MATH 105B: LAPLACE'S EQUATION IN SPHERICAL COORDINATES

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We investigate the two problems: the "interior problem"

$$\begin{cases} \Delta u = 0 & \textit{inside} \text{ the sphere of radius } R \\ u = f & \text{on the boundary of the sphere.} \end{cases}$$

and the "exterior problem"

$$\begin{cases} \Delta u = 0 & \textit{outside} \text{ the sphere of radius } R \\ u = f & \text{on the boundary of the sphere.} \end{cases}$$

Because we are looking at a boundary condition on a sphere, it is simplest to use spherical coordinates.

Laplace's equation $\Delta u = 0$ in spherical coordinates is:

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}\right] = 0.$$

As usual, we first look for solutions of the form

$$u(r,\theta,\phi) = G(r)W(\theta)H(\phi).$$

We plug this into Laplace's equation and multiply by

$$\frac{r^2\sin^2\phi}{GWH}$$

to get:

$$(*) \qquad \frac{\sin^2\phi}{G(r)}\frac{\partial}{\partial r}\left(r^2G'(r)\right) + \frac{\sin\phi}{H(\phi)}\frac{\partial}{\partial\phi}\left(\sin\phi H'(\phi)\right) = -\frac{W''(\theta)}{W(\theta)}.$$

We see that the left side is independent of θ, and the right side is independent of (r,ϕ), so this quantity is a *constant*, which we call β.

In particular, we have the ODE

$$W''(\theta) + \beta W(\theta) = 0.$$

If $\beta = 0$, this has the general solution $W(\theta) = a\theta + b$. However, W must be a well-defined function on the circle: $W(\theta + 2\pi) = W(\theta)$ for all θ. We can use this to show that W must then be a constant. We can similarly look at the cases when $\beta > 0$ or $\beta < 0$. In the end, the only nontrivial case is when

$$\beta = m^2, \qquad m \text{ an integer.}$$

So we get the solutions

$$W_m(\theta) = C_1 \cos(m\theta) + C_2 \sin(m\theta).$$

(Note that when $m = 0$ we have that W_0 is a constant.)

Returning to the equation $(*)$, we have

$$\frac{1}{G(r)} \frac{\partial}{\partial r} \left(r^2 G'(r) \right) = \frac{m^2}{\sin^2\phi} - \frac{1}{\sin\phi H(\phi)} \frac{\partial}{\partial\phi} \left(\sin\phi H'(\phi) \right).$$

Since the left side is independent of ϕ and the right side is independent of r, this quantity is a constant, which we call $n(n+1)$ for reasons that will be clear later.

Thus we have the two ODE:

$$\frac{d}{dr}\left(r^2 G'(r) \right) - n(n+1)G(r) = 0$$

$$\frac{1}{\sin\phi} \frac{d}{d\phi} \left(\sin\phi H'(\phi) \right) + \left[n(n+1) - \frac{m^2}{\sin^2\phi} \right] H(\phi) = 0.$$

The equation for G is an "Euler-Cauchy equation" (see §2.5, §5.3, or Math 45) and has the two linearly independent solutions

$$G_n(r) = r^n \qquad \text{and} \qquad G_n^*(r) = \frac{1}{r^{n+1}}.$$

To solve the equation for H, we let $w = \cos\phi$. Then by the Chain Rule

$$\frac{1}{\sin\phi} \frac{d}{d\phi} = -\frac{d}{dw},$$

so we can rewrite the equation for H as:

$$0 = \frac{d}{dw}\left((1 - w^2) \frac{d}{dw} H \right) + \left[n(n+1) - \frac{m^2}{1 - w^2} \right] H = 0.$$

This is "the generalized Legendre equation," and its solutions are "Legendre functions."

Our book only considers the special case when u is independent of θ (the case of "azimuthal symmetry"). Then we have $m = 0$, and the equation for H becomes the ordinary Legendre equation of §5.2. When $n = 0, 1, 2, \ldots$, one of its solutions is the "Legendre polynomial"

$$H = P_n(w) = P_n(\cos\phi).$$

To summarize: If we have azimuthal symmetry, then we have special solutions of the form

$$u_n(r,\phi) = A_n r^n P_n(\cos\phi) \qquad \text{and} \qquad u_n^*(r,\phi) = \frac{B_n}{r_{n+1}} P_n(\cos\phi).$$

To solve the "interior problem" stated at the beginning of the notes, we look for a solution of the form

$$u(r,\phi) = \sum_{n=0}^{\infty} u_n(r,\phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi).$$

To solve the "exterior problem," we look for a solution of the form

$$u(r,\phi) = \sum_{n=0}^{\infty} u_n^*(r,\phi) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\phi).$$

We can use the orthogonality of the Legendre polynomials (see §11.6) to satisfy the boundary condition: we can find formulas for the coefficients A_n and B_n. See the book for details.

The rest of the week: more examples and review.