formulas (5.2) and (5.3) tell us that $f_0, v_0, v_1, \ldots, v_k$ form a basis of vector fields, namely the partial derivatives with respect to the coordinate system $\mathbf{r}_0, \mathbf{n}_0, \ldots, \mathbf{n}_k$. Formula (5.8) tells us how to express f_k as a linear combination of this basis, and definitions (5.6) and (5.7) guarantee that the coefficients in this linear combination are monomials with positive coefficients. □

Turning to Lie brackets, we note that we will make frequent use of the product rule for a function a and a pair w, z of vector fields:

$$[w, az] = a[w, z] + w(a)z\,. \tag{6.2}$$

Looking at our standard vector fields, we first present an example.

Example 6. Continuing Example 2, here is a table of Lie brackets in the chart $\mathcal{C}(ooioii)$, for which IP $= \{3, 5, 6\}$. The row heading indicates the vector field used in the left slot, e.g., $[v_1, f_1] = v_0$.

	f_0	f_1	f_2	f_3	f_4	f_5	f_6
v_0	0	0	0	0	0	0	0
v_1	0	v_0	v_0	$\mathbf{n}_3 v_0$	$\mathbf{n}_3 v_0$	$\mathbf{n}_3\mathbf{n}_5 v_0$	$\mathbf{n}_3\mathbf{n}_5\mathbf{n}_6 v_0$
v_2	0	0	v_1	$\mathbf{n}_3 v_1$	$\mathbf{n}_3 v_1$	$\mathbf{n}_3\mathbf{n}_5 v_1$	$\mathbf{n}_3\mathbf{n}_5\mathbf{n}_6 v_1$
v_3	0	0	0	f_2	f_2	$\mathbf{n}_5 f_2$	$\mathbf{n}_5\mathbf{n}_6 f_2$
v_4	0	0	0	0	v_3	$\mathbf{n}_5 v_3$	$\mathbf{n}_5\mathbf{n}_6 v_3$
v_5	0	0	0	0	0	f_4	$\mathbf{n}_6 f_4$
v_6	0	0	0	0	0	0	f_5
f_0	0	0	0	0	0	0	0
f_1	0	0	$-\mathbf{n}_2 v_0$	$-\mathbf{n}_2\mathbf{n}_3 v_0$	$-\mathbf{n}_2\mathbf{n}_3 v_0$	$-\mathbf{n}_2\mathbf{n}_3\mathbf{n}_5 v_0$	$-\mathbf{n}_2\mathbf{n}_3\mathbf{n}_5\mathbf{n}_6 v_0$
f_2	0	$\mathbf{n}_2 v_0$	0	$-v_1$	$-v_1$	$-\mathbf{n}_5 v_1$	$-\mathbf{n}_5\mathbf{n}_6 v_1$
f_3	0	$\mathbf{n}_2\mathbf{n}_3 v_0$	v_1	0	$-\mathbf{n}_4 f_2$	$-\mathbf{n}_4\mathbf{n}_5 f_2$	$-\mathbf{n}_4\mathbf{n}_5\mathbf{n}_6 f_2$
f_4	0	$\mathbf{n}_2\mathbf{n}_3 v_0$	v_1	$\mathbf{n}_4 f_2$	0	$-v_3$	$-\mathbf{n}_6 v_3$
f_5	0	$\mathbf{n}_2\mathbf{n}_3\mathbf{n}_5 v_0$	$\mathbf{n}_5 v_1$	$\mathbf{n}_4\mathbf{n}_5 f_2$	v_3	0	$-f_4$
f_6	0	$\mathbf{n}_2\mathbf{n}_3\mathbf{n}_5\mathbf{n}_6 v_0$	$\mathbf{n}_5\mathbf{n}_6 v_1$	$\mathbf{n}_4\mathbf{n}_5\mathbf{n}_6 f_2$	$\mathbf{n}_6 v_3$	f_4	0

We now explain how to obtain the sort of explicit formulas appearing in Example 6. To begin, an elementary computation shows that, for each j,

$$[v_0, f_j] = [f_0, f_j] = 0. \tag{6.3}$$

The following two lemmas give the other formulas.

Lemma 7. *For $i > j$ we have*

$$[v_i, f_j] = 0. \tag{6.4}$$

For $1 \le i \le j \le k$,

$$[v_i, f_j] = \begin{cases} a_{ij} f_{i-1} & \text{if } i \in \mathrm{IP} \\ a_{ij} v_{i-1} & \text{if } i \notin \mathrm{IP}. \end{cases} \tag{6.5}$$

Proof. In (5.8) we see that f_j is a linear combination of f_0 and $v_0, \ldots, v_{j-1}$, from which (6.4) is immediate.

We prove the formulas of (6.5) by induction on j. In the inductive step, we first suppose that $j \in \mathrm{IP}$. Here (5.5) and the product rule (6.2) give us that

$$[v_i, f_j] = [v_i, \mathbf{n}_j f_{j-1} + v_{j-1}] = v_i(\mathbf{n}_j) f_{j-1} + \mathbf{n}_j [v_i, f_{j-1}]. \tag{6.6}$$

If $i < j$, then the first term on the right of (6.6) vanishes, and using (5.6) we find that

$$[v_i, f_j] = \mathbf{n}_j [v_i, f_{j-1}] = \begin{cases} \mathbf{n}_j a_{i,j-1} f_{i-1} = a_{ij} f_{i-1} & \text{if } i \in \mathrm{IP} \\ \mathbf{n}_j a_{i,j-1} v_{i-1} = a_{ij} v_{i-1} & \text{if } i \notin \mathrm{IP}. \end{cases}$$

In the remaining case $i = j$, the last term on the right of (6.6) vanishes (as a consequence of (6.4)), and thus we have

$$[v_j, f_j] = f_{j-1} = a_{jj} f_{j-1}.$$

Now suppose that $j \notin \mathrm{IP}$. Here (5.4) and the product rule (6.2) give us that

$$[v_i, f_j] = [v_i, f_{j-1} + \mathbf{n}_{j-1} v_{j-1}] = [v_i, f_{j-1}] + v_i(\mathbf{n}_j) v_{j-1}. \tag{6.7}$$

If $i < j$, then the second term on the right of (6.7) vanishes, so that

$$[v_i, f_j] = [v_i, f_{j-1}] = \begin{cases} a_{i,j-1} f_{i-1} = a_{ij} f_{i-1} & \text{if } i \in \mathrm{IP} \\ a_{i,j-1} v_{i-1} = a_{ij} v_{i-1} & \text{if } i \notin \mathrm{IP}. \end{cases}$$

In the remaining case $i = j$, the first term on the right of (6.7) vanishes, and thus we have

$$[v_j, f_j] = v_{j-1} = a_{jj} v_{j-1}.$$

□

Lemma 8. *For $1 \le i < j \le k$*

$$[f_i, f_j] = \begin{cases} -b_{ij} f_{i-1} & \text{if } i \in \mathrm{IP} \\ -b_{ij} v_{i-1} & \text{if } i \notin \mathrm{IP}. \end{cases} \tag{6.8}$$

For $0 \le i < j \le k$

$$\begin{cases} \mathbf{n}_{i+1}[f_i, f_j] + [v_i, f_j] = 0 & \text{if } i+1 \in \mathrm{IP} \\ [f_i, f_j] + \mathbf{n}_{i+1}[v_i, f_j] = 0 & \text{if } i+1 \notin \mathrm{IP}. \end{cases} \tag{6.9}$$

Proof. In view of (6.5), and using the definitions (5.6) and (5.7), one sees that formulas (6.8) and (6.9) are equivalent for each $i \ge 1$. We prove them by induction on i. To provide a base case for the induction, we remark that when $i = 0$, formula (6.9) is a consequence of (6.3).

Here is the inductive step of the argument: If $i \in \mathrm{IP}$, then by (5.5) we have

$$\begin{aligned} [f_i, f_j] &= [\mathbf{n}_i f_{i-1} + v_{i-1}, f_j] \\ &= -f_j(\mathbf{n}_i) f_{i-1} + \mathbf{n}_i [f_{i-1}, f_j] + [v_{i-1}, f_j]. \end{aligned}$$

The inductive hypothesis tells us that the sum of the last two terms vanishes. Thus, by Lemma 4,

$$[f_i, f_j] = -b_{ij} f_{i-1}.$$

If $i \notin \mathrm{IP}$, then by (5.4) we have

$$\begin{aligned}[f_i, f_j] &= [f_{i-1} + \mathbf{n}_i v_{i-1}, f_j] \\ &= [f_{i-1}, f_j] - f_j(\mathbf{n}_i) v_{i-1} + \mathbf{n}_i [v_{i-1}, f_j].\end{aligned}$$

The inductive hypothesis tells us that the sum of the first and last terms vanishes. Thus, by Lemma 4,

$$[f_i, f_j] = -b_{ij} v_{i-1}.$$

□

Lemma 9. *On each standard chart of $S(k)$, there is an ordered basis $g_0, g_1, \ldots, g_{k+1}$ of sections of the tangent bundle with the following properties:*

(1) The first two elements are the standard focal and vertical vector fields: $g_0 = f_k$ and $g_1 = v_k$.

(2) For $2 \le i \le k+1$ we have

$$g_i = \begin{cases} \pm v_{k-i+1} & \text{if } k-i+2 \notin \mathrm{IP} \\ \pm f_{k-i+1} & \text{if } k-i+2 \in \mathrm{IP}, \end{cases}$$

with the appropriate sign being determined by property (5). The vector field g_i generates the rank one quotient sheaf Δ_i/Δ_{i-1}.

(3) The vector fields $g_0, g_1, \ldots, g_i$ form a basis of sections of Δ_i.

(4) If a is a monomial in the standard coordinates, then $g_0(a)$ and $g_1(a)$ are linear combinations of monomials with positive coefficients.

(5) For $1 \le i \le k$ we have

$$[g_0, g_i] = \left(\prod_{\substack{h=k-i+3 \\ h \in \mathrm{IP}}}^{k} \mathbf{n}_h \right) g_{i+1}\,.$$

Proof. We define g_0 and g_1 as required by property (1), and define (for $g \ge 1$)

$$g_{i+1} = \left(\prod_{\substack{h=k-i+3 \\ h \in \mathrm{IP}}}^{k} \mathbf{n}_h \right)^{-1} [g_0, g_i]\,.$$

Although it seems that there is a denominator, we proceed to demonstrate that we obtain the formulas of property (2), as follows.

Starting with the formula in property (5) and making the substitutions of property (2), we find that the two formulas of property (2) are equivalent to the following four displayed formulas; thus these formulas provide an inductive proof of property (2). Here are the four formulas:

$$[f_k, v_{k-i+1}] = \pm \left(\prod_{\substack{h=k-i+3 \\ h \in \mathrm{IP}}}^{k} \mathbf{n}_h \right) v_{k-i} \qquad \text{if } k-i+2 \notin \mathrm{IP} \quad \text{and } k-i+1 \notin \mathrm{IP}$$

$$[f_k, v_{k-i+1}] = \pm \left(\prod_{\substack{h=k-i+3 \\ h\in \mathrm{IP}}}^{k} \mathbf{n}_h \right) f_{k-i} \qquad \text{if } k-i+2 \notin \mathrm{IP} \quad \text{and } k-i+1 \in \mathrm{IP}$$

$$[f_k, f_{k-i+1}] = \pm \left(\prod_{\substack{h=k-i+3 \\ h\in \mathrm{IP}}}^{k} \mathbf{n}_h \right) v_{k-i} \qquad \text{if } k-i+2 \in \mathrm{IP} \quad \text{and } k-i+1 \notin \mathrm{IP}$$

$$[f_k, f_{k-i+1}] = \pm \left(\prod_{\substack{h=k-i+3 \\ h\in \mathrm{IP}}}^{k} \mathbf{n}_h \right) f_{k-i} \qquad \text{if } k-i+2 \in \mathrm{IP} \quad \text{and } k-i+1 \in \mathrm{IP}$$

The first two formulas are special cases (with indeterminate signs) of (6.5); observe that to verify this we use the condition $k-i+2 \notin \mathrm{IP}$ to infer that $a_{k-i+1,k} = a_{k-i+2,k}$. The last two formulas are special cases of (6.8); to verify this, we use the condition $k-i+2 \in \mathrm{IP}$, which tells us that in (5.7) we should read the top line to infer that $b_{k-i+1,k} = a_{k-i+2,k}$.

To obtain property (2), we compare the basis

$$f_{k-i+2}, v_{k-i+2}, \ldots, v_k$$

for Δ_{i-1} obtained from (5.9) with the alternative basis for Δ_i exhibited in Lemma 3. Property (3) is now immediate, and property (4) is a restatement of Lemma 5. □

Example 10. On chart $\mathcal{C}(ooioii)$, using the results from Example 6, we have

$$\begin{aligned}
g_0 &= f_6 \\
g_1 &= v_6 \\
g_2 &= [g_0, g_1] = -f_5 \\
g_3 &= [g_0, g_2] = -f_4 \\
g_4 &= \tfrac{1}{\mathbf{n}_6}[g_0, g_3] = -v_3 \\
g_5 &= \tfrac{1}{\mathbf{n}_5\mathbf{n}_6}[g_0, g_4] = f_2 \\
g_6 &= \tfrac{1}{\mathbf{n}_5\mathbf{n}_6}[g_0, g_5] = v_1 \\
g_7 &= \tfrac{1}{\mathbf{n}_3\mathbf{n}_5\mathbf{n}_6}[g_0, g_6] = -v_0.
\end{aligned}$$

From these explicit calculations on each standard chart, we can now draw a global conclusion. Recall from Section 9 of [2] that the monster space $S(k)$ carries *divisors at infinity* I_j for $2 \le j \le k$. Let $\mathcal{I}_j$ denote the ideal sheaf of I_j.

Corollary 11. *For each i with $3 \le i \le k$ we have*

$$[\Delta, \Delta_i] = \Delta_i + \left(\prod_{j=k-i+3}^{k} \mathcal{I}_j \right) \Delta_{i+1}. \tag{6.10}$$

Proof. Consider a chart specified by a word in the symbols o and i. In this chart the divisor at infinity I_j is the hyperplane $\mathbf{n}_j = 0$ if $j \in \mathrm{IP}$, but if $j \notin \mathrm{IP}$ then I_j does not meet the chart. Therefore in this chart the ideal appearing in (6.10) is

$$\prod_{\substack{j \in \mathrm{IP} \\ k-i+3 \leq j \leq k}} \mathcal{I}_j \tag{6.11}$$

and it is a principal ideal generated by

$$\prod_{\substack{j \in \mathrm{IP} \\ k-i+3 \leq j \leq k}} \mathbf{n}_j \,.$$

We know that $[\Delta, \Delta_{i-1}]$ is a subsheaf of $\Delta_i = [\Delta_{i-1}, \Delta_{i-1}]$, and thus to understand $[\Delta, \Delta_i]$ it suffices to know the Lie brackets of the two generators g_0, g_1 of Δ with the single generator g_i of the quotient sheaf Δ_i/Δ_{i-1}. By definition $g_1 = v_k$, and (6.4) tells us that $[g_1, g_i] = 0$. Thus by property (5) of Lemma 9 we obtain (6.10). □

7. Focal order and vertical order

The notion of focal order which we lay out here is a special case of the notion of order of a function with respect to a nonholonomic system, as explicated in Section 2.1.1 of [4] and in Section 4 of [1]. Our starting point and application are quite different, however; we do not employ a metric, nor do we make use of privileged coordinates. Thus we offer here a self-contained treatment, confined to the case of the Goursat distribution Δ on a space in the monster tower.

Given a point $p \in S(k)$, a smooth function a defined in some neighborhood, and a parameterized smooth focal curve germ C through p, let $\#(a \cdot C)$ be the intersection number. Concretely, if $\gamma\colon \Gamma \to C$ is the parametrization and t is the parameter (so that $\gamma(0) = p$), then $\#(a \cdot C)$ is the order of vanishing of $a \circ \gamma$ as function of t. If the focal curve lies within the hypersurface $a = 0$, then the intersection number is $+\infty$.

We define *the focal order* of a to be

$$o(a) = \min\{\#(a \cdot C)\},$$

the minimum over all possible focal curve germs through p. The focal order is a valuation, but here the value $+\infty$ is impossible, since there are no integral hypersurfaces for the focal distribution. Note that in a family of curves, the intersection number can only increase or stay the same under specialization; in this sense the order is the generic value of the intersection number.

Example 12. Working at the point $p = (0, 0; 0, 0, 0, 1) \in \mathcal{C}(ooio)$ on $S(4)$, and using the coordinates x, y, $y' = dy/dx$, $y'' = dy'/dx$, $x' = dx/dy''$, $x'' = dx'/dy''$, consider the function $a = y - \frac{1}{15}(y'')^{15}$. We write

$$y'' = A_1 t + A_2 t^2 + \cdots$$
$$x'' = 1 + B_1 t + B_2 t^2 + \cdots$$

where the ellipses indicate higher-order terms. By repeated integration we find that

$$
\begin{aligned}
x' &= \int x''dy'' = A_1 t + \left(A_2 + \tfrac{1}{2}A_1 B_1\right) t^2 + \cdots \\
x &= \int x'dy'' = \tfrac{1}{2}{A_1}^2\, t^2 + \left(A_1 A_2 + \tfrac{1}{6}{A_1}^2\, B_1\right) t^3 + \cdots \\
y' &= \int y''dx = \tfrac{1}{3}{A_1}^3\, t^3 + \left({A_1}^2\, A_2 + \tfrac{1}{8}{A_1}^3\, B_1\right) t^4 + \cdots \\
y &= \int y'dx = \tfrac{1}{15}{A_1}^5\, t^5 + \left(\tfrac{1}{3}{A_1}^4\, A_2 + \tfrac{7}{144}{A_1}^5\, B_1\right) t^6 + \cdots \\
y - \tfrac{1}{15}(y'')^5 &= \tfrac{7}{144}{A_1}^5\, B_1 t^6 + \cdots
\end{aligned}
$$

so that the intersection number $\#(a \cdot C)$ is 6 whenever A_1 and B_1 don't vanish, and larger otherwise. Thus $o(a) = 6$. (For those familiar with the notion of nonholonomic privileged coordinates, this is the last coordinate in a system of such coordinates; as already indicated, however, we make no use of this concept.)

The notion of order is easily extended to exact 1-forms: we define $o(da) = o(a - a(p))$. Conversely, if we know $o(da)$ then we can compute $o(a)$: if a vanishes at p we have $o(a) = o(da)$, and otherwise $o(a) = 0$. Using this extension, we can easily compute the focal orders of the standard coordinates in a standard chart on a monster space.

Example 13. We use the same chart $\mathcal{C}(ooioii)$ as in Examples 2, 6, and 10. Working at the point $(x, y; y', y'', x', x'', y^{(3)}, x^{(3)}) = (0, 0; 0, 0, 0, 1, 0, 0)$, begin with the fact that the focal order of each active coordinate is 1, and then compute as follows:

$$
\begin{aligned}
o(dy^{(3)}) &= 1 & o(y^{(3)}) &= 1 \\
o(dx^{(3)}) &= 1 & o(x^{(3)}) &= 1 \\
o(dx'') &= o(x^{(3)}) + o(dy^{(3)}) = 2 & o(x'') &= 0 \\
o(dy'') &= o(y^{(3)}) + o(dx'') = 3 & o(y'') &= 3 \\
o(dx') &= o(x'') + o(dy'') = 3 & o(x') &= 3 \\
o(dx) &= o(x') + o(dy'') = 6 & o(x) &= 6 \\
o(dy') &= o(y'') + o(dx) = 9 & o(y') &= 9 \\
o(dy) &= o(y') + o(dx) = 15 & o(y) &= 15
\end{aligned}
$$

The RVT code word of this point is $RRVRVV$.

Example 14. If instead we work at the origin of this chart, whose code word is $RRVTVV$, we find that

$$\begin{aligned} o(dy^{(3)}) &= o(y^{(3)}) = 1 \\ o(dx^{(3)}) &= o(x^{(3)}) = 1 \\ o(dx'') &= o(x'') = 2 \\ o(dy'') &= o(y'') = 3 \\ o(dx') &= o(x') = 5 \\ o(dx) &= o(x) = 8 \\ o(dy') &= o(y') = 11 \\ o(dy) &= o(y) = 19 \end{aligned}$$

The general procedure is as follows.

(1) The order of the differential of each active coordinate is 1.
(2) Recursively, using the definitions of the coordinates in backwards order:
   (a) use the definition of the coordinate to infer the order of vanishing of the differential of a prior coordinate;
   (b) if the prior coordinate vanishes at p, then its order agrees with the order of its differential, and otherwise is 0.

Given a parameterized focal curve germ C through p, locally there is a focal vector field v_C whose integral curve through p is C; it is not unique, but its restriction to C is unique. For a smooth function germ a in a neighborhood of p, we consider the Lie derivative $v_C(a)$.

Lemma 15. *Suppose that* $\#(a \cdot C) > 0$. *Then*

$$\#(v_C(a) \cdot C) = \#(a \cdot C) - 1.$$

Proof. This follows from the basic fact that for a curve germ parameterized by $\gamma \colon \Gamma \to C$, we have

$$\tfrac{d}{dt}(a \circ \gamma) = v_C(a) \circ \gamma,$$

and that differentiation reduces the order of vanishing by 1. □

Corollary 16. *If* $o(a) > 0$, *then* $o(v_C(a)) = o(a) - 1$.

This leads to an alternative characterization of the focal order. For each focal vector field w, we can differentiate a repeatedly with respect to this vector field until we obtain a function with nonzero value at p; call this value the *order of a with respect to w*. (If we never reach such a function, then we say the order is $+\infty$.) Then $o(a)$ is the minimal order of a, taking the minimum over all focal vector fields. One can even vary the choice of vector field at each step: $o(a)$ is also the least number of focal derivatives required to take a to a function with nonzero value. This is the definition that one finds in [1] and [4].

For $2 \leq j \leq k$, consider the divisor at infinity I_j. Given a point $p \in S(k)$, we define its *vertical order* VO_j to be the focal order of the function defining I_j, i.e., the minimum intersection number of a regular focal curve germ with this divisor; this is independent of the choice of chart. The *vertical orders vector* is

$$\mathrm{VO}(p) = (\mathrm{VO}_2, \mathrm{VO}_3, \ldots, \mathrm{VO}_k).$$

If $p \notin I_j$, then $\mathrm{VO}_j = 0$; in general we can compute it by computing the focal order of the standard coordinate function defining I_j.

Example 17. Referring to Example 13, we observe that $(0,0;0,0,0,1,0,0)$ does not meet I_2 or I_4. Furthermore the divisors I_3, I_5, and I_6 are defined by the vanishing, respectively, of x', $y^{(3)}$, and $x^{(3)}$. Thus the vertical orders vector at p is

$$(0,3,0,1,1).$$

To relate this notion to our earlier paper [2], observe that there we associated a vertical orders vector to a curve germ C on the base surface S (and more generally to a focal curve germ at some other level). Starting with a curve C that lifts to a regular curve germ through $p \in S(k)$, we see that the two notions agree. For a more extensive example, we refer to Example 31 of [2], which shows that the vertical orders vector associated to a point with RVT code word $RVVVRVT$ is $(6,3,3,0,2,0)$; since the word is not a Goursat word, the first entry is nonzero.

8. Sections of the small growth sheaves

Suppose that $k \geq 3$. Here we consider the small growth sheaves Δ^h on $S(k)$. In Theorem 19 we will give a description of their sections; it involves bounds on certain focal orders. In the subsequent section, we will show that these bounds are sharp, and this will allow us to give new characterizations of the small growth invariants.

Choosing a point $p \in S(k)$ and using the entries in its vertical orders vector, for $h \geq 2$ and $2 \leq i \leq \min\{h, k+1\}$ we define

$$e_{hi} = \begin{cases} -(h-i) + \displaystyle\sum_{j=k-i+4}^{k} (i+j-k-3)\,\mathrm{VO}_j & \text{if this quantity is nonnegative,} \\ 0 & \text{otherwise.} \end{cases} \tag{8.1}$$

The sum in (8.1) is zero if $i = 2$ or 3; hence $e_{h2} = e_{h3} = 0$ in all cases. Note that all the values e_{hi} depend only on the Goursat code word of the specified point; thus they are Goursat invariants. It is helpful to arrange these invariants in a table, as in the following example.

Example 18. For a point with Goursat word $RRVTVV$, such as the point in Example 14, here is a table of values for e_{hi}. To the right, we present the table of values for a point whose Goursat word is $RRRVV$. In Example 21 we explain why certain rows are colored in red; in Section 9 we will explain how we know the values of the small growth vector listed in the rightmost column. The juxtaposition of the two tables hints at a recursion for computing them; we will explain this in Section 11, where we introduce the notion of *lifted Goursat word.*

h \ i	2	3	4	5	6	7	SG_h
2	0						3
3	0	0					4
4	0	0	1				4
5	0	0	0	3			5
6	0	0	0	2	5		5
7	0	0	0	1	4	12	5
8	0	0	0	0	3	11	6
9	0	0	0	0	2	10	6
10	0	0	0	0	1	9	6
11	0	0	0	0	0	8	7
12	0	0	0	0	0	7	7
13	0	0	0	0	0	6	7
14	0	0	0	0	0	5	7
15	0	0	0	0	0	4	7
16	0	0	0	0	0	3	7
17	0	0	0	0	0	2	7
18	0	0	0	0	0	1	7
19	0	0	0	0	0	0	8

h \ i	2	3	4	5	6	SG_h
2	0					3
3	0	0				4
4	0	0	1			4
5	0	0	0	3		5
6	0	0	0	2	5	5
7	0	0	0	1	4	5
8	0	0	0	0	3	6
9	0	0	0	0	2	6
10	0	0	0	0	1	6
11	0	0	0	0	0	7

Here are some elementary properties:

$$\max\{e_{hi} - 1, 0\} = e_{h+1,i}\,. \tag{8.2}$$

$$e_{hi} \geq e_{h+1,i} \tag{8.3}$$

$$e_{h,i+1} \geq e_{hi} \tag{8.4}$$

$$e_{hi} + \sum_{j=k-i+3}^{k} \mathrm{VO}_j \geq e_{h+1,i+1}, \quad \text{with equality when } h = i\,. \tag{8.5}$$

As remarked in Section 3, the subsheaves Δ_h and Δ^h of the tangent sheaf $\Theta_{S(k)}$ are equal for $h = 1, 2, 3$. To describe the other sheaves in the small growth sequence, we employ the ordered basis $g_0, g_1, \ldots, g_{k+1}$ developed in Lemma 9.

Theorem 19. *Consider a standard chart of $S(k)$ containing the point p. For each $h \geq 3$, each section of the sheaf Δ^h in this chart may be written in the form*

$$w + \sum_{i=4}^{\min\{h,k+1\}} c_{hi} g_i, \tag{8.6}$$

where w is a section of Δ_3, and each c_{hi} is a function whose focal order $o(c_{hi})$ at p is at least e_{hi}.

Proof. We prove the theorem by induction on h. For the base case $h = 3$, the sum in (8.6) is empty and the assertion is that $\Delta^3 = \Delta_3$, which we already know. The theorem thus being clear for $k < 3$, we henceforth assume that $k \geq 3$.

Consider a section of Δ^h; by the inductive hypothesis it may be written as in (8.6). Bracketing this section with a focal vector field z (i.e., a section of Δ) yields

$$[z, w] + \sum_{i=4}^{\min\{h,k+1\}} \left(c_{hi}[z, g_i] + z(c_{hi}) g_i \right) . \tag{8.7}$$

Note we have used the product rule (6.2). We will separately analyze the three types of terms in (8.7), showing that each contribution satisfies the required inequality on the orders of the coefficients. To begin, by Corollary 11 the vector field $[z, w]$ is a section of $\Delta_3 + \mathcal{I}_k \Delta_4$, and each section of $\mathcal{I}_k$ on our chart is a function with focal order at least $\mathrm{VO}_k \geq e_{h,4}$.

Assume that $h < k+1$. We now deal with the second sort of term in (8.7). By Corollary 11, each $[z, g_i]$ is a section of

$$\Delta_i + \left(\prod_{j=k-i+3}^{k} \mathcal{I}_j \right) \Delta_{i+1} ,$$

and thus may be written as

$$w_i + \left(\prod_{j=k-i+3}^{k} c_j \right) g_{i+1} ,$$

where w_i is a section of Δ_i and each function c_j is a section of $\mathcal{I}_j$. Writing

$$w_i = w_3 + \sum_{j=4}^{i} a_j g_j$$

with w_3 a section of Δ_3, we remark that

$$o(c_{hi} a_j) \geq o(c_{hi}) \geq e_{hi} \geq e_{h+1,j}$$

for all j; here we have used the inductive hypothesis together with properties (8.3) and (8.4). Observe that by (8.5) we know that

$$o\left(c_{hi} \prod_{j=k-i+3}^{k} c_j \right) \geq e_{hi} + \sum_{j=k-i+3}^{k} \mathrm{VO}_j \geq e_{h+1,i+1} .$$

For the third type of term in (8.7) we observe that

$$o(z(c_{hi})) \geq \max\{o(c_{hi}) - 1, 0\} \geq \max\{e_{hi} - 1, 0\} = e_{h+1,i}$$

using the inductive hypothesis and property (8.2).

If $h \geq k+1$, the same arguments apply, except that $[z, g_{k+1}]$ is already a section of the full tangent bundle Δ_{k+1}. □

Example 20. As in Example 18, we consider a point $p \in S(6)$ in chart $\mathcal{C}(ooioii)$ whose code word is $RRVTVV$. Applying Theorem 19 with $h = 7$, we see that in this chart each section of Δ^7 may be written as

$$w + c_{74} g_4 + c_{75} g_5 + c_{76} g_6 + c_{77} g_7,$$

where w is a section of Δ_3, and with $o(c_{75}) \geq 1$, $o(c_{76}) \geq 4$, and $o(c_{77}) \geq 12$.

For each i with $2 \le i \le k+1$, let b_i be the smallest integer for which $e_{b_i,i} = 0$. Using the definition in (8.1), we see that

$$b_i = i + \sum_{j=k-i+4}^{k} (i+j-k-3)\,\mathrm{VO}_j\,. \tag{8.8}$$

We call $(b_2, b_3, \ldots, b_{k+1})$ the *b vector*.

Example 21. Consider the table on the left in Example 18. To directly use the definition of b_i we look in each column for the first appearance of a zero, concluding that $b_2 = 2$, $b_3 = 3$, $b_4 = 5$, $b_5 = 8$, $b_6 = 11$, and $b_7 = 19$. Alternatively, using formula (8.8), we have, e.g.,

$$b_7 = 7 + \mathrm{VO}_3 + 2\,\mathrm{VO}_4 + 3\,\mathrm{VO}_5 + 4\,\mathrm{VO}_6 = 7 + 5 + 0 + 3 + 4 = 19.$$

In both tables, the rows indexed by b_i's are highlighted in red.

Corollary 22.

(1) For each $h \ge 2$,

$$\mathrm{SG}_h \le 2 + \textit{the number of zero entries in row } h \textit{ of the table of } e_{hi} \textit{ values.}$$

(2) For $2 \le i \le k+1$ we have $\beta_{i+1} \ge b_i$.

Proof.

(1) Consider a section of Δ^h, as in display (8.6). When we evaluate this section at p, any coefficient c_{hi} for which $e_{hi} > 0$ will vanish.

(2) This is now immediate from the definitions of β_i and b_i.

□

9. Calculation pathways

In this section, we will show that the inequalities in Corollary 22 are actually equalities. To do this, we produce some specific sections of the sheaves Δ^h, by following certain calculation pathways.

Theorem 23. *Assume the circumstances of Theorem 19. For each $h \ge 3$ and for each i with $3 \le i \le \min\{h, k+1\}$, there is a section f_{hi} of Δ^h for which, when we write it as in (8.6), the coefficient c_{hi} has focal order exactly e_{hi}.*

Proof. We first remark that it suffices to produce f_{hi} when the quantity on the top line of (8.1) is nonnegative, since in the remaining cases we may set $f_{hi} = f_{h-1,i}$. We will obtain each of the other required sections by beginning with $f_{33} = g_3$ and repeatedly Lie-bracketing on the left by either g_0 or g_1, with the exact sequence of calculations to be specified below. The basis $g_0, g_1, \ldots, g_{k+1}$ has been chosen so that the results of these calculations have only positive coefficients, and thus it suffices to exhibit a single term of f_{hi} of the form ag_i in which a has the required focal order e_{hi}, since there is no possibility of cancellation.

To produce f_{ii}, we use only g_0:

$$\begin{aligned} f_{33} &= g_3 \\ f_{44} &= [g_0, f_{33}] \\ f_{55} &= [g_0, f_{44}] \\ &\text{etc.} \end{aligned}$$

We claim that f_{ii} includes the term

$$t_i := \left(\prod_{\substack{j=k-i+4 \\ j \in \mathrm{IP}}}^{k} \mathbf{n}_j^{i+j-k-3} \right) g_i \tag{9.1}$$

which visibly has the required focal order

$$e_{ii} = \sum_{j=k-i+4}^{k} (i+j-k-3)\,\mathrm{VO}_j\,.$$

This is clear by induction, beginning with $f_{33} = g_3$ (with the product being interpreted by the usual convention to be 1). In the inductive step we see that if we apply the product rule (6.2) to $[g_0, t_i]$ we obtain two terms, one of which is

$$\begin{aligned} \left(\prod_{\substack{j=k-i+4 \\ j \in \mathrm{IP}}}^{k} \mathbf{n}_j^{i+j-k-3} \right) [g_0, g_i] &= \left(\prod_{\substack{j=k-i+4 \\ j \in \mathrm{IP}}}^{k} \mathbf{n}_j^{i+j-k-3} \right) \left(\prod_{\substack{j=k-i+3 \\ j \in \mathrm{IP}}}^{k} \mathbf{n}_j \right) g_{i+1} \\ &= \left(\prod_{\substack{j=k-(i+1)+4 \\ j \in \mathrm{IP}}}^{k} \mathbf{n}_j^{(i+1)+j-k-3} \right) g_{i+1}\,. \end{aligned}$$

(Note that we have used property (5) of Lemma 9 in this computation.)

In general, we define the vector field f_{hi} by fixing i and inducting on h, beginning with f_{ii}. Having singled out a term

$$t_h = a g_i \tag{9.2}$$

in f_{hi} with $o(a) = e_{hi}$, we examine the coefficient a to see whether it includes the coordinate function $\mathbf{n}_k$. If so, we define $f_{h+1,i} = [g_1, f_{hi}]$; then by the product rule $f_{h+1,i}$ includes the term

$$t_{h+1} := g_1(a) g_i = v_k(a) g_i = \frac{\partial}{\partial \mathbf{n}_k}(a) g_i\,,$$

in which the coefficient has focal order $o(a) - 1 = e_{hi} - 1 = e_{h+1,i}$. If the coefficient a in (9.2) does not include $\mathbf{n}_k$, then we define $f_{h+1,i} = [g_0, f_{hi}]$. By the product rule $f_{h+1,i}$ includes $g_0(a) g_i = f_k(a) g_i$, which may consist of multiple terms. There is, however, at least one term, and each one is of the form $a' g_i$, where the coefficient has focal order $o(a') = o(a) - 1 = e_{hi} - 1 = e_{h+1,i}$. Select one of these terms and call it t_{h+1}. This induction continues until we reach the first zero value for e_{hi}, i.e., until we reach $h = b_i$. □

Corollary 24. *The inequalities of Corollary 22 are in fact equalities:*

(1) For each $h \geq 2$,

$$\mathrm{SG}_h = 2 + \textit{the number of zero entries in row } h \textit{ of the table of } e_{hi} \textit{ values.}$$

(2) For $2 \leq i \leq k+1$ *we have* $\beta_{i+1} = b_i$.

Example 25. As in Example 18, we work in chart $\mathcal{C}(ooioii)$. For the $RRVTVV$ point at the origin, here is a calculation leading to the vector field $f_{19,7}$.

vector field	term in this vector field	focal order of its coefficient
$f_{33} = g_3$	$t_3 = g_3$	0
$f_{44} = [g_0, f_{33}]$	$t_4 = \mathbf{n}_6 g_4$	1
$f_{55} = [g_0, f_{44}]$	$t_5 = \mathbf{n}_5 {\mathbf{n}_6}^2 g_5$	3
$f_{66} = [g_0, f_{55}]$	$t_6 = {\mathbf{n}_5}^2 {\mathbf{n}_6}^3 g_6$	5
$f_{77} = [g_0, f_{66}]$	$t_7 = \mathbf{n}_3 {\mathbf{n}_5}^3 {\mathbf{n}_6}^4 g_7$	12
$f_{87} = [g_1, f_{77}]$	$t_8 = 4\, \mathbf{n}_3 {\mathbf{n}_5}^3 {\mathbf{n}_6}^3 g_7$	11
$f_{97} = [g_1, f_{87}]$	$t_9 = 12\, \mathbf{n}_3 {\mathbf{n}_5}^3 {\mathbf{n}_6}^2 g_7$	10
$f_{10,7} = [g_1, f_{97}]$	$t_{10} = 24\, \mathbf{n}_3 {\mathbf{n}_5}^3 \mathbf{n}_6 g_7$	9
$f_{11,7} = [g_1, f_{10,7}]$	$t_{11} = 24\, \mathbf{n}_3 {\mathbf{n}_5}^3 g_7$	8
$f_{12,7} = [g_0, f_{11,7}]$	$t_{12} = 24\, \mathbf{n}_4 {\mathbf{n}_5}^4 \mathbf{n}_6 g_7$	7
$f_{13,7} = [g_1, f_{12,7}]$	$t_{13} = 24\, \mathbf{n}_4 {\mathbf{n}_5}^4 g_7$	6
$f_{14,7} = [g_0, f_{13,7}]$	$t_{14} = 24\, {\mathbf{n}_5}^4 \mathbf{n}_6 g_7$	5
$f_{15,7} = [g_1, f_{14,7}]$	$t_{15} = 24\, {\mathbf{n}_5}^4 g_7$	4
$f_{16,7} = [g_0, f_{15,7}]$	$t_{16} = 96\, {\mathbf{n}_5}^3 g_7$	3
$f_{17,7} = [g_0, f_{16,7}]$	$t_{17} = 288\, {\mathbf{n}_5}^2 g_7$	2
$f_{18,7} = [g_0, f_{17,7}]$	$t_{18} = 576\, \mathbf{n}_5 g_7$	1
$f_{19,7} = [g_0, f_{18,7}]$	$t_{19} = 576\, g_7$	0

(By a laborious calculation one discovers that in fact $f_{19,7} = t_{19}$.)

10. Relating the structural and small growth invariants

Corollary 24 provides a bridge between the structural invariants and the small growth invariants, which we now explain further. Looking at a point $p \in S(k)$, we consider three structural invariants:

(1) The b vector

$$(b_2, b_3, \ldots, b_{k+1})$$

has been defined in Section 8.

(2) From the multiplicity sequence $m_0, m_1, \ldots$ of any curve on S whose lift is a regular focal curve through p, we extract the entries m_1 through m_{k-1}, writing them in reverse order. We call

$$(m_{k-1}, m_{k-2}, \ldots, m_1)$$

the *multiplicity vector*. For the definition of the multiplicity sequence, see Section 13 of [2] or Section 3.5 of [8]. The omitted value m_0 is not a Goursat invariant, and at the other end we have $m_i = 1$ for $i \geq k$.

(3) We consider the restricted vertical orders vector

$$(\mathrm{VO}_k, \mathrm{VO}_{k-1}, \ldots, \mathrm{VO}_3),$$

writing the entries in reverse order from the convention used in [2].

There are three corresponding small growth invariants, namely the beta vector and its first two derived vectors, denoted der and der^2.

By Theorem 33 of [2], we have $m_i - m_{i+1} = \mathrm{VO}_{i+2}$; this tells us that the (reversed) restricted vertical orders vector is the derived vector of the multiplicity vector, i.e., its vector of successive differences.* Going in the opposite direction, we see that the vertical orders are obtained from the multiplicities by accumulation:

$$m_i = 1 + \sum_{j=i+2}^{k} \mathrm{VO}_j \tag{10.1}$$

(since we automatically have $m_{k-1} = 1$). Using the entries in the b vector as specified by (8.8), we compute that

$$b_{i+1} - b_i = 1 + \sum_{j=k-i+3}^{k} \mathrm{VO}_j$$

and then compare with (10.1) to conclude that $b_{i+1} - b_i = m_{k-i+1}$. This tells us that the multiplicity vector is the derived vector of the b vector. From these remarks and Corollary 24 we obtain the following result.

Theorem 26. *The b vector is obtained from the beta vector by removing its initial entry. Therefore the multiplicity vector is obtained from* der*, the derived vector of the beta vector, by removing its initial entry, and the (reversed) restricted vertical orders vector is obtained from* der^2 *by removing its initial entry. Conversely one obtains β,* der*, and* der^2 *from the b vector, the multiplicity vector, and the reversed restricted vertical orders vector by inserting the initial entries 1, 1, 0 (respectively).*

Example 27. For a point with RVT code word $RRVTVV$, we have $k = 6$ and

$$(\beta_2, \ldots, \beta_8) = (1, 2, 3, 5, 8, 11, 19) \qquad (b_2, \ldots, b_7) = (2, 3, 5, 8, 11, 19)$$
$$(\mathrm{der}_3, \ldots, \mathrm{der}_8) = (1, 1, 2, 3, 3, 8) \qquad (m_5, \ldots, m_1) = (1, 2, 3, 3, 8)$$
$$({\mathrm{der}^2}_4, \ldots, {\mathrm{der}^2}_8) = (0, 1, 1, 0, 5) \qquad (\mathrm{VO}_6, \ldots, \mathrm{VO}_3) = (1, 1, 0, 5).$$

11. Recursions

Here we address the issue of effective calculations. If we know any one of the six invariants of Section 10, we can easily obtain the other five; thus the issue is how to gain the first toehold.

In Section 8 we presented a method for determining the vertical orders: working in a chart containing our selected point $p \in S(k)$, calculate the focal orders of the standard coordinates; each vertical order equals one of these focal orders or is zero. This is a rather cumbersome method, however, involving parallel calculations for the focal orders of both the coordinates and their differentials.

The proximity diagram of Section 13 of [2] allows one to easily compute the multiplicity sequence. This seems to be the most efficient method for obtaining all six of our invariants. (Regarding proximity diagrams, see also Section 3.5 of [8], but note that Wall does not use code words.)

Implicit in the use of the proximity diagram is a front-end recursion, meaning a recursion for computing an invariant that uses alterations to the front end of the code word. Given a Goursat word W, we obtain its *lifted Goursat word* $\mathrm{LG}(W)$ by the following procedure:

- Remove the first symbol R.

*We take the opportunity to correct a slight error in the range of applicability of Theorem 33 in that earlier paper; the appropriate range there should be $k \leq j \leq r - 2$.

- If the new second symbol is V (i.e., if the truncated word fails to be Goursat), replace it by R, and likewise replace any immediately succeeding T's by R's. Stop when you reach the next R, the next V, or the end of the word.

The terminology comes from a consideration of focal curves on the monster spaces and their lifts into the monster tower. We introduced a similar procedure in Section 10.5 of [2], where we defined the lifted word associated to an arbitrary RVT code word; here, by contrast, we begin and end with Goursat words.

The proximity diagram for W is obtained from the proximity diagram for $\mathrm{LG}(W)$ as follows:

- Put a new vertex to the left (in position 0) and draw a horizontal edge to the vertex in position 1.
- From the vertex in position 1, draw edges rightward to the vertices whose labels have changed (if any); these vertices will be labeled by a block VT^{τ}.

This procedure is shown in Figure 3, which also illustrates the recursive rule for calculating the multiplicity m_i: sum the multiplicities on the vertices proximate to vertex i. (The leftmost vertex is vertex 0.) Appending m_1 to the end of the derived vector for $\mathrm{LG}(W)$ gives us the derived vector for W. Similarly, one obtains each of our five other invariants for a Goursat word W by appending the appropriate value to the end of the same invariant for $\mathrm{LG}(W)$; the easiest way to find this value seems to be to first work out the multiplicities.

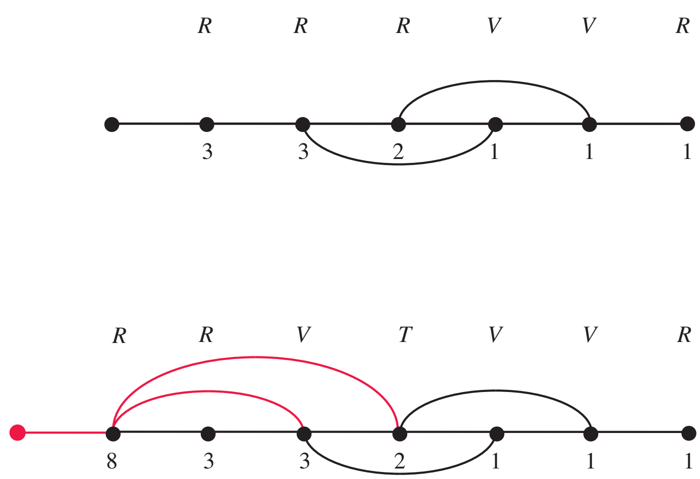


FIGURE 3. Obtaining the proximity diagram for the Goursat word $RRVTVVR$ from the proximity diagram for its lifted Goursat word $RRRVVR$; the new multiplicity is $m_1 = 3 + 3 + 2$.

Example 28. Here is the front-end calculation of the derived vector of $RRVTRRRVTTTV$:

$$\begin{aligned}
\operatorname{der}(RR) &= (1,1) \\
\operatorname{der}(RRV) &= (1,1,2) \\
\operatorname{der}(RRRRRV) &= (1,1,2,2,2,2) \\
\operatorname{der}(RRVTTTV) &= (1,1,2,2,2,2,9) \\
\operatorname{der}(RRRRRRVTTTV) &= (1,1,2,2,2,2,9,9,9,9,9) \\
\operatorname{der}(RRVTRRRVTTTV) &= (1,1,2,2,2,2,9,9,9,9,9,27)
\end{aligned}$$

There are also back-end recursions for our invariants, dating back to the work of Jean [3], who stated such a recursion for the beta vector. He stated and proved it while studying the configuration space for a physical system: a truck pulling multiple trailers. This was before the later work of Montgomery and Zhitomirskii [5], who showed that the truck-and-trailers configuration space is a universal space for Goursat distributions. More precisely, the configuration space for a truck with n trailers is the oriented version of the monster space $\mathbb{R}^2(n+1)$ over the real plane; it is thus an unramified cover of degree 2^{n+1} over $\mathbb{R}^2(n+1)$. Jean's recursion is stated in terms of special angles between trailers, and his proof of the recursion is achieved via an elaborate recursive calculation in trigonometry. Jean's work also predates the introduction of code words for coarse classification of Goursat distributions; see [6] for an early instance.

Here we will show that Jean's back-end recursion is a consequence of our front-end recursion, thus extending its validity to all Goursat distributions in all three settings of Section 2.

Theorem 29. *For every nonempty Goursat word $\beta_2(W) = 1$, and $\beta_3(W) = 2$ for every word of length at least 2. Letting X represent any single symbol, we have these recursive formulas:*

(1) $\beta_j(WR) = 1 + \beta_{j-1}(W)$,
(2) $\beta_j(WXV) = \beta_{j-1}(WX) + \beta_{j-2}(W)$,
(3) $\beta_j(WXT) = 2\beta_{j-1}(WX) - \beta_{j-2}(W)$.

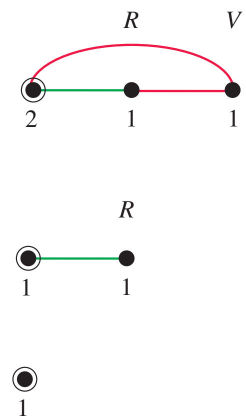
Proof. We first observe that Jean's recursions are equivalent to the following recursions for computing the derived vector, which we proceed to prove:

(1) $\operatorname{der}_j(WR) = \operatorname{der}_{j-1}(W)$,
(2) $\operatorname{der}_j(WXV) = \operatorname{der}_{j-1}(WX) + \operatorname{der}_{j-2}(W)$,
(3) $\operatorname{der}_j(WXT) = 2\operatorname{der}_{j-1}(WX) - \operatorname{der}_{j-2}(W)$.

Using the derived vector to label the vertices of the proximity diagram, let $\operatorname{der}_v(W)$ indicate the component of $\operatorname{der}_v(W)$ labeling vertex v. Note that der is essentially the multiplicity vector; thus we use the same rules to compute it from the diagram. One obtains the proximity diagram of WR from that of W by adding a single vertex at the right, together with a single horizontal edge. The label at this vertex is 1, and the label at each other vertex is unchanged. This establishes (1).

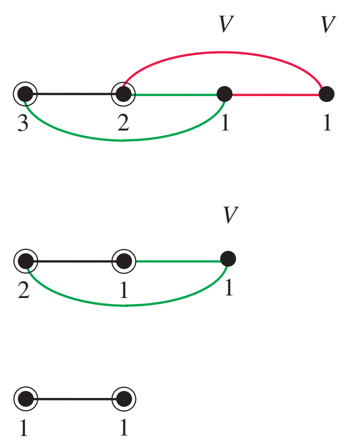
To establish (2), we consider three cases.

- Compare the right ends of the proximity diagrams of WRV, WR, and W:



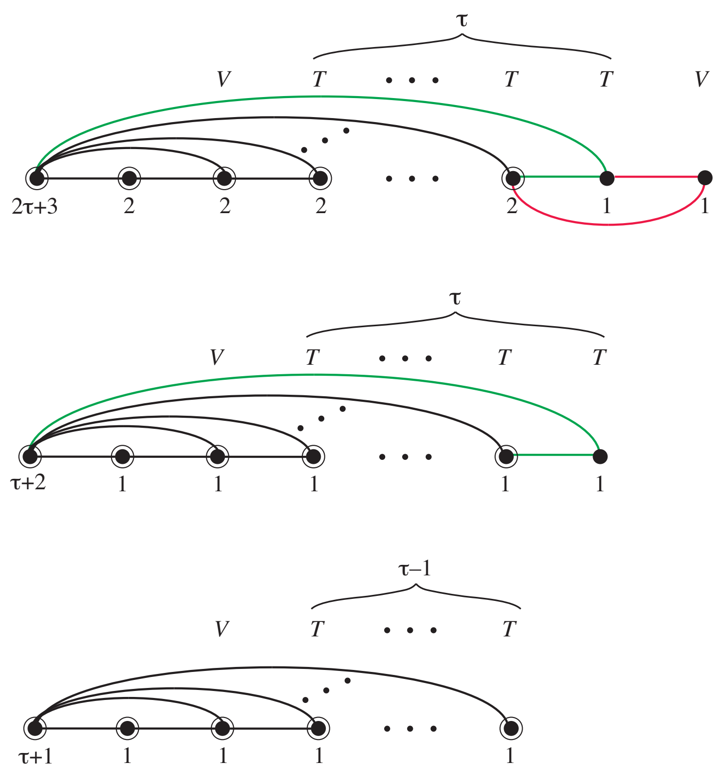
Observe that at the circled vertex we have $\mathrm{der}_v(WRV) = \mathrm{der}_v(WR) + \mathrm{der}_v(W)$. All the undrawn edges are common to all three diagrams; thus the same equation holds for all vertices further to the left.

- Compare the right ends of the proximity diagrams of WVV, WV, and W:



Note that we show only those edges that begin and end within the indicated portion of the diagrams; there may or may not be an additional edge coming in from the left, but it is common to all three diagrams. Observe the equality $\mathrm{der}_v(WVV) = \mathrm{der}_v(WV) + \mathrm{der}_v(W)$ at the circled vertices. This must persist for all vertices further to the left of these.

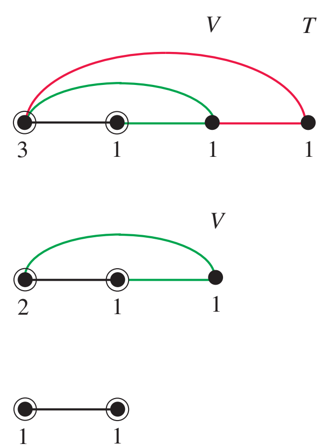
- Compare the right ends of the proximity diagrams of WTV, WT, and W:



Again we show only those edges that begin and end within the indicated portion of the diagrams. Observe that W ends with $VT^{\tau-1}$ for some positive integer τ. At the circled vertices we have $\mathrm{der}_v(WTV) = \mathrm{der}_v(WT) + \mathrm{der}_v(W)$; thus the same equation must hold at every vertex further to the left.

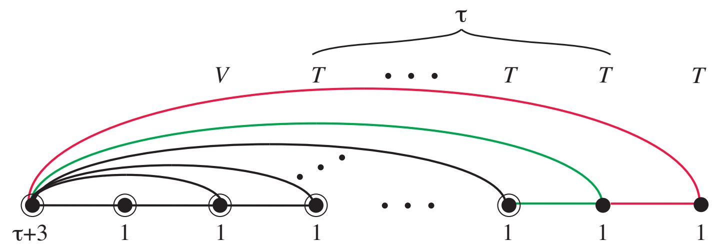
To establish (3), we consider two cases.

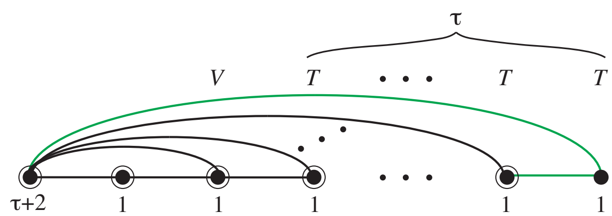
- Compare the right ends of the proximity diagrams of WVT, WV, and W:

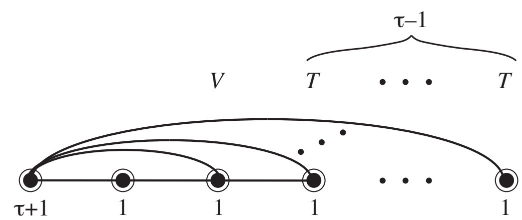


The equality $\mathrm{der}_v(WVT) = 2\,\mathrm{der}_v(WV) - \mathrm{der}_v(W)$ holds at the two circled vertices, and thus at every vertex further to the left.

- Compare the right ends of the proximity diagrams of WTT, WT, and W:







The word W ends with $VT^{\tau-1}$ for some positive integer τ. At the circled vertices we have $\mathrm{der}_v(WTT) = 2\,\mathrm{der}_v(WT) - \mathrm{der}_v(W)$; thus the same equation must hold at every vertex further to the left.

□

Example 30. Here is the back-end calculation of the derived vector of $RRVTRRRVTTTV$:

$$\begin{aligned}
\mathrm{der}(RR) &= (1,1)\\
\mathrm{der}(RRV) &= (1,1,2)\\
\mathrm{der}(RRVT) &= (1,1,1,3)\\
\mathrm{der}(RRVTR) &= (1,1,1,1,3)\\
\mathrm{der}(RRVTRR) &= (1,1,1,1,1,3)\\
\mathrm{der}(RRVTRRR) &= (1,1,1,1,1,1,3)\\
\mathrm{der}(RRVTRRRV) &= (1,1,2,2,2,2,2,6)\\
\mathrm{der}(RRVTRRRVT) &= (1,1,1,3,3,3,3,3,9)\\
\mathrm{der}(RRVTRRRVTT) &= (1,1,1,1,4,4,4,4,4,12)\\
\mathrm{der}(RRVTRRRVTTT) &= (1,1,1,1,1,5,5,5,5,5,15)\\
\mathrm{der}(RRVTRRRVTTTV) &= (1,1,2,2,2,2,9,9,9,9,9,27)
\end{aligned}$$

For completeness, here is the back-end recursion for the second derived vector (i.e., for the vertical orders). To begin, we have

$$(\mathrm{der}^2{}_4(W), \mathrm{der}^2{}_5(W)) = \begin{cases} (0,1) & \text{if } W \text{ ends with } V \\ (0,0) & \text{otherwise.} \end{cases}$$

For the other entries use these formulas, identical to those for the derived vector:

(1) $\mathrm{der}^2{}_j(WR) = \mathrm{der}^2{}_{j-1}(W)$,
(2) $\mathrm{der}^2{}_j(WXV) = \mathrm{der}^2{}_{j-1}(WX) + \mathrm{der}^2{}_{j-2}(W)$,
(3) $\mathrm{der}^2{}_j(WXT) = 2\,\mathrm{der}^2{}_{j-1}(WX) - \mathrm{der}^2{}_{j-2}(W)$.

12. Degree of nonholonomy via the Puiseux characteristic

We end with an observation concerning the degree of nonholonomy. For a point $p \in S(k)$, let W be the RVT code word at p. As we know, W determines the beta vector of the focal focal distribution $\Delta(k)$ at p. We denote it by $\beta(W) = (\beta_2, \dots, \beta_{k+2})$; its last entry β_{k+2} is the degree of nonholonomy. Let $\mathrm{PC}(W) = [\lambda_0; \lambda_1, \dots, \lambda_g]$ be the Puiseux characteristic, as defined in Section 12.5 of [2]. Recall that this is the Puiseux characteristic associated to any regular focal curve germ passing through p, which in turn is the Puiseux characteristic associated to the (usually singular) curve germ on S obtained by projecting. Its initial entry λ_0 is the multiplicity m_0 of this curve. Note that m_0 is not a Goursat invariant, as the following example illustrates.

Example 31. The germ of the focal distribution $\Delta(5)$ at a point with code word $RVTRV$ is equivalent to the focal distribution at some other point with Goursat code word $RRRRV$. The associated Puiseux characteristics are $[6; 8, 9]$ and $[2; 9]$.

Theorem 32.

(1) If W ends with a critical symbol V or T, then the last entry in the Puiseux characteristic is the degree of nonholonomy: $\lambda_g = \beta_{k+2}$.
(2) More generally, if W ends with a critical symbol followed by a string of r occurrences of R, then $\lambda_g + r = \beta_{k+2}$.

Proof. To prove (1), we first observe that if W is RVT^τ, then the Puiseux characteristic is $[\tau+2; \tau+3]$ and the beta vector is $(1, 2, \dots, \tau+3)$. These words provide the base cases for a recursion. If W is not of this form, then its lifted word $L(W)$ again ends in a critical symbol. Since the beta vector is the accumulation vector for the multiplicities, we know

$$\beta_{k+2}(W) = m_0(L(W)) + \beta_{k+1}(L(W)). \tag{12.1}$$

There is a similar result for the last entry of the Puiseux characteristic, according to Theorem 24 of [2]; in all three cases of that theorem we see that

$$\lambda_g(W) = \lambda_0(L(W)) + \lambda_{\text{last}}(L(W)).$$

(Here λ_{last} is either λ_g or λ_{g-1}.) Thus if the desired equation holds for $L(W)$ it likewise holds for W.

Part (2) follows from the observation that adding an R to the end of a word doesn't alter the Puiseux characteristic, while increasing the degree of holonomy by 1. $\square$

The proof is elementary, but it may raise a question. As we have said, m_0 is not a Goursat invariant, but it appears in equation (12.1) as the difference of two Goursat invariants; how can this be? The apparent contradiction is resolved by observing that all the RVT code words associated to a particular Goursat distribution have the same lifted word; thus although the distribution does not determine $m_0(W)$ it does determine $m_0(L(W))$. To say this another way, the quantity $m_0(L(W))$ is the same as $m_1(W)$, and we know that m_1 is a Goursat invariant.

Theorem 32 is implicit in [7]. Our discussion here concerns just the initial entry and last entry of the Puiseux characteristic; the cited paper also treats the more intricate relations between the other entries of the Puiseux characteristic and the multiplicity sequence.

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