COMPLEX ANALYSIS REVIEW PROBLEMS

(1) Recall that a function $u(x, y)$ is *harmonic* if
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x, y) = 0.$$
Prove that $u(x, y) = 2x(1 - y)$ is harmonic, find v so that $f(z) = u + iv$ is holomorphic, and write $f(z)$ in terms of z. Now do the same for $u(x, y) = x^2 - y^2 - 2xy - 2x + 3y$.

(2) Find a Möbius transformation that maps the circle $|z-1| = 2$ onto the line $x+y = 1$.

(3) Describe what $f(z) = \frac{z+1}{z-1}$ does to the real axis, the imaginary axis, the line $x = y$, and a circle of radius $r > 0$ centered at $a \in \mathbb{R}$.

(4) A mapping is called *involutory* if $f(f(z)) = z$. Find conditions on a, b, c, d so that $f(z) = \frac{az+b}{cz+d}$ is involutory.

(5) Derive from scratch the formulas for stereographic projection.

(6) Describe what $f(z) = 1/\overline{z}$ does to the Riemann sphere.

(7) Find a Möbius transformation that maps the vertices $1 + i, -i, 2 - i$ of a triangle T of the z-plane into the points $0, 1, i$ of the w-plane. Sketch the image of the triangle T.

(8) Let $f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$. Show that if $a, b, c, d \in \mathbb{R}$, then f maps the real axis onto the real axis. Conversely, if f maps the real axis onto the real axis, is it true that $a, b, c, d \in \mathbb{R}$?

(9) Prove that for fixed $z_0 \in \mathbb{C}$ with $|z_0| \neq 1$ and fixed $\phi \in \mathbb{R}$,
$$f(z) = e^{i\phi} \frac{z - z_0}{1 - z\overline{z}_0}$$
is a Möbius transformation that maps the unit circle onto the unit circle. Find such a Möbius transformation that takes $z = 0$ to $w = \frac{1}{2}(1 + i)$.

(10) Prove that a Möbius transformation
$$f(z) = \frac{\alpha z + \overline{\gamma}}{\gamma z + \overline{\alpha}}$$
maps the unit circle onto the unit circle. What is the difference between $|\alpha|^2 - |\gamma|^2 = 1$ and $|\alpha|^2 - |\gamma|^2 = -1$? Find such a Möbius transformation that takes $z = 0$ to $w = \frac{1}{2}(1 + i)$.