Star coloring planar graphs from small lists

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Abstract

A *star coloring* of a graph is a proper vertex-coloring such that no path on four vertices is 2-colored. We prove that the vertices of every planar graph of girth 6 (respectively 7,8) can be star colored from lists of size 8 (respectively 7,6). We give an example of a planar graph of girth 5 that requires 6 colors to star color.

1 Introduction

A *proper coloring* of a graph is an assignment of colors to the vertices of the graph such that adjacent vertices are assigned different colors. In 1973, Grünbaum [4] introduced acyclic colorings and star colorings. An *acyclic coloring* of a graph is a proper coloring such that no cycle is 2-colored. A *star coloring* of a graph is a proper coloring such that no path on four vertices is 2-colored, or equivalently, the union of any two color classes is a star forest. The fewest number of colors needed to properly (resp. acyclically, star) color a graph G is the *chromatic* (resp. *acyclic chromatic, star chromatic*) *number* of G.

Grünbaum [4] showed that every planar graph can be acyclically colored with 9 colors, and conjectured that any planar graph can by acyclically colored with 5 colors. In 1979, Borodin [2] confirmed Grünbaum's conjecture. This upper bound is best possible since there are planar graphs that require 5 colors to acyclically color (see [6] for example). Grünbaum noted that the star chromatic number of a graph is bounded by a function of its acyclic chromatic number. This observation, along with Borodin's 5-coloring result, implies that any planar graph can be star colored with 80 colors.

In 2003, Nešetřil and Ossona de Mendez [7] made a significant improvement upon this upper bound by proving that any planar graph has a star coloring with 30 colors. Additionally, they proved that any bipartite planar graph has a star coloring with 18 colors. Further improvements were made by Albertson et

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al. [1] who proved that any planar graph has a star coloring with 20 colors, and gave an example of a planar graph that required 10 colors to star color. They also showed that planar graphs of girth 5 (resp. 7) can be star colored with 16 (resp. 9) colors. Furthermore, they observed that the graph C_n^+ (an n-cycle with a leaf vertex added to each vertex of the cycle) has star chromatic number 4 when n is not divisible by 3. Thus there are planar graphs of arbitrarily high girth that require 4 colors to star color. Recently Kierstead et al. [5] showed that bipartite planar graphs can be (list) star colored with 14 colors, and gave an example of a bipartite planar graph that requires 8 colors to star color. For low girth planar graphs, the bounds proved in [1] and [5] are the best known.

For high girth planar graphs, the bounds are closer together. Bu et al. [3] proved that planar graphs of girth 13 can be star colored with 4 colors, while in [10] it is shown that planar graphs of girth 9 can be star colored with 5 colors, and an example of a planar graph of girth 7 that requires 5 colors to star color is given.

In this paper we improve upon the upper bounds on the star chromatic number for the families of planar graphs of girth in the intermediate range 6 to 8. In particular, we show that planar graphs of girth 6 (resp. 7,8) can be star colored from lists of size 8 (resp. 7,6) assigned to each vertex. Bu, et al. [3] use a partitioning approach to show that planar graphs of girth 7 (resp. 8) can be star colored with 7 (resp. 6) colors. The results in the current paper have the advantage of being list-coloring results.

We conclude this paper by giving an example of a girth 5 planar graph whose star chromatic number is at least 6. This construction and weaker versions of Theorems 4.1 and 5.1 can also be found in [9].

2 Preliminaries

A k-vertex is a vertex of degree k. A k^--vertex is a vertex of degree at most k, and k^+-vertex is a vertex of degree at least k. A $k(d)$-vertex is a k-vertex adjacent to d 2-vertices. C_n and P_n denote a cycle and a path, respectively, on n vertices. If H is a subgraph of G, then $N_H(v)$ is the set of neighbors of v in H.

An *in-coloring* of a directed graph $\vec{G}$ is a proper coloring of the underlying graph G such that any 2-colored P_3 has its edges oriented towards the middle vertex. We say that $\vec{G}$ is *k-in-colorable* if $\vec{G}$ has an in-coloring with at most k colors. In-colorings were used implicitly by Nešetřil and Ossona de Mendez [7], and explicitly by Albertson et al. [1], who formalized the connection to star coloring in the following lemma. We include the proof for completeness.

Lemma 2.1 *A coloring of the vertices of G is a star coloring if and only if it is an in-coloring for some orientation $\vec{G}$ of G.*

Proof. In a star coloring of G the subgraph induced by the union of any two color classes is a star forest, and we simply orient each edge towards the center of the 2-colored star containing it to obtain the desired orientation of G.

Conversely, an in-coloring of $\vec{G}$ is a star coloring of G: Suppose $xyzt$ is a path in G, and that yz is oriented into z. Then three colors must appear on xyz so that $xyzt$ is not 2-colored. ■

In order to control the number of colors used in an in-coloring it is useful to bound the maximum outdegree of the orientation $\vec{G}$. For instance in [5], it is shown that any bipartite planar graph G has a star-coloring with 14 colors by using a specific orientation with maximum outdegree 2. Tarsi [8] showed that a graph has an orientation with maximum outdegree at most d if and only if $\text{mad}(G) \leq 2d$, where $\text{mad}(G)$ denotes the maximum possible average degree over all subgraphs of G. A planar graph G of girth g has $\text{mad}(G) < \frac{2g}{g-2}$, so it has an orientation of outdegree at most 2 as long as $g \geq 4$. It is important to note that in this paper **all orientations have maximum outdegree at most 2.**

We say that a graph G is *k-star-choosable*, if for any assignment of lists of size k to the vertices of G we can find a star-coloring of G where the color of each vertex is chosen from its list. Similarly, a directed graph $\vec{G}$ is *k-in-choosable* if for any assignment of lists of size k to the vertices of $\vec{G}$ we can find an in-coloring of $\vec{G}$ in which the color of each vertex is chosen from its list. A graph G is *k-in-choosable* if some orientation $\vec{G}$ of G with maximum outdegree 2 is k-in-choosable.

3 Reducible Configurations

To prove the upper bounds, we use discharging. In this section we collect the reducible configurations that we will need.

A graph G is a *k-obstruction* if every proper subgraph of G is k-in-choosable, but G itself is not k-in-choosable. Observe that k-obstructions are always connected. K_2 is the only 1-obstruction, whereas the 2-obstructions are C_3 and P_4. The graphs C_n^+ (with n not divisible by 3) and C_5 are 3-obstructions.

A *configuration* is a connected graph H in which each vertex v has a label $l(v) \in \{0, 1\}$. If $l(v) = 0$, then v is called an *interior* vertex, and if $l(v) = 1$, then v is called a *boundary* vertex. We say that a graph G *contains* H, if H is an induced subgraph of G with $\deg_G(v) = \deg_H(v) + l(v)$. A *good orientation* $\vec{H}$ of H, is an orientation in which each vertex v has outdegree at most $2 - l(v)$. We say that H is *k-reducible* if it has a good orientation $\vec{H}$, that is in-choosable when each vertex has a list of size $k - m(v)$, where $m(v) = 3l(v) + \sum_{w \in N_H(v)} l(w)$.

Lemma 3.1 *A k-obstruction contains no k-reducible configuration.*

Proof. Let G be a k-obstruction which contains a k-reducible configuration H. We combine an orientation of $G - V(H)$, which is k-in-choosable, together with a good orientation of H to obtain an orientation $\vec{G}$ of G by orienting all edges with exactly one endpoint in $V(H)$ away from H. We claim that $\vec{G}$ is k-in-choosable, so let an assignment of lists $L(v)$ (with $|L(v)| = k$) to $v \in V(G)$ be given.

Color $G-V(H)$ from these lists. We now color $V(H)$. Each vertex $v \in V(H)$ with $l(v) = 1$ has a unique neighbor $v' \notin V(H)$ which is already colored, and we remove the colors of v' and its up to two outneighbors from $L(v)$. Furthermore, if a vertex $v \in V(H)$ has a neighbor u with $l(u) = 1$, then we also remove the color of u' from $L(v)$. Thus each $v \in V(H)$ has a list of size at least $k - m(v)$ remaining, and $\vec{H}$ can be in-colored from these lists.

To see that we obtained an in-coloring of $\vec{G}$, observe that every vertex receives a color different from its neighbors, so suppose that uvw is a 2-colored path with at least one vertex in $G - V(H)$, and another in $V(H)$. If $v \in V(H)$ then, since $\vec{H}$ is in-colored, it suffices to consider $u = v'$ i.e. $u \in G - V(H)$. Then $w \in V(H)$ and since vv' is oriented into v', w receives a color different from v'. If $u, v \notin V(H)$ and $w \in V(H)$, then w receives a color different from u whenever uv is oriented into u. Finally, since edges are oriented away from H, we observe that for $u, w \in V(H)$ and $v \notin V(H)$, both uv and wv are directed towards v. ∎

We specify a configuration H as a graph and indicate which, if any, vertices are interior. The remaining vertices may be assumed to be boundary vertices, since that only makes reducing the configuration harder. If H has no interior vertex, then $m(v) = \deg_H(v) + 3$, and an orientation of H is good if and only if each vertex has outdegree at most 1. Specifically, if H is a tree then a good orientation is obtained by picking a root vertex u and orienting each edge towards u. This leads to the following useful tool.

Lemma 3.2 *Let k be a positive integer, and H be a configuration that is a tree. If there is a vertex u such that $m(u) \le k-1$, $m(v) \le k-2$ for each $v \in N_H(u)$, and $m(w) \le k-3$ for each other vertex, then H is k-reducible.*

Proof. Let $\vec{H}$ be obtained by orienting all edges towards u. We in-color $\vec{H}$ from the given lists by coloring vertices in increasing order of their distance from u. Since $k - m(u) \ge 1$ we can first color u. Then we can color each neighbor v using a color different from u, since $k - m(v) \ge 2$. For each remaining vertex w we must avoid the color of its parent, and its grandparent, which can be done since $k - m(w) \ge 3$. ∎

Combining these results we immediately obtain.

Lemma 3.3 *Let G be a k-obstruction.*

1. *If $k \ge 4$, then G contains no 1-vertex.*
2. *If $k \ge 6$, then G contains no $d(d-1)$-vertex or $d(d)$-vertex for $2 \le d \le k-3$.*
3. *If $k \ge 7$, then G contains no adjacent 3(1)-vertices.*
4. *If $k \ge 7$, then G contains no 3-vertex adjacent to three 3(1)-vertices.*

Proof. Given a k-obstruction G with such vertices we find a k-reducible configuration H as an induced subgraph, a contradiction.

1. If u is a 1-vertex, then $V(H) = \{u\}$ satisfies $m(u) = 3 \leq k - 1$.

2. If u, v are adjacent 2-vertices, then let $V(H) = \{u, v\}$ and observe that $m(u) = m(v) = 4 \leq k - 2$. For $d \geq 3$ let u be a d-vertex with 2-vertex neighbors $v_1, v_2, \ldots, v_{d-1}$. Since we have no adjacent 2-vertices, $V(H) = \{u, v_1, \ldots, v_{d-1}\}$ induces a star, and it suffices to observe that $m(u) = d + 2 \leq k - 1$ and $m(v_i) = 4 \leq k - 2$.

3. If u, v are adjacent 3(1)-vertices, with respective 2-vertex neighbors u', v', then let H be induced by $\{u, v, u', v'\}$. If $v' \neq u'$, then H is a P_4 (u', v' are non-adjacent by 3.3.2) with $m(u) = 5 \leq k - 1$, $m(v) = 5 \leq k - 2$, $m(u') = 4 \leq k - 2$, and $m(v') = 4 \leq k - 3$. If $v' = u'$, then H is a K_3 in which v' is interior, and thus $7 - m(u) = 7 - m(v) = 3$ colors are available for u, v and $7 - m(v') = 5$ colors are available for v'. Orienting K_3 cyclically yields a good orientation, and each vertex can be colored differently.

4. Let u be a 3-vertex, with 3-vertex neighbors v_1, v_2, v_3, which in turn have 2-vertex neighbors w_1, w_2, w_3 respectively. We obtain H by letting $V(H) = \{u, v_1, v_2, v_3, w_1, w_2, w_3\}$ and observe that u is interior. By 3.3.3 there is no edge $v_i v_j$ and by 3.3.2 there is no edge $w_i w_j$. Hence, if the w_i are distinct then H induces a subdivision of $K_{1,3}$ with $m(u) = 3 \leq k - 1$, $m(v_i) = 4 \leq k - 2$ and $m(w_i) = 4 \leq k - 3$. If, say $w_1 = w_2$, then w_1 is interior and H contains the 4-cycle u, v_1, w_1, v_2, u. We orient both edges incident to w_1 out of w_1 and all edges towards u. Now, first color u ($k - m(u) \geq 1$), then v_i ($k - m(v_i) \geq 2$) followed by w_3 ($k - m(w_3) \geq 3$) as before. For w_1 we have $k - m(w_1) \geq 5$ colors available and we must only avoid the colors given to u, v_1, v_2.

■

An induced path $P = x_1, x_2, \ldots, x_m$ is called *removable* if $m \geq 4$, x_1, x_m are 2-vertices, and the remaining vertices are 3-vertices, except possibly one 4-vertex x_i which is adjacent to a 2-vertex y with $y \neq x_{i-1}, x_{i+1}$.

Lemma 3.4 *If $k \geq 8$, then a k-obstruction contains no removable path.*

Proof. Let P be a removable path. If P contains no 4-vertex, then let $u = x_2$ and $H = P$. If P contains a 4-vertex x_i, then let $u = x_i$ and include y in $V(H)$ as well.

If H is a tree, then we are done since $m(u) \leq 6 \leq k - 1$, and every other vertex satisfies $m(v) \leq 5 \leq k - 3$. If H is not a tree, then H must contain a cycle that passes through y as $x_1, x_2, \ldots, x_m$ induces a path. Assume that y is adjacent to some x_t where, without loss of generality, $t < i$. Orient $x_t y$ towards x_t and all other edges towards x_i to obtain a good orientation of $\vec{H}$. We in-color $\vec{H}$ from the given lists by coloring the vertices in the order $x_i, x_{i-1}, x_{i-2}, \ldots, x_1, x_{i+1}, \ldots, x_m, y$. For each x_j with $1 \leq j \leq m$ we must avoid the color of its parent and grandparent in H (if they have been colored) along with the colors of the vertices defined by $m(x_j)$. We can color each x_j, since $k - m(x_j) \geq 8 - 5 = 3$. For y, we have $m(y) = 1$ and we need only avoid the colors assigned to x_t, x_{t+1}, and x_i so that y can be colored.

■

4 Graphs with mad$(G) < \frac{8}{3}$

Theorem 4.1 *Every graph G with mad(G) $< \frac{8}{3}$ has an orientation of maximum outdegree at most 2 which is 6-in-choosable.*

Proof. Suppose, aiming for a contradiction, that there is a 6-obstruction G with $\mathrm{mad}(G) < 8/3$. Assign an initial charge of $\deg(v)$ to v. The charges are now redistributed in such a way that the net charge assigned to G is preserved. The rule for redistribution is:

1. Each 2(0)-vertex receives charge $\frac{1}{3}$ from each neighbor.

The net charge on the vertices of G after the redistribution has taken place is now calculated. Let $v \in V(G)$. By Lemma 3.3.1, $\deg(v) \geq 2$.

If v is a 2-vertex, then by Lemma 3.3.2, v must be a 2(0)-vertex. The charge of v after redistribution is $2+2\left(\frac{1}{3}\right) = \frac{8}{3}$. If v is a 3-vertex, then by Lemma 3.3.2, v is a 3(1)-vertex or a 3(0)-vertex. Then v sends out at most charge $\frac{1}{3}$ to a 2-vertex. The charge of v after redistribution is at least $3 - \frac{1}{3} = \frac{8}{3}$. If v is a 4^+-vertex, then the charge of v after redistribution is at least $\deg(v) - \frac{1}{3}\deg(v) = \frac{2}{3}\deg(v) \geq \frac{8}{3}$.

This implies $\mathrm{mad}(G) \geq \frac{8}{3}$, which gives the needed contradiction. ■

Using $\mathrm{mad}(G) < \frac{2g}{g-2}$ we obtain

Corollary 4.2 *Every planar graph of girth at least 8 is 6-star choosable.*

Since C_8^+ requires 4 colors to star color, the star chromatic number for the family of planar graphs of girth 8 is between 4 and 6.

5 Graphs with mad$(G) < \frac{14}{5}$

Theorem 5.1 *Every graph G with mad(G) $< \frac{14}{5}$ has an orientation of maximum outdegree at most 2 which is 7-in-choosable.*

Proof. Suppose, aiming for a contradiction, that there is a 7-obstruction G with $\mathrm{mad}(G) < 14/5$. Assign an initial charge of $\deg(v)$ to v. The charges are now redistributed in such a way that the net charge assigned to G is preserved. The rules for redistribution are:

1. Each 2-vertex receives charge $\frac{2}{5}$ from each neighbor.

2. Each 3(1)-vertex receives charge $\frac{1}{10}$ from each neighbor that is a 3^+-vertex.

The net charge on the vertices of G after the redistribution has taken place is now calculated. Let $v \in V(G)$. By Lemma 3.3.1, $\deg(v) \geq 2$.

If v is a 2-vertex, then by Lemma 3.3.2, v is a 2(0)-vertex. The charge of v after redistribution is $2 + 2\left(\frac{2}{5}\right) = \frac{14}{5}$. If v is a 3-vertex, then by Lemma 3.3.2, v is a 3(1)-vertex or a 3(0)-vertex. If v is a 3(1)-vertex, then by Lemma 3.3.3, v is not adjacent to any other 3(1)-vertex. The charge of v after redistribution is $3 - \frac{2}{5} + 2\left(\frac{1}{10}\right) = \frac{14}{5}$. If v is a 3(0)-vertex, then by Lemma 3.3.4, v is adjacent to at most two 3(1)-vertices. The charge of v after redistribution is at least $3 - 2\left(\frac{1}{10}\right) = \frac{14}{5}$. If v is a 4-vertex, then by Lemma 3.3.2, v is adjacent to at most two 2-vertices. The charge of v after redistribution is at least $4 - 2\left(\frac{2}{5}\right) - 2\left(\frac{1}{10}\right) = \frac{15}{5}$. If v is a 5^+-vertex, then the charge of v after redistribution is at least $\deg(v) - \frac{2}{5}\deg(v) = \frac{3}{5}\deg(v) \geq \frac{15}{5}$.

This implies $\mathrm{mad}(G) \geq \frac{14}{5}$, which gives the needed contradiction. ∎

Corollary 5.2 *Every planar graph of girth at least 7 is 7-star choosable.*

An example of a planar graph of girth 7 that requires 5 colors to star color is given in [10], so that the star chromatic number for the family of planar graphs of girth 7 is between 5 and 7.

6 Planar graphs of girth 6

Theorem 6.1 *Every planar graph of girth at least 6 has an orientation of maximum outdegree at most 2 which is 8-in-choosable.*

Proof. Suppose there is a planar 8-obstruction G of girth at least 6. By Lemma 3.3.1, G has no 1-vertex so that G has minimum degree 2 and must contain a cycle. Observe that Euler's Formula implies that a planar graph on n vertices of girth g with $6 \leq g < \infty$ has at most $\frac{3}{2}(n-2)$ edges.

For each vertex v in G, we assign an initial charge of $\deg_G(v) - 3$ to v. Then

$$\sum_{v \in V} (\deg_G(v) - 3) = -3n + \sum_{v \in V} \deg_G(v) \leq -3n + 3n - 6 = -6.$$

so that the total charge is at most -6. Let H be the subgraph of G induced by all 3^--vertices.

A component of H is called *weak* if it contains a 2-vertex, and the vertices of a weak component are called *weak vertices.* By Lemmas 3.3.2 (with $d = 2, 3$) and 3.4, every weak component of H contains precisely one 2-vertex.

Now we redistribute the charges according to the following rules:

1. Each 4(1)-vertex adjacent to three weak 3-vertices that are all in the same weak component C sends charge 1/2 to its own 2-vertex neighbor, and charge 1/2 to the 2-vertex in C.

2. Each other 4^+-vertex v sends charge $1/2$ to each adjacent 2-vertex, and for each of its weak 3-vertex w neighbors v sends charge $1/4$ to the 2-vertex in the weak component containing w.

Let $\mu^*(v)$ denote the charge of vertex v after the redistribution. Observe that only 2-vertices receive additional charges, and only 4^+-vertices send charges. We will show that the 2-vertices and 4^+-vertices now have non-negative charge, so that the total charge of G is non-negative, a contradiction.

Consider the 2-vertex v in the weak component C of H. If $|C| = 1$, then v receives charge $1/2$ from each neighbor, and so $\mu^*(v) = 0$. If $|C| = 2$, then the second vertex of C is a weak 3(1)-vertex w, and v receives charge $1/2$ from its other neighbor. Also, v receives charge $2\left(\frac{1}{4}\right)$ via w by Rule 2, so that $\mu^*(v) = 0$. For $|C| = m \geq 3$ we have

$$\begin{aligned} 3(m-1)+2 &= \sum_{u \in C} \deg_G(u) = \sum_{u \in C} (\deg_G(u) - \deg_H(u)) + \sum_{u \in C} \deg_H(u) \\ &\leq \sum_{u \in C} (\deg_G(u) - \deg_H(u)) + 2(m-1), \end{aligned}$$

where the last step follows since C has at least $m-1$ edges. Thus the number of edges going from $V(G) - C$ to C is $\sum_{u \in C} (\deg_G(u) - \deg_H(u)) \geq m+1 \geq 4$. So v receives a charge of at least $4(\frac{1}{4})$ unless (by Rule 1) there is a 4(1)-vertex w with three 3-vertex neighbors in C. If there are two such vertices w, then v receives charge $2(\frac{1}{2})$. If there is one such vertex w, then $m \geq 4$ and v receives a charge of $\frac{1}{2} + (m+1-3)\frac{1}{4} \geq 1$. Either way $\mu^*(v) \geq 0$.

Thus each 2-vertex has non-negative charge and it remains to check $\mu^*(v) \geq 0$ for all 4^+ vertices v. If v is a 6^+-vertex, then $\mu^*(v) \geq (\deg_G(v) - 3) - \deg_G(v)/2 = (\deg_G(v) - 6)/2 \geq 0$. If v is a 5-vertex, then by Lemma 3.3.2, v is not a 5(4)- or 5(5)-vertex, and thus $\mu^*(v) \geq (5-3) - 3\left(\frac{1}{2}\right) - 2\left(\frac{1}{4}\right) = 0$. Suppose v is a 4-vertex. By Lemma 3.3.2, v is not a 4(3)- or 4(4)-vertex. If v is a 4(2)-vertex, then v can't be adjacent to a 3-vertex u in a weak component C, since otherwise there is a removable path starting at the 2-vertex w in C via u, v and ending at a 2-vertex neighbor of v different from w, and thus $\mu^*(v) = (4-3) - 2\left(\frac{1}{2}\right) = 0$. If v is a 4(0)-vertex, then $\mu^*(v) \geq (4-3) - 4\left(\frac{1}{4}\right) = 0$. If v is a 4(1)-vertex adjacent to at most three weak vertices, then $\mu^*(v) \geq (4-3) - \frac{1}{2} - 2\left(\frac{1}{4}\right) = 0$. So suppose v is a 4(1)-vertex adjacent to 4 weak vertices. If v is as in Rule 1, then $\mu^*(v) = (4-3) - 1 = 0$, so suppose not. Thus v has weak 3-vertex neighbors x, y from different weak components C_x, C_y. But in that case we obtain a removable path starting at the 2-vertex in C_x via x, v, y and ending at the 2-vertex in C_y, a contradiction. ■

Corollary 6.2 *Every planar graph of girth at least 6 is 8-star choosable.*

Thus we have that the star chromatic number for the family of planar graphs of girth 6 is between 5 and 8. The planar graph of girth 7 constructed in [10] provides the lower bound here.

In general it seems difficult to construct high girth planar graphs requiring many colors to star color. The next section suggests a way of constructing such graphs. This "cluster method" was introduced in [9], and was used in [5] and [10].

7 A girth 5 planar graph requiring 6 colors

In this section we give an example of a planar graph of girth 5 that requires 6 colors to star color. We begin with two definitions that play a key role in the construction.

Definition 7.1 *A* k-cluster with center v *is a graph* C *together with a star coloring* c *such that:*

1. *C has vertex set $\{v, u_1, u_2, \ldots, u_k, u'_1, u'_2, \ldots, u'_k\}$ where the u'_i's need not be distinct*
2. *v has k distinct neighbors $u_1, u_2, \ldots, u_k$*
3. *each neighbor u_i of v is adjacent to a vertex $u'_i \neq v$ with $c(u'_i) = c(v)$*

Call the k neighbors $u_1, u_2, \ldots, u_k$ of v the special neighbors of v. *The edge $u_i u'_i$ is said to be a* leg *of the k-cluster.*

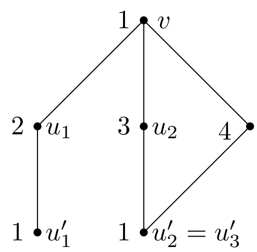

Figure 1: A 3-cluster with center v and legs $u_1u'_1, u_2u'_2$, and $u_3u'_3$

The main idea of the construction is to build a sequence of girth 5 planar graphs G_1, G_2, G_3, and G_4 such that every 5-star coloring of G_k contains a k-cluster ($1 \leq k \leq 3$), whereas G_4 can't be 5-star colored at all. Set $G_1 = K_{1,5}$.

Lemma 7.2 *Every 5-star coloring of G_1 contains a 1-cluster.*

Proof. In any proper coloring of $K_{1,5}$ with 5 colors, at least one color must be used twice on the 1-vertices, say on v and u'. Then v is the center of a 1-cluster with leg uu' where u is the 5-vertex in $K_{1,5}$. ■

To construct G_k ($2 \le k \le 4$) we will attach a separate copy $H_k(v)$ of a graph H_k to every vertex v in G_{k-1}, and use $H_k(v)$ to show that the $(k-1)$-cluster C centered at v in G_{k-1} forces a k-cluster in the subgraph $H_k(v) \cup C$ of G_k.

First we construct G_2. Let $y_1 \dots y_{13}$ be a path on 13 vertices and make vertex z adjacent to each vertex on the path. For $1 \le i \le 13$, subdivide zy_i with x_i, and add edges y_1y_5, y_5y_9, and y_9y_{13}. Let this be H_2. For every vertex v in G_1, attach a copy of H_2 to v by identifying z in H_2 with v in G_1. Let $H_2(v)$ be the copy of H_2 attached to v in G_1. This completes the construction of G_2.

It is easy to see that a 5-cycle is not 3-star colorable. This fact will be used in the proofs of Lemmas 7.3 and 7.4, and Theorem 7.5.

Lemma 7.3 *Every 5-star coloring of G_2 contains a 2-cluster.*

Proof. Suppose c is a 5-star coloring of G_2 such that there is no 2-cluster in G_2. By Lemma 7.2, there is a 1-cluster, say with center v, in the subgraph G_1 of G_2. Suppose $c(v) = 1$ and $c(u_1) = 2$, where u_1 is the special neighbor of v (see Figure 2).

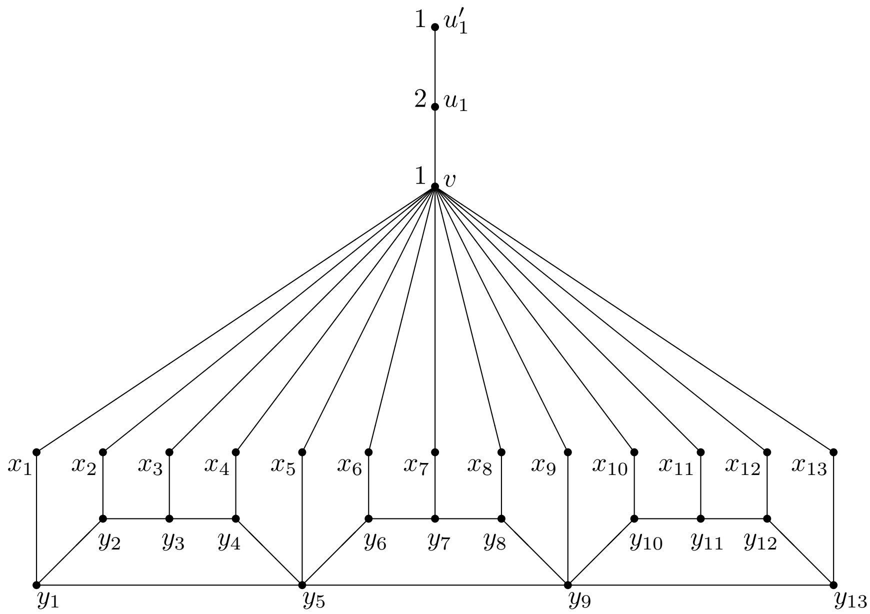

Figure 2: $H_2(v)$ in G_2 with 1-cluster centered at v

Observe that $c(x_i) \notin \{1, 2\}$ for $1 \le i \le 13$. If $c(y_i) = 1$, then v is the center of a 2-cluster with legs x_iy_i and u_1u_1'; so $c(y_i) \neq 1$ for $1 \le i \le 13$. One of the colors 2, 3, 4, or 5 must be used twice on the 5-cycle $y_iy_{i+1}y_{i+2}y_{i+3}y_{i+4}y_i$ where $i \in \{1, 5, 9\}$. First suppose color 3 is repeated on two vertices of such a 5-cycle. Two cases are considered depending on where the color 2 appears on this cycle.

Case 1: Color 2 is assigned to the vertex that is adjacent to the two vertices on the 5-cycle that have been assigned color 3.

Relabel the vertices on the cycle so that $c(y_j) = c(y_{j+2}) = 3$, $c(y_{j+1}) = 2$ and y_{j+3} and y_{j+4} are the other vertices on the 5-cycle. Without loss of generality, assume $c(y_{j+3}) = 4$ and $c(y_{j+4}) = 5$. If $c(x_{j+3}) = 3$, then y_{j+2} is the center of a 2-cluster with legs $y_{j+3}x_{j+3}$ and $y_{j+1}y_j$; so $c(x_{j+3}) \neq 3$. Similarly, $c(x_{j+4}) \neq 3$. Clearly $c(x_{j+3}) \neq 4$ and $c(x_{j+4}) \neq 5$, so we must have $c(x_{j+3}) = 5$ and $c(x_{j+4}) = 4$; but then $x_{j+3}y_{j+3}y_{j+4}x_{j+4}$ is 2-colored.

Case 2: Color 2 is assigned to a vertex adjacent to exactly one of the vertices on the 5-cycle that has been assigned color 3.

Relabel the vertices on the 5-cycle so that $c(y_j) = c(y_{j+2}) = 3$, $c(y_{j+3}) = 2$ and the other two vertices on the 5-cycle are y_{j+1} and y_{j+4}. Without loss of generality, assume $c(y_{j+1}) = 4$ and $c(y_{j+4}) = 5$. If $c(x_j) = 4$, then $x_jy_jy_{j+1}y_{j+2}$ is 2-colored; so $c(x_j) = 5$. Similarly $c(x_{j+2}) = 5$. Then x_j is the center of a 2-cluster with legs vx_{j+2} and y_jy_{j+4}, a contradiction.

We conclude that color 3 may not be assigned to two vertices on any 5-cycle of the form $y_iy_{i+1}y_{i+2}y_{i+3}y_{i+4}y_i$, where $i \in \{1, 5, 9\}$. A similar argument applies to colors 4 and 5. Hence color 2 must be used on twice on each one of these cycles.

Consider the cycle $y_5y_6y_7y_8y_9y_5$, and suppose $c(y_5) = 2$. One of y_2 or y_3 must be assigned color 2, and one of y_7 or y_8 must be assigned color 2. Regardless of where color 2 is assigned, y_5 is the center of a 2-cluster since each of y_2, y_3, y_7 and y_8 are second neighbors of y_5. This shows that color 2 may not be assigned to y_5. Similarly $c(y_9) \neq 2$, so we must have $c(y_6) = c(y_8) = 2$.

If $c(y_1) = 2$, then y_6 is the center of a 2-cluster with legs y_5y_1 and y_7y_8. If $c(y_4) = 2$, then y_6 is the center of a 2-cluster with legs y_5y_4 and y_7y_8. Thus the only vertices in the cycle $y_1y_2y_3y_4y_5y_1$ that can be assigned color 2 are y_2 and y_3, a contradiction. ■

We now construct G_3. Let x and y be the two 8-vertices in $K_{2,8}$, and let $b_1, \ldots, b_8$ be the 2-vertices. For $1 \leq i \leq 7$, add edge b_ib_{i+1}, and for $1 \leq i \leq 8$, subdivide edges xb_i and yb_i with a_i and d_i respectively. Add edge xy. Call this graph $H(x, y)$. Denote a copy of $H(x, y)$ between x and y by a double edge (see Figure 3).

Let $y_1y_2y_3y_4y_5y_1$ be a 5-cycle and make each y_i adjacent to vertex x_i. For $1 \leq i \leq 5$, add $H(z, x_i)$ between z and x_i. Call this graph H_3 (see Figure 4).

For every vertex v in G_2, attach a copy of H_3 to v by identifying z in H_3 with v in G_2. Let $H_3(v)$ be the copy of H_3 attached to v in G_2. This completes the construction of G_3

Lemma 7.4 *Every 5-star coloring of G_3 contains a 3-cluster.*

Proof. Suppose c is a 5-star coloring of G_3 such that G_3 has no 3-cluster. By Lemma 7.3, there is a 2-cluster in the subgraph G_2 of G_3. Let v be the center of this 2-cluster and assume v has been assigned color 1, and the special neighbors u_1 and u_2 of v have been assigned colors 2 and 3 respectively.

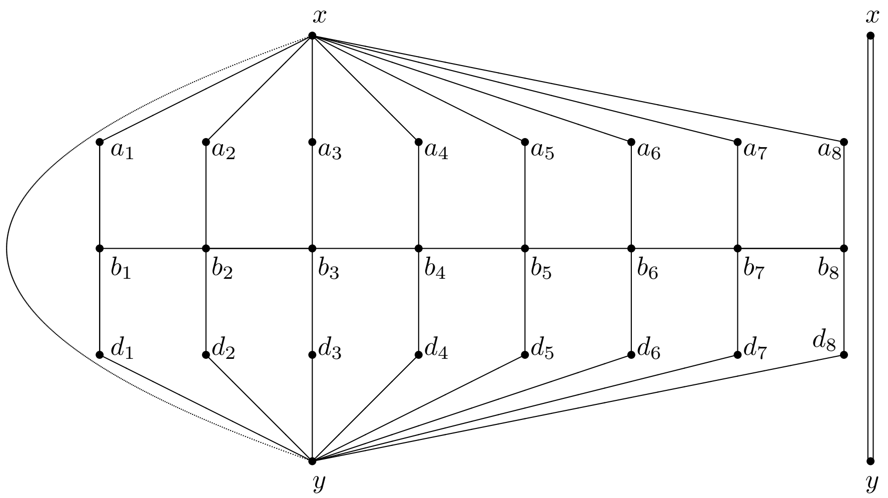

Figure 3: $H(x, y)$

Consider $H_3(v)$. Observe $c(x_i) \notin \{1, 2, 3\}$ and $c(y_i) \neq 1$ for $1 \leq i \leq 5$. Furthermore, each of the colors 2, 3, 4, and 5 must be used on the cycle $y_1y_2y_3y_4y_5y_1$. If $c(x_i) = 4$ for $1 \leq i \leq 5$, then only colors 2, 3, and 5 can be used on the cycle $y_1y_2y_3y_4y_5y_1$. Therefore, not all of the x_i's may be assigned the same color and so we assume $c(x_1) = c(x_2) = 4$ and $c(x_3) = 5$. Two cases are now considered depending on how x_4 is colored.

Case 1: $c(x_4) = 4$

Since color 4 must be used on the cycle $y_1y_2y_3y_4y_5y_1$, we may assume that $c(y_3) = 4$ for if $c(y_5) = 4$, then $c(x_5) = 5$ and by symmetry we can argue similarly. If $c(y_2) = 5$, then $x_2y_2y_3x_3$ is 2-colored, so assume $c(y_2) = 2$ and consider $H(x_2, v)$ (see Figure 5).

If $c(d_i) = 1$ or $c(d_i) = 2$, then $d_ix_2vx_1$ or $d_ix_2y_2y_3$, respectively, is 2-colored. If $c(b_i) = 1$, then v is the center of a 3-cluster with legs a_ib_i, u_1u_1', and u_2u_2'. If $c(b_i) = 4$, then x_2 is the center of a 3-cluster with legs d_ib_i, y_2y_3, and vx_1. Thus for $1 \leq i \leq 8$, $c(a_i) \in \{4, 5\}$, $c(b_i) \in \{2, 3, 5\}$, and $c(d_i) \in \{3, 5\}$.

Color 3 must be used on at least one of the vertices b_2, b_3, b_4, or b_5; so assume $c(b_i) = 3$ where $i \in \{2, 3, 4, 5\}$. This forces $c(d_i) = 5$. If $c(b_{i+1}) = 5$, then $c(d_{i+1}) = 3$ and $d_ib_ib_{i+1}d_{i+1}$ is 2-colored; so we must have $c(b_{i-1}) = c(b_{i+1}) = 2$. This forces $c(d_{i+1}) = c(b_{i+2}) = 5$, which then forces $c(b_{i+3}) = 3$. But then $c(d_{i+2}) = 3$ and $c(d_{i+3}) = 5$ and $d_{i+3}b_{i+3}b_{i+2}d_{i+2}$ is 2-colored.

Case 2: $c(x_4) = 5$

Without loss of generality, assume $c(x_5) = 5$. Color 5 must be assigned to at least one vertex on the cycle $y_1y_2y_3y_4y_5y_1$, so assume $c(y_2) = 5$. If $c(y_3) = 4$, then $x_3y_3y_2x_2$ is 2-colored; so assume $c(y_3) = 2$ and consider $H(x_3, v)$. By a similar argument as used in Subcase 1.1 with the roles of colors 4 and 5 interchanged, $H(x_3, v)$ cannot be colored without creating a 3-cluster or a 2-

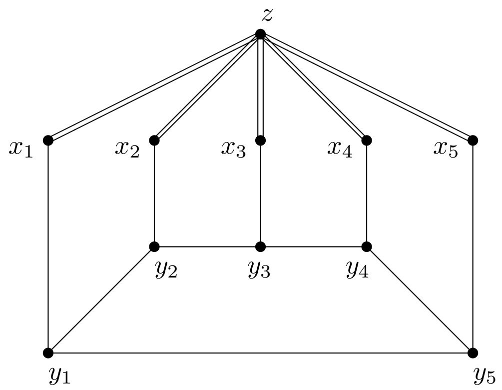

Figure 4: H_3

colored P_4.

■

We now construct G_4. Let $y_1y_2y_3y_4y_5y_1$ be a 5-cycle and make a vertex z adjacent to each vertex on the cycle. For $1 \leq i \leq 5$, subdivide zy_i with x_i. Call this graph H_4. For every vertex v in G_3, attach a copy of H_4 to v by identifying z in H_4 with v in G_3. Let $H_4(v)$ be the copy of H_4 attached to v in G_3. This completes the construction of G_4.

Theorem 7.5 *G_4 is a planar graph of girth 5 that is not 5-star colorable.*

Proof. Suppose c is a 5-star coloring of G_4. By Lemma 7.4, there is a 3-cluster in the subgraph of G_4. Let v be the center of this 3-cluster and assume v has been assigned color 1, and the special neighbors of v have been assigned colors 2, 3, and 4. Consider $H_4(v)$. To avoid a 2-colored P_4, we must have $c(x_i) = 5$ for all i. Then color 5 cannot be used on any vertex of the 5-cycle $y_1y_2y_3y_4y_5y_1$. If $c(y_i) = 1$ for some i, then $y_ix_ivx_{i+1}$ is 2-colored. This implies that the 5-cycle $y_1y_2y_3y_4y_5y_1$ can be star colored with just 3 colors, a contradiction. ■

In [1] it is shown that planar graphs of girth 5 are star colorable with 16 colors, so that the maximum star chromatic number for the family of planar graphs of girth 5 is between 6 and 16. We believe the lower bound is much closer to the truth.

8 Acknowledgments

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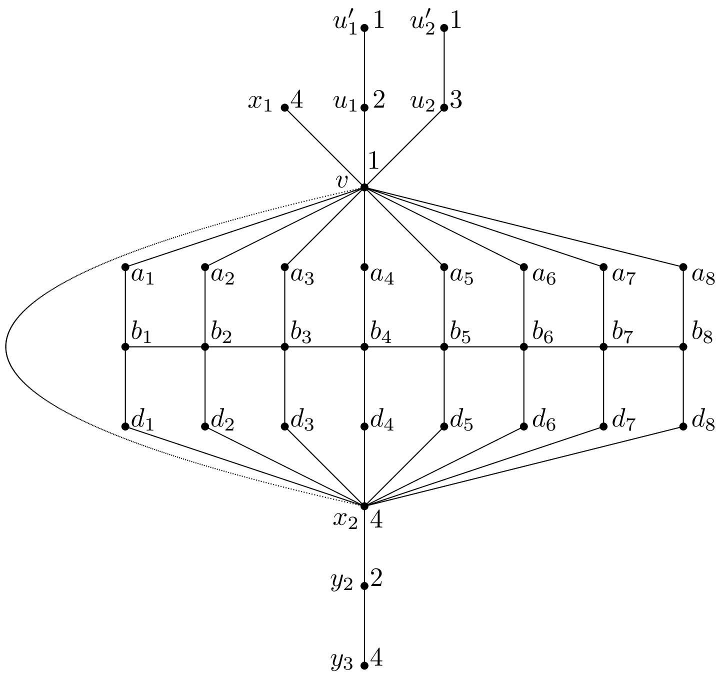

Figure 5: $H(x_2, v)$

References

[1] M.O. Albertson, G.G. Chappell, H.A. Kierstead, A. Kündgen, R. Ramamurthi, Coloring with no 2-colored P_4's, *Electron. J. Combin.* **11** (2004), #R26.

[2] O.V. Borodin, On acyclic colorings of planar graphs, *Discrete Math.* **25** (1979), 211–236.

[3] Y. Bu, D. Cranston, M. Montassier, A. Raspaud, W. Wang, Star coloring of sparse graphs, *J. Graph Theory*, to appear.

[4] B. Grünbaum, Acyclic colorings of planar graphs, *Israel J. Math.* **14** (1973), 390–408.

[5] H.A. Kierstead, A. Kündgen, C. Timmons, Star coloring bipartite planar graphs, *J. Graph Theory* **60** (2009), 1-10.

[6] A.V. Kostochka, L.S. Melnikov, Note to the paper of Grünbaum on acyclic colorings, *Discrete Math.* **14** (1976), 403–406.

[7] J. Nešetřil and P. Ossona de Mendez, Colorings and homomorphisms of minor closed classes, *Discrete and Computational Geometry: The Goodman-*

Pollack Festschrift (ed. B. Aronov, S. Basu, J. Pach, M. Sharir), Springer-Verlag 2003, 651–664.

[8] M. Tarsi, On the decomposition of a graph into stars, *Discrete Math.* **36** (1981), 299–304.

[9] C. Timmons, Star coloring planar graphs, Master's Thesis, California State University San Marcos (2007).

[10] C. Timmons, Star coloring high girth planar graphs, *Electron. J. Combin.* **15** (2008), #R124.