COMPLEX ANALYSIS REVIEW PROBLEMS

(1) Recall that a function $u(x, y)$ is *harmonic* if

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x, y) = 0.$$

Prove that $u(x, y) = 2x(1 - y)$ is harmonic, find v so that $f(z) = u + iv$ is holomorphic, and write $f(z)$ in terms of z. Now do the same for $u(x, y) = x^2 - y^2 - 2xy - 2x + 3y$.

(2) Find a Möbius transformation that maps the circle $|z-1| = 2$ onto the line $x+y = 1$.

(3) Describe what $f(z) = \frac{z+1}{z-1}$ does to the real axis, the imaginary axis, the line $x = y$, and a circle of radius $r > 0$ centered at $a \in \mathbb{R}$.

(4) A mapping is called *involutory* if $f(f(z)) = z$. Find conditions on a, b, c, d so that $f(z) = \frac{az+b}{cz+d}$ is involutory.

(5) Derive from scratch the formulas for stereographic projection.

(6) Describe what $f(z) = 1/\overline{z}$ does to the Riemann sphere.

(7) Find a Möbius transformation that maps the vertices $1 + i, -i, 2 - i$ of a triangle T of the z-plane into the points $0, 1, i$ of the w-plane. Sketch the image of the triangle T.

(8) Let $f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$. Show that if $a, b, c, d \in \mathbb{R}$, then f maps the real axis onto the real axis. Conversely, if f maps the real axis onto the real axis, is it true that $a, b, c, d \in \mathbb{R}$?

(9) Prove that for fixed $z_0 \in \mathbb{C}$ with $|z_0| \neq 1$ and fixed $\phi \in \mathbb{R}$,

$$f(z) = e^{i\phi} \frac{z - z_0}{1 - z\overline{z}_0}$$

is a Möbius transformation that maps the unit circle onto the unit circle. Find such a Möbius transformation that takes $z = 0$ to $w = \frac{1}{2}(1 + i)$.

In the first draft, I forgot to say $|z_0| \neq 1$ (which you need in order to have "$ad - bc \neq 0$"). Then just show that

$$|z - z_0|^2 = |1 - z\overline{z}_0|^2 \qquad \text{when } |z| = 1.$$

Since $\frac{1}{2}|1 + i| \neq 1$, we can take $z_0 = \frac{1}{2}(1 + i)$, and we might as well take $\phi = 0$. Then consider

$$g(z) = \frac{z - \frac{1}{2}(1 + i)}{1 - \frac{1}{2}(1 - i)z}.$$

The transformation

$$f(z) = g^{-1}(z) = \frac{z + \frac{1}{2}(1+i)}{\frac{1}{2}(1-i)z + 1}$$

is what we want.

(10) Prove that a Möbius transformation

$$f(z) = \frac{\alpha z + \overline{\gamma}}{\gamma z + \overline{\alpha}}$$

maps the unit circle onto the unit circle. What is the difference between $|\alpha|^2 - |\gamma|^2 = 1$ and $|\alpha|^2 - |\gamma|^2 = -1$? Find such a Möbius transformation that takes $z = 0$ to $w = \frac{1}{2}(1+i)$.

First show that $|\alpha z + \overline{\gamma}|^2 = |\gamma z + \overline{\alpha}|^2$ when $|z| = 1$. This shows that f maps the unit circle onto the unit circle. Note that $|f(0)| = |\gamma|/|\alpha|$. The condition $|\alpha|^2 - |\gamma|^2 = 1$ implies that $|f(0)| < 1$, so the inside of the circle gets mapped to the inside of the circle. In the other case, the inside of the circle gets mapped to the outside of the circle. We note that

$$f(0) = \overline{\gamma}/\overline{\alpha}.$$

We can take $\gamma = \frac{1}{2}(1-i)$ and $\alpha = 1$ to get the same example as in the previous problem.