

Star coloring bipartite planar graphs

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Abstract

A *star coloring* of a graph is a proper vertex-coloring such that no path on four vertices is 2-colored. We prove that the vertices of every bipartite planar graph can be star colored from lists of size 14, and we give an example of a bipartite planar graph that requires at least 8 colors to star color.

1 Introduction

A proper coloring of a graph G is an assignment of colors to the vertices of G such that adjacent vertices receive different colors. In 1973, Grünbaum [4] introduced *acyclic colorings*, that is proper colorings such that all 2-colored subgraphs are acyclic. Grünbaum proved that any planar graph can be acyclically colored with nine colors, and conjectured that any planar graph can be acyclically colored with five colors. This was confirmed by Borodin [2] in 1979. Kostochka and Melnikov [5] gave examples of a bipartite planar graph that requires 5 colors to acyclically color so that Borodin's result is optimal, even when restricted to bipartite planar graphs.

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Grünbaum [4] also suggested the notion of what has become known as *star colorings*, i.e. proper colorings such that no path on four vertices is 2-colored, in other words all 2-colored subgraphs are star-forests. Any star coloring is also an acyclic coloring so that star colorings typically require more colors than acyclic colorings, and less is known about star colorings of planar graphs. Albertson, Chappell, Kierstead, Kündgen, and Ramamurthi [1] established that the minimum number of colors sufficient for star coloring arbitrary planar graphs is between 10 and 20. A corollary to a result of Nešetřil and Ossona de Mendez [6] shows that any bipartite planar graph can be star colored with 18 colors, but the best published constructions of bipartite planar graphs require only 5 colors to star color: Fertin, Raspaud and Reed [3] establish this for grid graphs of size at least 4×4 , whereas Ramamurthi and Sanders [7] show that bipartite outerplanar graphs can be 5-star colored and that this is best possible by exhibiting a small outerplanar grid-like graph requiring 5 colors. Of course the examples of Kostochka and Melnikov [5] also require at least 5 colors to star color: for example, the graph obtained from the double-wheel $C_5 \vee \overline{K}_2$ by replacing each edge uv of the 5-cycle with a copy of $K_{2,4}$ so that u, v are the degree 4 vertices.

In Section 3 we prove that every bipartite planar graph can be star colored from lists of size 14, and in Section 4 we give an example of a bipartite planar graph that requires 8 colors to star color. We begin by collecting some relevant definitions and notations.

2 Preliminaries

All graph considered have no loops or multiple edge. We denote the vertex set of a graph G by $V(G)$ and denote the edge set of G by $E(G)$. If G has vertex set $V(G)$ and edge set $E(G)$ we may write $G = (V, E)$. An *orientation* of a graph G is a digraph $\vec{G}$ obtained by assigning one of two directions to each edge of G . If $\vec{G}$ is a digraph, we may write $G = (V, \vec{E})$. If edge e has endpoints x and y we may write xy to denote e . If e is an oriented edge that points away from x and towards y then we may write (x, y) or $x \rightarrow y$ to denote e and its orientation.

Let $d_G(x)$ denote the *degree* of x in G and let $\Delta(G)$ denote the maximum degree of G . The *outdegree* of a vertex v in a digraph is the number of edges adjacent to v that are oriented away from v . Let $d_G^+(v)$ denote the outdegree of v in $\vec{G}$. The maximum outdegree of a digraph $\vec{G}$ is denoted by $\Delta^+(\vec{G})$.

A graph is *k-choosable* if given any assignment of lists of size k to the vertices of G , G has a proper coloring in which the colors are chosen from the lists. Similarly, a graph is *k-star choosable* if given any assignment of lists of size k to the vertices of G , there is a star coloring in which the colors are chosen from the lists. It is known that if every subgraph $H \subseteq G$ has average (and thus minimum) degree less than k , then G is *k-choosable*. Of course *k-star choosable* graphs are *k-star colorable*.

3 The upper bound

In this section we prove that bipartite planar graphs are 14-star choosable. A useful equivalent notion to star coloring introduced (in an equivalent form) by Nešetřil and Ossona de Mendez [6] and used by Albertson, Chappell, Kierstead, Kündgen, Ramamurthi [1] is that of an in-coloring. An *in-coloring* of a digraph $\vec{G}$ is a proper coloring of $\vec{G}$ such that any 2-colored path on three vertices has its edges oriented towards the center vertex. We repeat the following proof for the sake of completeness.

Lemma 3.1 *A coloring of G is a star coloring if and only if it is an in-coloring for some orientation $\vec{G}$ of G .*

Proof. If G has been assigned a star coloring, then we may orient each edge towards the center of each star in each star forest induced by the union of two color classes. Assume $\vec{G}$ has an in-coloring. If $xyzt$ is a path on four vertices in G then we may assume yz is oriented into z . Then three colors must appear on x, y , and z . ■

For any graph $G = (V, E)$ and orientation $\vec{G} = (V, \vec{E})$, define a supergraph $\vec{G}^* = (V, \vec{E}^*)$ by $\vec{E}^* = \vec{E} \cup O(\vec{G}) \cup T(\vec{G})$, where

$$O(\vec{G}) = \{xz : xz \notin E \wedge x \neq z \wedge \exists y(y \rightarrow x \wedge y \rightarrow z)\} \text{ and} \\ T(\vec{G}) = \{xz : xz \notin E \wedge \exists y(x \rightarrow y \wedge y \rightarrow z)\}.$$

At times we may abuse notation by also writing $O(\vec{G})$ and $T(\vec{G})$ for the graphs they induce.

Since an in-coloring of $\vec{G}$ corresponds to a proper coloring of $\vec{E}^*$ we immediately obtain the following result, which is essentially Corollary 3 of [6].

Lemma 3.2 *A coloring of G is a star coloring if and only if it is a proper coloring of $\vec{G}^*$ for some orientation $\vec{G}$ of G .*

The following observation will be crucial.

Lemma 3.3 *Every bipartite planar graph G can be partitioned into two edge-disjoint forests.*

Proof. Every subgraph $H \subseteq G$ is bipartite and planar. Using Euler's formula H has at most $2|V(H)| - 4$ edges. By the Nash-Williams Theorem, the edges of G can be partitioned into two forests. ■

Theorem 3.4 *Every bipartite planar graph is 14-star choosable.*

Proof. Let $G = (V, E)$ be a bipartite planar graph. Using Lemma 3.3, let $G = F_1 \cup F_2$, where F_1 and F_2 are edge-disjoint forests. Orient each F_i , $i = 1, 2$, as $\vec{F}_i$ so that $\Delta^+(\vec{F}_i) \leq 1$ and let $\vec{G} = (V, \vec{E}) = \vec{F}_1 \cup \vec{F}_2$. Denote $(x, y) \in E(\vec{F}_i)$ by $x \rightarrow_i y$. By Lemma 3.2 it suffices to show that $\vec{G}^*$ is 14-choosable.

Define graphs $H = (V, A)$ and $H' = (V, A')$ and digraph $\vec{H}' = (V, \vec{A}')$ by

$$\begin{aligned} A &= \{xz : xz \notin E \wedge x \neq z \wedge \exists y(xy \in E \wedge y \rightarrow_1 z)\}, \\ \vec{A}' &= \{(x, z) : xz \notin E \wedge \exists y(x \rightarrow y \wedge y \rightarrow_2 z)\} \text{ and} \\ A' &= \{xz : (x, z) \in \vec{A}'\}. \end{aligned}$$

First we show that $O(\vec{G}) \subseteq A$: Suppose $x_1x_2 \in O(\vec{G})$. Then there exists $y \in V$ such that $y \rightarrow x_1$ and $y \rightarrow x_2$. Since $d_{\vec{F}_2}^+(y) \leq 1$, there exists $i \in \{1, 2\}$ such that $y \rightarrow_i x_i$. Thus $x_1x_2 \in A$.

Next we show that $T(\vec{G}) \subseteq A \cup A'$: Suppose $x_1x_2 \in T(\vec{G})$. Then there exists $y \in V$ and $i \in \{1, 2\}$ such that $x_{3-i} \rightarrow y$ and $y \rightarrow x_i$. If $y \rightarrow_1 x_i$ then $x_1x_2 \in A$; otherwise $y \rightarrow_2 x_i$ and $x_1x_2 \in A'$.

It now follows that $G^* \subseteq G \cup H \cup H'$, so to prove that $\vec{G}^*$ is 14-choosable it suffices to show that the average degree of any subgraph of $G \cup H \cup H'$ is less than 14. Note that $\Delta^+(\vec{H}') \leq 2$. Since every edge in H goes between vertices of the same color class in G it follows that H consists of two disjoint subgraphs, both of which are minors of G , and thus H is planar. By hypothesis G is a bipartite planar graph. These properties are all preserved for subgraphs. The average degree of a bipartite planar graph is less than 4. So the average degree of any subgraph of $G \cup H \cup H'$ is less than $4 + 6 + 4 = 14$. ■

4 A construction

In this section we give an example of a bipartite planar graph that is not 7-star colorable.

The simple observation that in any 7-star coloring of $K_{2,7}$ the degree seven vertices must receive different colors will be useful. So if a given graph contains vertices x and y that are the degree 7 vertices of a $K_{2,7}$ -subgraph, then we say that x and y are *weakly adjacent*, and we speak of the *weak edge* xy . A weak edge between two vertices will be depicted by a double edge (see Figure 1). Weak edges can help us in bipartite constructions because they allow us to connect vertices in the same partition. For example, the Kostochka-Melnikov construction [5] is essentially a $K_{2,5}$ in which we add a 5-cycle of weak edges in the larger part. We can also use the Ramamurthi-Sanders construction [7] of a bipartite outerplanar graph requiring 5 colors to obtain a bipartite planar graph requiring 6 colors by adding a vertex in the outer face, and connecting it with edges to every vertex in the first part, and with weak edges to every vertex in the second part. However, we need to work harder than that to force more colors.

A *weak k -path* is a sequence of vertices $z_1, \dots, z_k$ such that z_i is weakly adjacent to z_{i+1} for $k = 1, \dots, k-1$. If x and y are two vertices each of which is adjacent to every vertex of a weak 7-path, then we say that $S(xz_1yz_7)$ is a *strong 4-cycle*, and $S(xz_1yz_7)$ will be depicted by a 4-cycle in which x, y are colored black and z_1, z_7 are colored white (See Figure 1). Observe that in this case x and y are weakly adjacent.

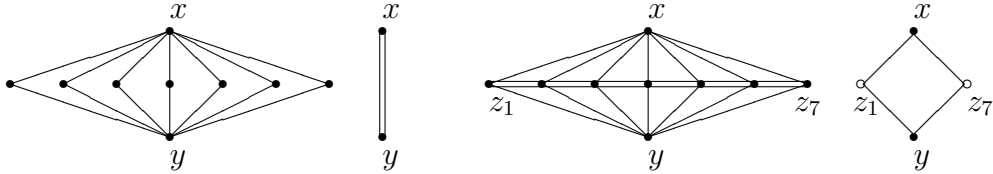


Figure 1: A weak edge

A strong 4-cycle

The next definition is crucial for our construction.

Definition 4.1 A k -cluster with center v is a graph C with vertex-set $V(C) = \{v, x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$, together with a star-coloring f such that:

1. v has k distinct neighbors $x_1, x_2, \dots, x_k$, called its special neighbors,

2. each x_i is adjacent to a vertex $y_i \neq v$ with $f(y_i) = f(v)$.

The edges $x_i y_i$ are said to be the legs of the k -cluster.

Observe that we require the x_i 's to be distinct and so they receive distinct colors, and none of these colors is $f(v)$. However some of the y_i 's may be identical. For instance every 7-star colored weak edge xy contains a 2-cluster with center different from x and y . The main idea of the construction is to build a sequence of bipartite planar graphs G_3, G_4, G_5 such that every 7-star coloring of G_k contains a k -cluster (for $k = 3, 4$), whereas G_5 can't be 7-star colored at all. To construct G_k we will attach a separate copy $H_k(v)$ of a graph H_k to every vertex v of G_{k-1} , and use $H_k(v)$ to show that the $(k-1)$ -cluster C in G_{k-1} centered at v forces a k -cluster in the subgraph $H_k(v) \cup C$ of G_k . Formally we say that a cluster C with center v is *attached* to a subgraph H if v is the only vertex shared by C and H . Figure 4 shows 2-clusters attached to v and w in a specific graph.

To start the construction we let $A(v, y)$ denote the graph consisting of the strong 4-cycles $S(ca^1 ya^7)$ and $S(cb^1 vb^7)$ (which only share c) together with the edges va^1 and va^7 . $A(v, y)$ will be denoted by a triple edge with an arrow pointing towards y (See Figure 2). Furthermore $B(v, w)$ denotes the bipartite planar graph obtained from $K_{2,8}$ with parts $\{v, w\}$ and $\{x_1, \dots, x_8\}$ by connecting each x_i to x_{i+1} with a path $x_i y_i x_{i+1}$ and then adding in $A(v, y_i)$ and $A(w, y_i)$ for all $1 \leq i \leq 7$ (See Figure 3). Finally, G_3 is obtained from a $K_{2,14}$ with parts $\{w_1, w_2\}$ and $\{u_1, \dots, u_{14}\}$ by adding $B(u_i, u_{i+1})$ for all $1 \leq i \leq 13$.

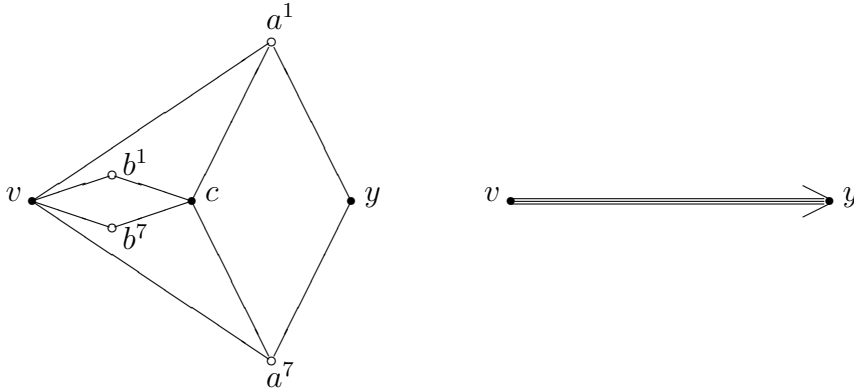


Figure 2: Graph $A(v, y)$

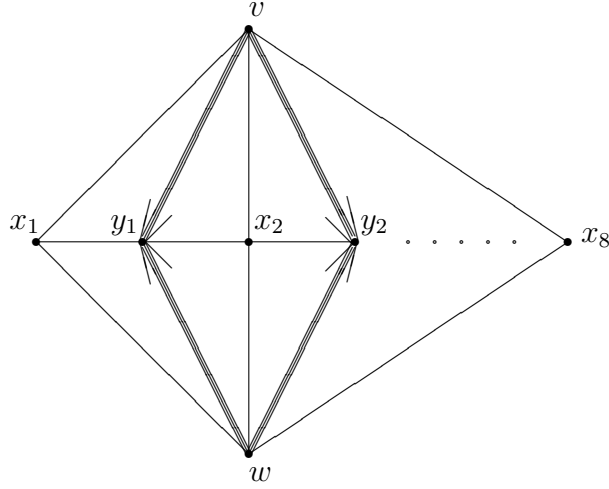


Figure 3: Graph $B(v, w)$

Lemma 4.2 *Every 7-star coloring of G_3 contains a 3-cluster.*

Proof. Let f be a 7-star coloring of G_3 , where we may assume that $f(w_1) = 1$ and $f(w_2) = 2$. Observe that $K_{2,14}$ must contain consecutive vertices u_i, u_{i+1} both of which are centers of 2-clusters in $K_{2,14}$ (any u_i whose color is used twice in $K_{2,14}$ is the center of a 2-cluster). Since $B(u_i, u_{i+1})$ contains a weak edge $u_i u_{i+1}$ we can assume that $f(u_i) = 3$, $f(u_{i+1}) = 4$. Observe that w_1 and w_2 are special neighbors of both u_i and u_{i+1} . Thus we have a copy of $B(v, w)$ with attached 2-clusters as in Figure 4, where the neighbors of v, w of colors 1, 2 represent w_1, w_2 respectively. We focus on this copy of $B(v, w)$ and relabel u_i as v and u_{i+1} as w .

Observe colors 3 and 4 are forbidden on each y_j for if $f(y_j) = 3$ or $f(y_j) = 4$ then v or w respectively is the center of a 3-cluster. If $f(x_j) = f(x_{j+1})$, then x_j is the center of a 3-cluster with legs vx_{j+1} , wx_{j+1} , and $y_j x_{j+1}$. Since each x_j is adjacent to both v and w , $f(x_j) \in \{5, 6, 7\}$ for each j . Since there are eight distinct x_j 's, we can find $j \neq k$ with $f(x_j) = f(x_k)$ and $f(x_{j+1}) = f(x_{k+1})$. Without loss of generality, we can assume that $f(x_j) = 5$ and $f(x_{j+1}) = 6$, and all vertices are named as in Figure 4.

Both a_j^1 and a_j^7 are adjacent to v so colors 1, 2, and 3 are forbidden on a_j^1 and a_j^7 . If either of these vertices are assigned colors 5 or 6, then x_j respectively x_{j+1} is the center of a 3-cluster with legs $y_j a_j^i, vx_k, wx_k$ respectively

$y_j a_j^i, vx_{k+1}, wx_{k+1}$ ($i = 1$ or $i = 7$). If a_j^1 and a_j^7 are assigned the same color, then a_j^1 is the center of a 3-cluster with legs $va_j^7, c_j a_j^7$, and $y_j a_j^7$. So, by symmetry of $A(v, y_j)$, we may assume $f(a_j^1) = 7$ and $f(a_j^7) = 4$. Similarly, we may assume $f(p_j^1) = 7$ and $f(p_j^7) = 3$. Applying the same argument with the roles of k and j interchanged we obtain $f(a_k^1) = 7, f(a_k^7) = 4, f(p_k^1) = 7$, and $f(p_k^7) = 3$. Only colors 1 and 2 are available for y_j , so without loss of generality $f(y_j) = 1$ (See Figure 4).

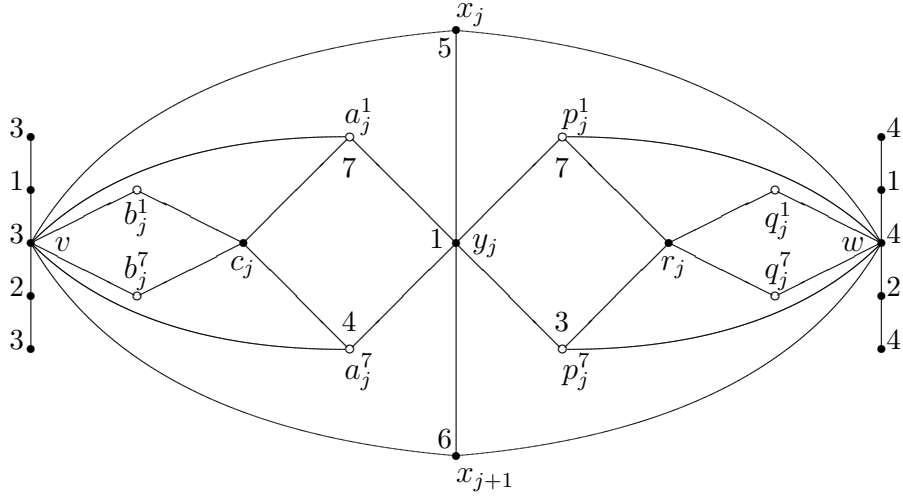


Figure 4: Partial Coloring of $B(v, w)$

Since each a_j^i is adjacent to y_j , color 1 is forbidden on each a_j^i . If a_j^i is assigned color 4, then a_j^7 is the center of a 3-cluster with legs $c_j a_j^i, y_j a_j^i$, and va_k^7 . Similarly, if a_j^i is assigned colors 5, 6, or 7, then x_j, x_{j+1} , or a_j^1 (respectively) is the center of a 3-cluster with legs $y_j a_j^i, vx_k, wx_k$ or $y_j a_j^i, vx_{k+1}, wx_{k+1}$ or $y_j a_j^i, c_j a_j^i, va_k^1$ (respectively). Therefore we may assume $f(a_j^s) = 2$ and $f(a_j^{s+1}) = 3$ for some $s \in \{2, 3\}$. By symmetry, we may assume $f(p_j^t) = 2$ and $f(p_j^{t+1}) = 4$ for some $t \in \{2, 3\}$.

Since each b_j^i is adjacent to v , colors 1, 2, and 3 are forbidden on b_j^i . If $f(b_j^i) = 4$ then a_j^7 is the center of a 3-cluster with legs $vb_j^i, c_j b_j^i$, and $y_j p_j^{t+1}$. If $f(b_j^i) = 7$ then a_j^1 is the center of a 3-cluster with legs $vb_j^i, c_j b_j^i$, and $y_j p_j^1$. Therefore colors 5 and 6 must be used on the weakly adjacent vertices b_j^1 and b_j^2 . Since c_j has neighbors of each color other than 1, it follows that $f(c_j) = 1$, but c_j is weakly adjacent to y_j . Thus there is no 7-star coloring

of G_3 without 3-clusters. ■

Let $H_4 = H_4(v)$ be the graph constructed as follows. Start with a $K_{2,7}$ with parts $\{v, w\}$ and $\{x_1, \dots, x_7\}$, and then do the following for all $1 \leq j \leq 6$: add the path $x_j y_j x_{j+1}$, the strong 4-cycles $S(va_j^1 y_j a_j^7)$ and $S(x_j b_j^1 x_{j+1} b_j^7)$, followed by the weak edges $x_j a_j^1$ and $x_{j+1} a_j^7$. Now let G_4 be obtained by identifying every vertex v of G_3 with its own copy $H_4(v)$ of H_4 at the vertex v .

Lemma 4.3 *Every 7-star coloring of G_4 contains a 4-cluster.*

Proof. Suppose f is a 7-star coloring of G_4 . Then by Lemma 4.2 G_3 contains a 3-cluster C centered at some vertex v , and we may assume that $f(v) = 1$, and the special neighbors of v have been assigned colors 2, 3, and 4. This 3-cluster is attached to $H_4(v)$, and we look for a 4-cluster in the subgraph $C \cup H_4(v)$ of G_4 .

Since each x_j is adjacent to v , $f(x_j) \in \{5, 6, 7\}$ for each j . Also, each x_j is weakly adjacent to x_{j+1} so that $f(x_j) \neq f(x_{j+1})$. We can find $2 \leq j \leq 5$ with $f(x_j) = f(x_k)$ and $f(x_{j+1}) = f(x_l)$ for some $k, l \in \{1, \dots, 7\}$, $k \neq j$ and $l \neq j+1$, and without loss of generality $f(x_j) = 5$ and $f(x_{j+1}) = 6$.

For each i, j , only colors 5, 6, and 7 are available for a_j^i since each a_j^i is adjacent to v . The weak path $a_j^1 \dots a_j^7$ requires 2 colors so that we may assume $f(a_j^m) = 6$ for some m (See Figure 5). Colors 1, 5, and 6 are forbidden on y_j since y_j is weakly adjacent to v , and is adjacent to x_j and x_{j+1} . If $f(y_j) = 7$, then since a_j^1 is weakly adjacent to x_j , we must have $f(a_j^1) = 6$ which forces $f(a_j^7) = 6$, a contradiction since a_j^7 is weakly adjacent to x_{j+1} . By symmetry of colors 2, 3, and 4, we may assume $f(y_j) = 2$ (See Figure 5).

If $f(a_{j+1}^i) = 6$ then x_{j+1} is the center of a 4-cluster with legs $y_{j+1} a_{j+1}^i$, $y_j a_j^m$, vx_l , and wx_l . Thus the weak path $a_{j+1}^1 \dots a_{j+1}^7$ must be colored using colors 5 and 7. If $f(y_{j+1}) = 2$ then the path $y_{j+1} x_{j+1} y_j a_j^m$ is 2-colored. By symmetry of colors 3 and 4, we may assume $f(y_{j+1}) = 3$.

Observe that each vertex in $X = \{x_{j+2}, w, b_{j+1}^1, b_{j+1}^2\}$ must receive a different color as x_{j+2} is adjacent to w, b_{j+1}^1 , and b_{j+1}^2 , the path $x_l w x_{j+1} b_{j+1}^i$, $i = 1, 2$, would be 2-colored if w and b_{j+1}^i receive the same color, and b_{j+1}^1 is weakly adjacent to b_{j+1}^2 . For a contradiction it suffices to show that colors 1, 2, 3, and 6 cannot be used on any vertex in X .

Colors 1, 2, and 3 cannot be assigned to x_{j+2} since x_{j+2} is adjacent to v . If color 1 is assigned to any vertex in $X - \{x_{j+2}\}$, then v is then center of a 4-cluster. If $f(z) = 2$ for some $z \in X - \{x_{j+2}\}$, then $zx_{j+1} y_j a_j^m$ is 2-colored.

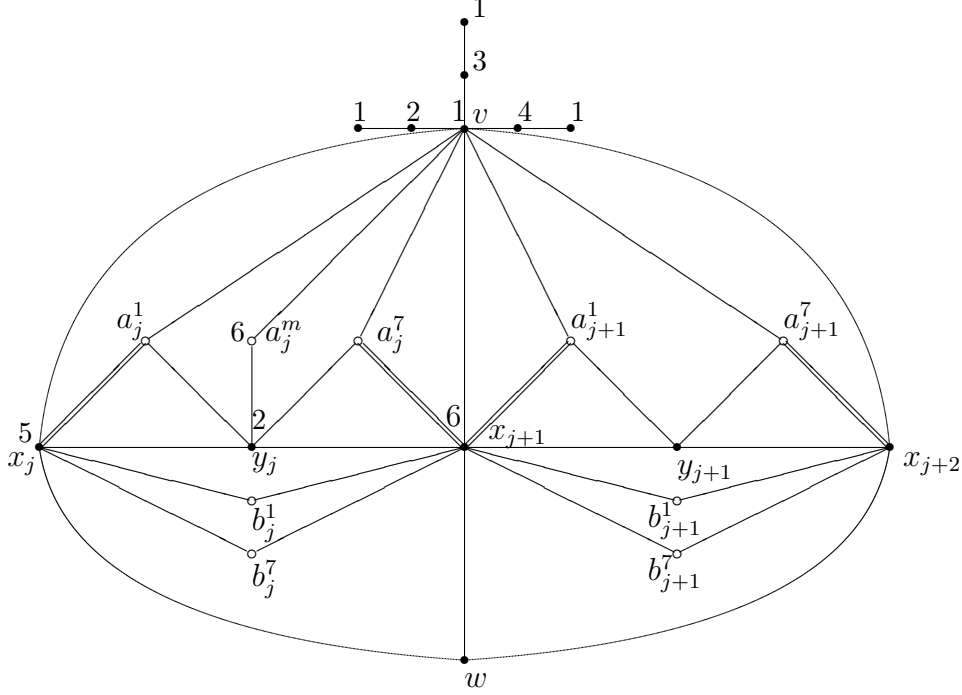


Figure 5: Partial coloring of H_4

If $f(w) = 3$ then $y_{j+1}x_{j+1}wx_l$ is 2-colored. Since x_{j+1} is weakly adjacent to x_{j+2} , it must be that $f(x_{j+2}) \in \{5, 7\}$. The weak path $a_{j+1}^1 \dots a_{j+1}^7$ is colored using colors 5 and 7 therefore, if $f(b_{j+1}^i) = 3$ with $i = 1, 2$, then $b_{j+1}^i x_{j+2} y_{j+1} a_{j+1}^n$ is 2-colored for some n . Since each vertex in X is adjacent or weakly adjacent to x_{j+1} , color 6 is forbidden on X . We conclude that no vertex in X may be assigned colors 1, 2, 3, or 6 which gives the needed contradiction. ■

Let $H_5(v)$ consist of a vertex v adjacent to the vertices x_1, x_2, x_3 of a triangle of weak edges. Obtain G_5 by attaching a separate $H_5(v)$ to each vertex v of G_4 .

Theorem 4.4 G_5 has no 7-star coloring.

Proof. Suppose f is a 7-star coloring of G_5 . Then by Lemma 4.3 G_4 contains a 4-cluster C centered at v and we may assume $f(v) = 1$ and the special neighbors of v have been assigned colors 2, 3, 4, and 5. This 4-cluster is

attached to $H_5(v)$ and colors 1,2,3,4, and 5 cannot be used on any x_i in $H_5(v)$, but each x_i must be assigned a distinct color. ■

5 Remarks

It is possible that neither the upper bound of 14, nor the lower bound of 8 is the correct answer for the minimum k such that each bipartite planar graph is k -star colorable or k -star choosable, but we believe that the answer for both questions is probably closer to the lower bound. To improve the lower bound from 8 to 9 the difficulty lies in forcing k -clusters for $3 \leq k \leq 5$ and then the last step, Theorem 4.4, would work similarly.

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