MATH 105B: THE FROBENIUS METHOD

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For an ODE of the form
$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0,$$
where the functions $b(x)$ and $c(x)$ are analytic at $x = 0$, we can use the *Frobenius Method* to find a basis of solutions $\{y_1(x), y_2(x)\}$:

(1) In all cases, we first use the ansatz[1]
$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r},$$
where r and the a_m are to be determined. (We may assume $a_0 \neq 0$.) We plug the ansatz into the ODE and group terms according to the power of x.

(2) Setting the coefficient of the lowest power of x equal to zero, we get *the indicial equation*. It has roots r_1 and r_2. We will only consider the case when r_1 and r_2 are real numbers, and we may assume $r_1 \geq r_2$.

(3) One solution is of the form
$$y_1(x) = x^{r_1} \sum_{m=0}^{\infty} a_m x^m,$$
where we can find a recurrence relation for the a_m.

(4) If $r_1 - r_2 \notin \{0, 1, 2, 3, \ldots\}$, then a second solution, linearly independent of y_1, is
$$y_2(x) = x^{r_2} \sum_{m=0}^{\infty} A_m x^m,$$
where we can find a recurrence relation for the A_m.

[1] Wikipedia says: "In physics and mathematics, an ansatz (German, meaning: 'initial placement of a tool at a work piece') is an educated guess that is verified later by its results."

(5) If $r_1 - r_2 \in \{0, 1, 2, 3, \ldots\}$:

(a) If y_1 has a nice closed form (as in the book's Examples 1, 2, and 3), then it might be easiest to use "reduction of order" directly: Seek a solution y_2 of the form

$$y_2(x) = u(x)y_1(x),$$

where $u(x)$ is a function to be determined.

(b) If y_1 does not have a nice closed form, and if $r_1 = r_2$, then we use the ansatz

$$y_2(x) = y_1(x)\ln x + x^{r_2}\sum_{m=1}^{\infty} A_m x^m.$$

We can find a recurrence relation for the A_m.

(c) If y_1 does not have a nice closed form, and if $r_1 - r_2 \in 1, 2, 3, \ldots$, then we use the ansatz

$$y_2(x) = \alpha y_1(x)\ln x + x^{r_2}\sum_{m=0}^{\infty} A_m x^m.$$

It might turn out that $\alpha = 0$. We can find a recurrence relation for the A_m

Notes on Kreyszig's Section 5.3 problems:

#4: I found a nice power series for $y_1(x)$ but no closed form. [In fact, it can be written in terms of Bessel functions.] According to (5c) above, we look for a second solution of the form

$$y_2(x) = \alpha y_1(x)\ln x + \sum_{m=0}^{\infty} A_m x^m. \qquad \text{(Here } r_2 = 0.\text{)}$$

I found a recurrence relation for the A_m but did not find a nice closed form. [Presumably it can be written in terms of Bessel functions of the second kind.]

#6: I only found a recurrence relation for $y_1(x)$ but did not find a closed form for its coefficients.

#8: I found a nice power series for $y_1(x)$ but no closed form. [In fact, it can be written in terms of *modified Bessel functions*. (See p. 200.)] I have a messy recurrence relation for the coefficients of $y_2(x)$. [Presumably it can be written in terms of modified Bessel functions of the third kind. (Again, see p. 200.)]

#10: Everything works out nicely. I found a closed form for $y_1(x)$ (it involves sine). I then used Reduction of Order directly to find a closed form for $y_2(x)$ (it involves cosine). [So this is the only exercise that works out as nicely as the examples.]