

MATH 105B, SPRING 2018: LECTURE 4

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Last time we looked at Legendre's equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

where n is a given constant. This differential equation comes up in science whenever we want to solve “Laplace's equation” in spherical coordinates. [We will do that later in the semester.] It has the power series solution $y(x) = \sum_{m=0}^{\infty} a_m x^m$ on $-1 < x < 1$. Here a_0 and a_1 can be anything, but then all the following coefficients are determined by the recurrence relation

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)}a_m \quad \text{for } m \geq 0.$$

The general solution is then of the form

$$y(x) = a_0 y_{\text{even}}(x) + a_1 y_{\text{odd}}(x)$$

where $y_{\text{even}}(x)$ has only even powers of x , and $y_{\text{odd}}(x)$ has only odd powers of x .

Note: If n is a nonnegative integer, then the recurrence relation gives $a_{n+2} = 0$, and therefore $a_{n+4} = 0$, etc., so we end up with a *polynomial* solution of Legendre's equation.

Example. If $n = 2$ and we take $a_0 = 1$ and $a_1 = 0$, then we get the solution $y(x) = 1 - 3x^2$. [Check that it solves Legendre's equation in this case.]

Example. If $n = 3$ and we take $a_0 = 0$ and $a_1 = 1$, then we get the solution $y(x) = x - \frac{5}{3}x^3$. [Check that it solves Legendre's equation with $n = 3$.]

It was decided that “Legendre polynomials” $P_n(x)$ must all satisfy $P_n(1) = 1$, so we get

$$P_2(x) = -\frac{1}{2}(1 - 3x^2) \quad \text{and} \quad P_3(x) = -\frac{3}{2}\left(x - \frac{5}{3}x^3\right), \quad \text{etc.}$$

We have the amazing and useful “Rodrigues’s formula” for Legendre polynomials (see homework):

$$P_0(x) = 1$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n].$$

Important: We can use *linear algebra* when studying these polynomials! Legendre polynomials are *orthogonal* on the interval $(-1, 1)$ with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx.$$

This is a “dot product” for functions! The “length” of a function on $(-1, 1)$ is:

$$\|f\| = \langle f, f \rangle^{1/2} = \sqrt{\int_{-1}^1 f(x)^2 dx}.$$

And Legendre polynomials are orthogonal to each other in the sense that

$$\langle P_n, P_m \rangle = 0 \quad \text{if } m \neq n.$$

(See §§ 11.5-6.) In fact, $P_n(x)$ is orthogonal to *all* polynomials of lower degree:

Fact. Let $0 \leq m < n$ be integers. Then $\int_{-1}^1 x^m P_n(x) dx = 0$.

Proof. Using Rodrigues’s formula, it suffices to show

$$\int_{-1}^1 x^m \frac{d^n}{dx^n} [(x^2 - 1)^n] dx = 0.$$

Step 1: If $k < n$, then note that $\left. \frac{d^k}{dx^k} \right|_{x=\pm 1} (x^2 - 1)^n = 0$.

Step 2: Now use integration by parts with:

$$u = x^m \quad dv = \frac{d^n}{dx^n} [(x^2 - 1)^n].$$

So we get (with no boundary terms because of Step 1)

$$\int_{-1}^1 x^m \frac{d^n}{dx^n} [(x^2 - 1)^n] dx = - \int_{-1}^1 m x^{m-1} \frac{d^{n-1}}{dx^{n-1}} [(x^2 - 1)^n] dx.$$

We repeat this process n times and get:

$$\dots = (-1)^n \int_{-1}^1 \left[\frac{d^n}{dx^n} (x^m) \right] (x^2 - 1)^n dx = 0.$$