EEE 180 Signals & Systems

Background

Prof. Pang

Real Number

$X - 5 = 0 \quad \Rightarrow \quad X = 5$

$X^2 - 4 = 0 \quad \Rightarrow \quad X = \mp 2$

Imaginary Number

$X^2 + 4 = 0$ => $X^2 = -4$

=> $X = \mp 2j$

or $X = \mp 2i$

Complex Number

2 + 3 j   => 2: real part, 3j: imaginary part

real part   imaginary part

$a + bj$

Complex Number

2 + 3 i => 2: real part, 3i : imaginary part

real part imaginary part

$$a + bi$$

Complex Number Arithmetic Example

$$a + bj + c + dj = (a + c) + (b + d)j$$

$$2 + 3j + 4 + 5j = (2 + 4) + (3 + 5)j$$

$$= 6 + 8j$$

Complex Number Arithmetic Example

$$a + b j - (c + d j) = (a - c) + (b - d) j$$

$$(4 + 8 j) - (2 + 3 j) = (4 - 2) + (8 - 3) j$$
$$= 2 + 5 j$$

Complex Number Arithmetic Example

$j * j = j^2 = -1$

$i * i = i^2 = -1$

Complex Number Arithmetic Example

$$(a + b\,j) * (c + d\,j)$$

$$= a * c + \underline{\mathbf{b\,j * c + a * d\,j}} + bj * (dj)$$

$$= (a * c - b * d) + (\underline{b * c + a * d})\,j$$

Complex Number Arithmetic Example

$(2 + 3j) * (4 + 5j)$

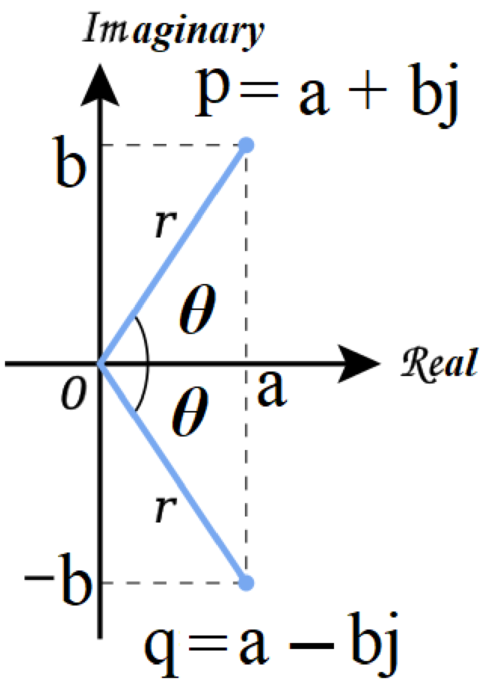
$= 2 * 4 + \underline{2 * (5j) + 3j * 4} + 3j * 5j$

$= (8 - 15) + (10 + 12)j$

$= -7 + 22j$

Complex Conjugate: $q = p^*$

- The complex conjugate of a complex number is the number with an equal real part and an imaginary party equal in magnitude but opposite in sign.
- The complex conjugate of $p = a + b\,j$ is: $q = a - b\,j$. $q = p^*$



Complex Number Arithmetic Example

$(a + b\,j) * (a - b\,j)$

$= a * a - \underline{a * (b\,j) + bj * a} - b\,j * b\,j$

$= (a^2 – b^2 \times (-1)) + 0\,j$

$= a^2 + b^2$

$(4 + 5\,j) * (4 - 5\,j)$

$= 4^2 + 5^2 = 16 + 25 = 41$

Complex Conjugate

- The complex conjugate of $P = a + b\,j$ is: $P^* = a - b\,j$.
- $P + P^* = (a + b\,j) + (a - b\,j) = 2\,a$
- $P\,P^* = (a + bj)(a - b\,j) = a^2 + b^2 = |\,P\,|^2$
- The magnitude of the complex number :

  $|\,P\,|$ = *square root of* ($P\,P^*$) = square root of $(a^2 + b^2)$.

Complex Number Arithmetic Example

$$(4 + 5j) * (4 - 5j) = 4^2 + 5^2 = 16 + 25 = 41$$

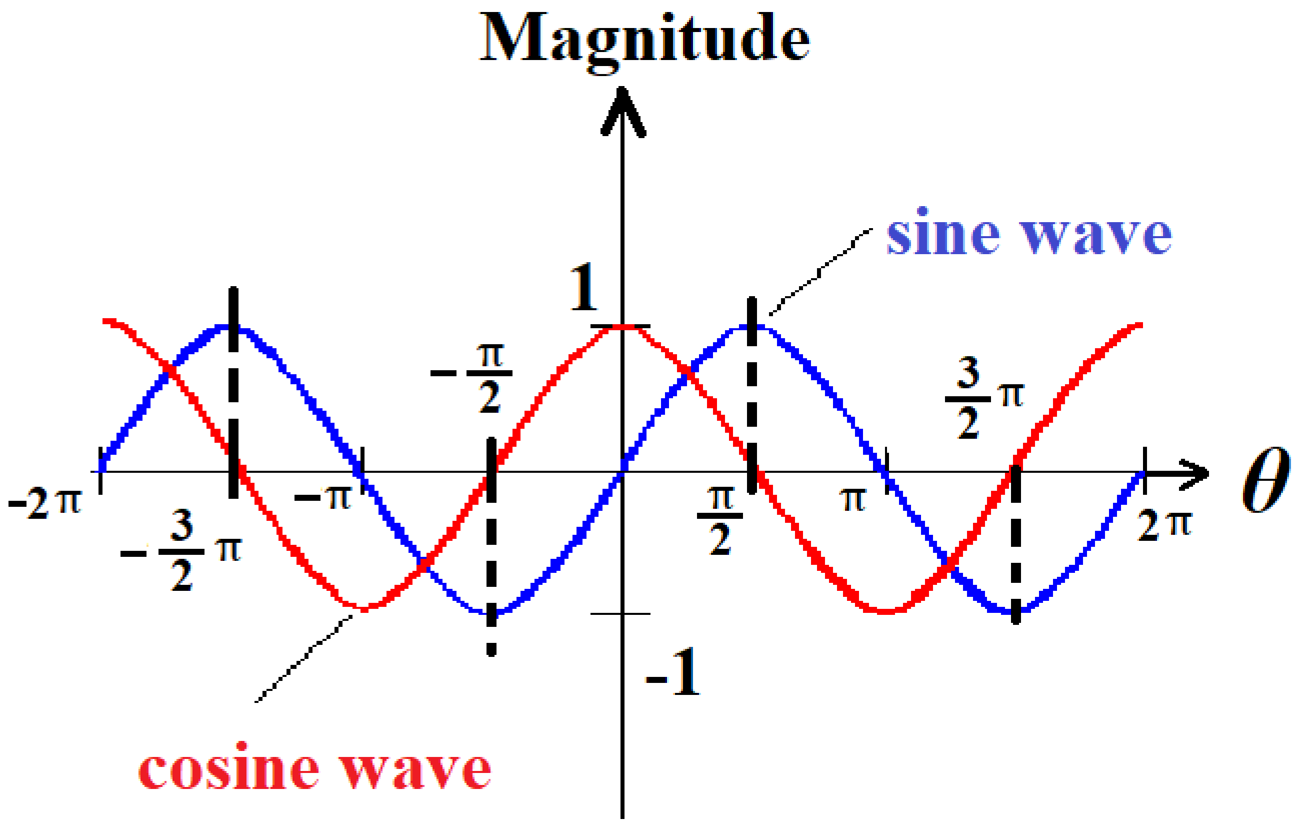
$$(2 + 3j) * (4 + 5j) = -7 + 22j$$

$$\frac{(2 + 3j)}{(4 - 5j)}$$

$$= \frac{(2+3j)\,\boxed{(4 + 5j)}}{(4 - 5j)\,\boxed{(4 + 5j)}}$$

$$= \frac{-7 + 22j}{41} = \frac{-7}{41} + \frac{22}{41} j$$

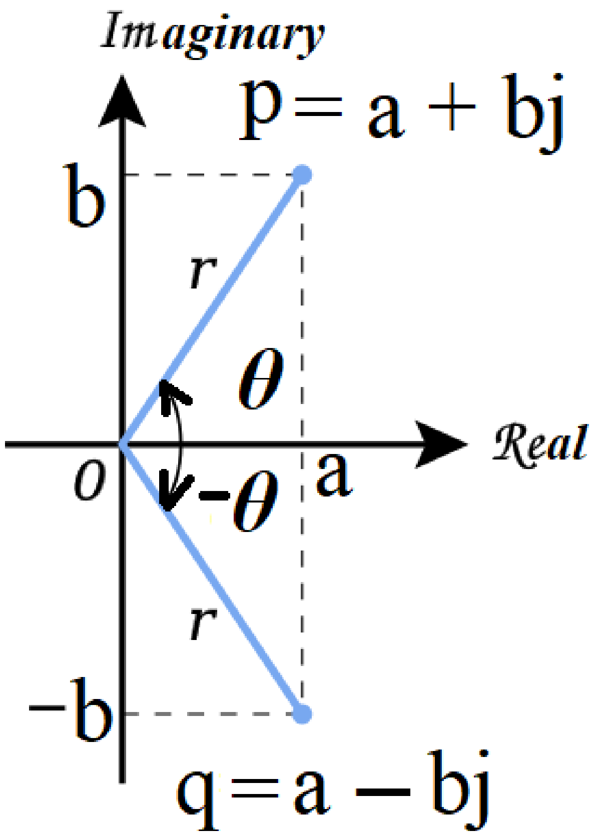
cosine and sine wave



$$\cos\theta = \cos(-\theta)$$

$$\sin\theta = -\sin(-\theta)$$

Polar Coordinates



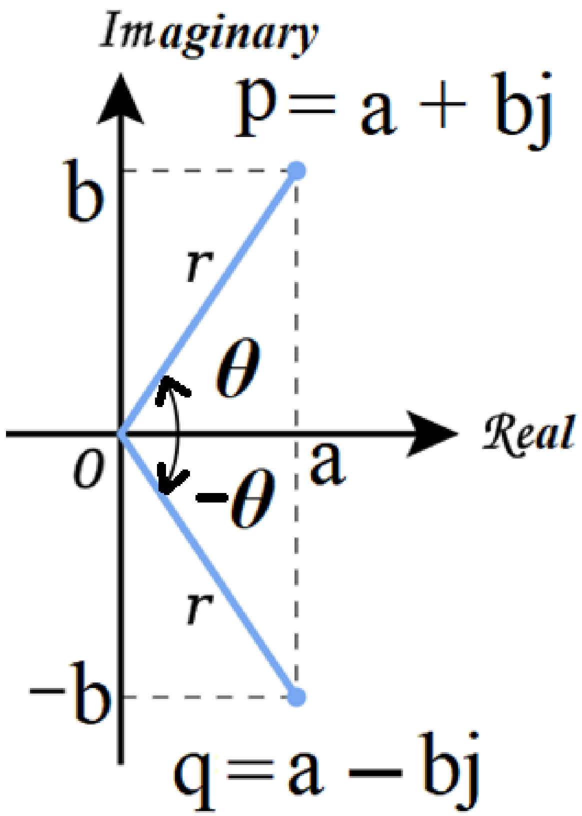
$$a = r\cos\theta$$

$$b = r\sin\theta$$

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Polar Coordinates



$$p = r\cos\theta + r\sin\theta \;\; \mathbf{j}$$

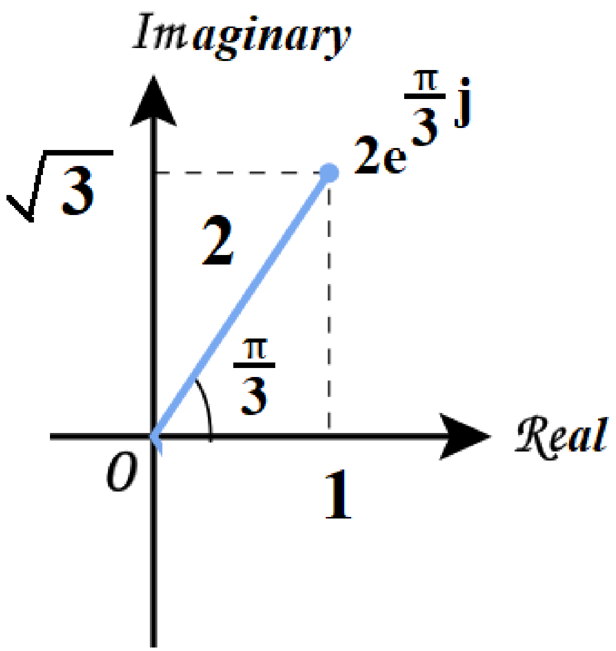
$$q = p^*$$

$$= r\cos\theta - r\sin\theta \;\; \mathbf{j}$$

$$|p| = |q| = \mathbf{r}$$

$$= \sqrt{\mathbf{a}^2 + \mathbf{b}^2}$$

Polar Coordinates



$$2e^{j\frac{\pi}{3}} = 2\left(\cos\frac{\pi}{3} + \sin\frac{\pi}{3}j\right)$$

$$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}j\right)$$

$$= 1 + \sqrt{3}j$$

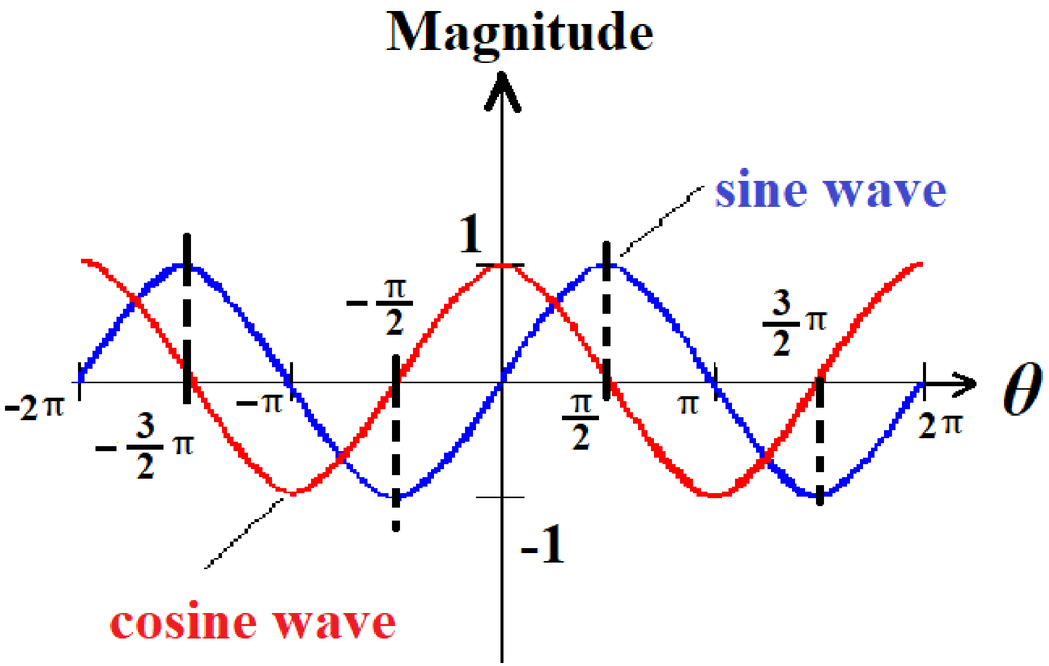
Euler Formula

$$\mathbf{e}^{\mathbf{j}\theta} = \cos\theta + \sin\theta\ \mathbf{j}$$

Euler Formula

$$e^{j\pi} = \cos\pi + \sin\pi\, j$$
$$= -1$$

$$e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + \sin\frac{\pi}{2} j$$
$$= j$$



Complex Number Arithmetic

Question: Suppose $p = r\, e^{j\theta}$,
determine p^n and $p^{1/n}$.

Solution:

$p^{n} = (\mathbf{r}\ \mathbf{e}^{\mathbf{j}\boldsymbol{\theta}})^n = r^n (\mathbf{e}^{\mathbf{j}\boldsymbol{\theta}})^n = r^n\, \mathbf{e}^{\mathbf{jn}\boldsymbol{\theta}}$

$P^{1/n} = (\mathbf{r}\ \mathbf{e}^{\mathbf{j}\boldsymbol{\theta}})^{1/n} = r^{1/n} (\mathbf{e}^{\mathbf{j}\boldsymbol{\theta}})^{1/n} = r^{1/n}\, \mathbf{e}^{\mathbf{j}\boldsymbol{\theta}/\mathbf{n}}$

Complex Number Arithmetic Example

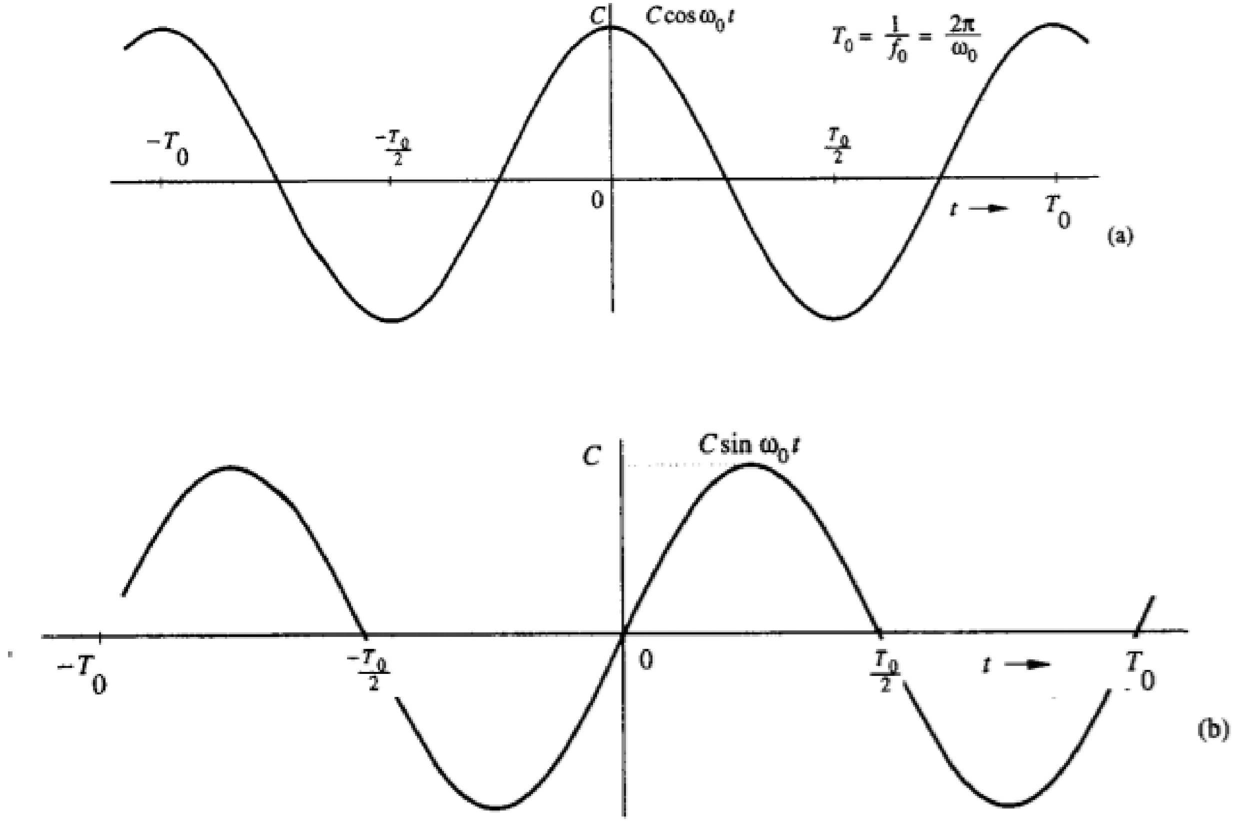
Question: Suppose $p = 8\, e^{\pi/3\, j}$, determine p^2 and $p^{1/3}$.

Solution:

p^{2} = $(8\, e^{\pi/3\, j})^2 = 8^2 (e^{\pi/3\, j})^2 = 64\, e^{2\pi/3\, j}$

$p^{1/3} = (8\, e^{\pi/3\, j})^{1/3} = 8^{1/3} (e^{\pi/3\, j})^{1/3} = 2\, e^{\pi/9\, j}$

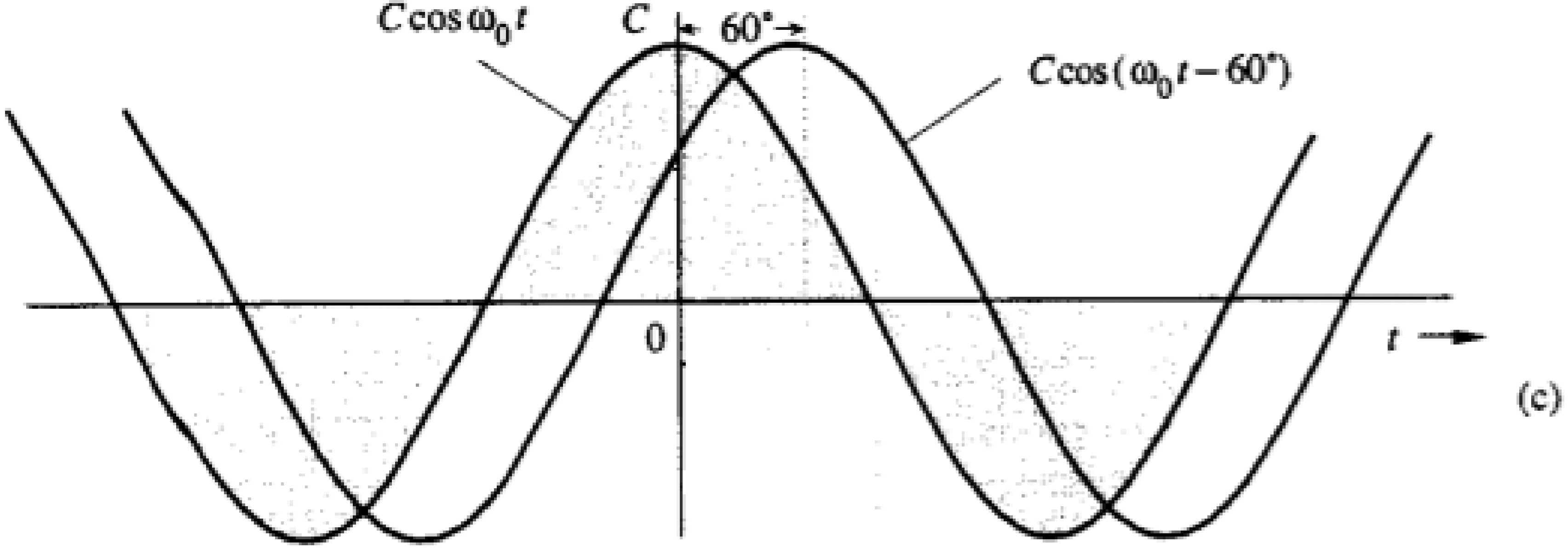
Sinusoids



$$f(t) = C\,cos(\omega t + \varphi)$$
$$= \mathrm{C}\cos(2\pi \mathrm{F}\,\mathrm{t} + \varphi)$$
$$= C\cos(\frac{2\pi}{T}t + \varphi)$$
$$T = \frac{1}{F}$$
$$\omega = 2\,\pi\,\mathrm{F}$$

$$C\ \sin(\omega t + \frac{\pi}{2}) = C\cos(\omega t)$$

Sinusoids



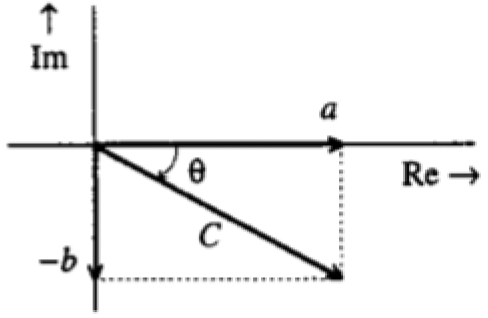
Trignometric Identity

$$C \cos(\omega_0 t + \theta) = C \cos\theta \cos\omega_0 t - C \sin\theta \sin\omega_0 t$$

$$= a \cos\omega_0 t + b \sin\omega_0 t$$

in which

$$a = C\cos\theta, \qquad b = -C\sin\theta$$



Phasor addition of sinusoids.

$$C = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{-b}{a}\right)$$

Addition of Sinusoids

$$a \cos \omega_0 t + b \sin \omega_0 t = C \cos (\omega_0 t + \theta)$$

Euler's Formula

$$\cos\varphi = \frac{1}{2}\left(e^{j\varphi} + e^{-j\varphi}\right)$$

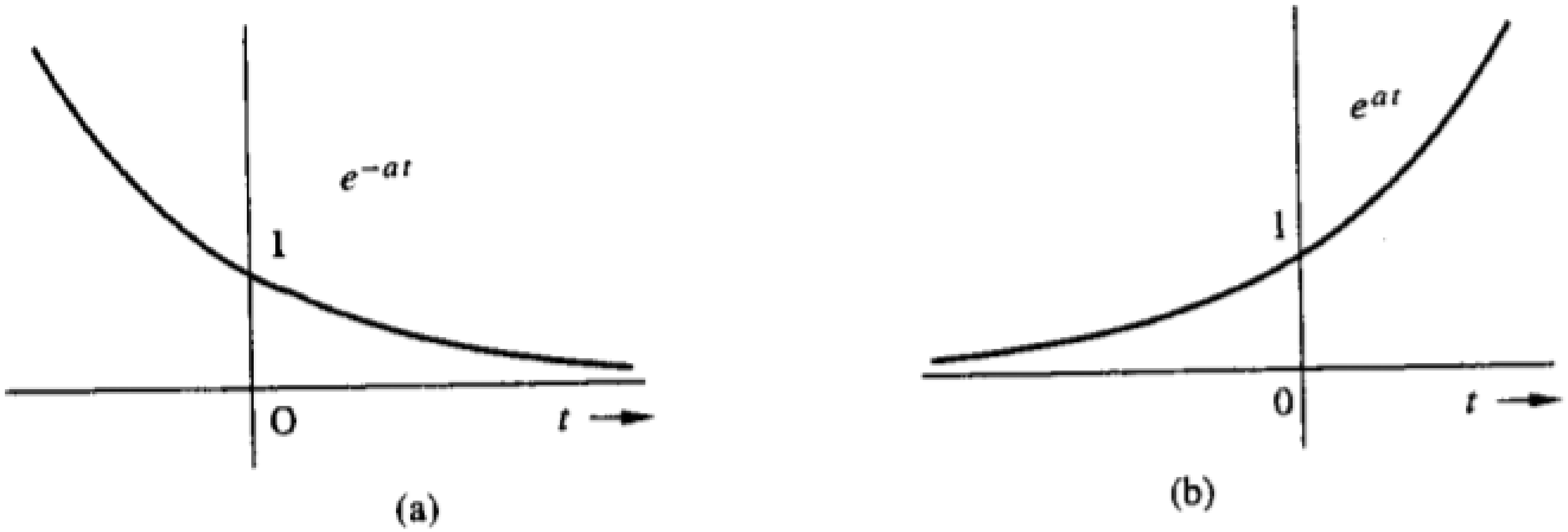
$$\sin\varphi = \frac{1}{2j}\left(e^{j\varphi} - e^{-j\varphi}\right)$$

$$e^{j\varphi} = \cos\varphi + j\sin\varphi$$

$$e^{-j\varphi} = \cos\varphi - j\sin\varphi$$

Monotonic Exponentials

a > 0



Monotonic exponentials.

Determinant

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

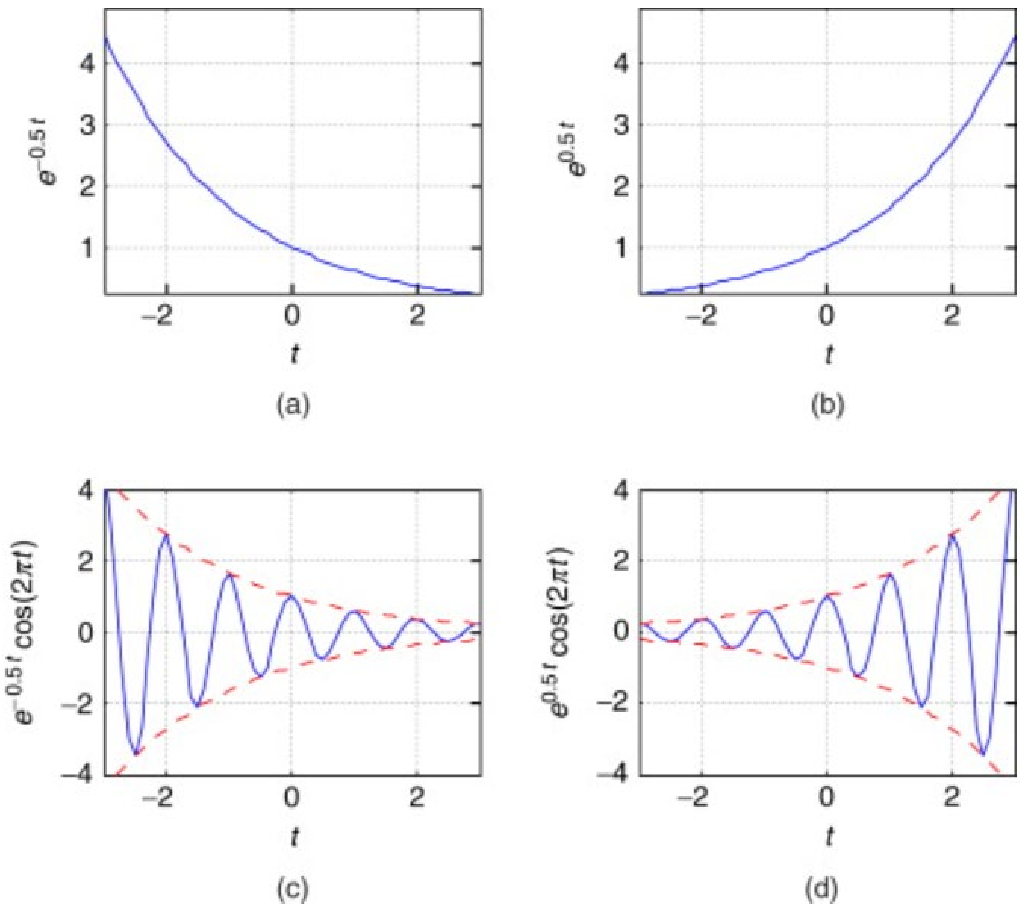
Determinant

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\det(A) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Sketching an Exponentially Varying Sinusoid



Cramer's Rule for Equation Solver

$$2x_1 + x_2 + x_3 = 3$$

$$x_1 + 3x_2 - x_3 = 7$$

$$x_1 + x_2 + x_3 = 1$$

In matrix form these equations can be expressed as

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$$

Here,

Determinant $|\mathbf{A}| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4$

$$x_1 = \frac{1}{|\mathbf{A}|} \begin{vmatrix} 3 & 1 & 1 \\ 7 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{8}{4} = 2$$

$$x_2 = \frac{1}{|\mathbf{A}|} \begin{vmatrix} 2 & 3 & 1 \\ 1 & 7 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{4}{4} = 1$$

$$x_3 = \frac{1}{|\mathbf{A}|} \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 7 \\ 1 & 1 & 1 \end{vmatrix} = \frac{-8}{4} = -2$$

Cramer’s Rule for Equation Solver

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \text{(B.30)}$$

We denote the matrix on the left-hand side formed by the elements a_{ij} as $\mathbf{A}$. The determinant of $\mathbf{A}$ is denoted by $|\mathbf{A}|$. If the determinant $|\mathbf{A}|$ is not zero, the set of equations (B.29) has a unique solution given by Cramer's formula

$$x_k = \frac{|\mathbf{D}_k|}{|\mathbf{A}|} \qquad k = 1, 2, \ldots, n \quad \text{(B.31)}$$

where $|\mathbf{D}_k|$ is obtained by replacing the kth column of $|\mathbf{A}|$ by the column on the right-hand side of Eq. (B.30) (with elements $y_1, y_2, \ldots, y_n$).

Matrix Transpose

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{then} \quad \mathbf{A}^T = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\mathbf{A} = (a_{ij})_{m \times n}$$

$$\mathbf{A}^T = (a_{ji})_{n \times m}$$

$$(\mathbf{A}^T)^T = \mathbf{A}$$

Switch the row elements of the original matrix to get the column elements of the transposed matrix.

Matrix Addition

$$\mathbf{G}=\begin{bmatrix}1 & 2\\ 3 & 4\\ 6 & 7\end{bmatrix},\quad \mathbf{F}=\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}$$

$$\mathbf{F+G}=\begin{bmatrix}1 & 2\\ 3 & 4\\ 6 & 7\end{bmatrix}+\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1+1 & 2+0\\ 3+1 & 4+1\\ 6+0 & 7+1\end{bmatrix}=\begin{bmatrix}2 & 2\\ 4 & 5\\ 6 & 8\end{bmatrix},$$

Matrix Multiplication

Matrix A · Matrix B

$$\begin{bmatrix} 1 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 5 & 8 \\ 7 & 9 \end{bmatrix}$$

$$(1 \cdot 2) + (4 \cdot 5) + (6 \cdot 7) = 64$$

Matrix A · Matrix B

$$\begin{bmatrix} 1 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 5 & 8 \\ 7 & 9 \end{bmatrix} \quad 89$$

$$(1 \cdot 3) + (4 \cdot 8) + (6 \cdot 9) = 89$$

Matrix A · Matrix B

$$\begin{bmatrix} 1 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 5 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 64 & 89 \end{bmatrix}$$

Identity Matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \dots,\ I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

2 x 2
Identiy Matrix

3 x 3
Identiy Matrix

n x n
Identiy Matrix

Matrix Inverse

The inverse of A is A^{-1} only when:

$$AA^{-1} = A^{-1}A = \mathbf{I}$$

Matrix Inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Matrix Inverse

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4\times6-7\times2}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

Matrix Inverse

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 4\times0.6+7\times-0.2 & 4\times-0.7+7\times0.4 \\ 2\times0.6+6\times-0.2 & 2\times-0.7+6\times0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4-1.4 & -2.8+2.8 \\ 1.2-1.2 & -1.4+2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$